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Existence of Commutativity in Parikh and Anti-Diagonal Matrices

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Abstract. Intensive investigation on various theoretical properties of Parikh matrices has been taken place in the theory of formal languages. This paper deals with the problem of finding properties of words so that their Parikh matrices and Anti-diagonal matrices commute.

Keywords: Parikh mapping, Parikh matrix, Anti-Diagonal matrix, ratio property, Subword.

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1. Introduction

In this article, the existence of commutative property in Parikh matrix is discussed. We introduce the transpose property of an Anti-diagonal matrix and check the transpose property for the same.

We assume that the reader is familiar with the basics of formal languages. Whenever necessary, [10] may be consulted. The number of occurrences of a word u as a subword in a word w, in symbols, $|w|_u$. The term subword means that w, as a sequence of letters, contains u as a subsequence. This means that there exist words $x_1, \dots, x_k$ and $y_0, \dots, y_k$ some of them possibly empty, such that $u = x_1, \dots, x_k$ and $w = y_0 x_1 y_1 \dots x_k y_k$. Subwords in this sense are often called scattered subwords.

The Parikh matrix of a word which has been recently introduced [6] as an extension of the notion of Parikh vector gives more numerical information about the word in terms of certain subwords (also called scattered subwords [9]) than given by the Parikh vector. Since the introduction of this notion of Parikh matrix of a word, there has been an intensive investigation on various theoretical properties of Parikh matrices (See for example [2,3,4,5,6,7]).

The Parikh matrix mapping introduced in [2] is a morphism ψ_{M_k}: $\Sigma^* \rightarrow M_{k+1}$ where M_{k+1} is a collection of $(k+1)$-dimensional upper-triangular matrices with nonnegative integral entries and unit diagonal. The classical Parikh vector $\psi(w)$ appears in the image matrix as the second diagonal.

An Anti-diagonal matrix introduced in [8] is a morphism δ_{M_k}: $\Sigma^* \rightarrow M_{k+1}$ where $M_{k+1}$is a collection of $(k+1)$-dimensional right lower-triangular matrices with nonnegative integral entries and right diagonal is unit diagonal. The classical Anti-diagonal vector $\delta(w)$ appears in the image matrix as the right lower second diagonal.

To get more information about a word, one has to focus the attention to subwords and factors. In this article, these notions are understood as follows.

2. Preliminaries

In this section, we recall subwords, Parikh matrices and anti-diagonal matrices of a word.

Subwords

Let Σ be an alphabet. The set of all words over Σ is denoted $\Sigma^*$and the empty word is λ. If $w \in \Sigma^*$, then $|w|$denotes the length of w.

Subword is denoted by $|w|_u$ the *number of occurrences* of word $u$as a subword in w, that is the number of mappings that can be defined with respect to the above definition.

For instance, $|abba|_{ba}$ = 2 and $|aabbc|_{abc}$ = 4.

Parikh matrices

The notion of Parikh matrix was introduced in [2]. All definitions and results presented in this subsection can be found in [1, 2, 3]. We recall the definition of a Parikh matrix mapping introduced and studied in [2]

Definition 2.1. Let $\Sigma = \{a_1 < a_2 < \cdots < a_k\}$ be an ordered alphabet. The *Parikh matrix mapping,* denoted $\psi_{\Sigma,\mathrm{k}}$, is the monoid morphism: $\psi_{\Sigma,\mathrm{k}}: (\Sigma^*, \cdot, \lambda) \rightarrow (M_{k+1}, \cdot, I_{k+1})$, defined by the condition: if $\psi_{\Sigma,\mathrm{k}}(a_q) = (m_{i,j})_{1 \le i,j \le (k+1)}$, then for each $1 \le i \le (k+1)$,

$m_{i,i} = 1$, $m_{q,q+1} = 1$, and all other elements of the matrix $\psi_{\Sigma,\mathrm{k}}(a_q)$are 0.

For the ordered alphabet $\Sigma = \{a_1 < a_2 < \cdots < a_k\}$, we denote by $a_{i,j}$ the word $a_i a_{i+1} \ldots a_j$, where $1 \le i \le j \le k$.

The following theorem characterizes the entries of the Parikh matrix:

Theorem 2.1. Let $\Sigma = \{a_1 < a_2 < \cdots < a_k\}$be an ordered alphabet and $w \in \Sigma^*$. The matrix $\psi_\Sigma(w) = (m_{i,j})_{1 \le i,j \le (k+1)}$, has the following properties:

- $m_{i,j} = 0$, for all $1 \le j < i \le (k+1)$,
- $m_{i,i} = 1$, for all $1 \le i \le (k+1)$,
- $m_{i,j+1} = |w|_{a_{i,j}}$, for all $1 \le i \le j \le k$.

Let $M = (m_{i,j})_{1 \le i,j \le k}$be a triangular matrix. The *alternate matrix of* M, denoted by $\bar{M}$, is the matrix $\bar{M} = (m'_{i,j})_{1 \le i,j \le k}$, where $m'_{i,j} = (-1)^{i+j}(M_{i,j})$ for all $1 \le i,j \le k$. The *reverse of* M, denoted by $M^{(rev)}$, is the matrix $M^{(rev)} = (m''_{i,j})_{1 \le i,j \le k}$, where $m''_{i,j} = m_{k+1-j,k+1-i}$, for all $1 \le i < j \le k$. (The entries below the main diagonal are the same in M and $M^{(rev)}$.)

Definition 2.2. [9] Two words w_1, w_2 over $\Sigma = \{a < b < c\}$ are said to satisfy the **ratio property**, written $w_1 \sim_r w_2$, if

$$\psi_3(w_1) = \begin{pmatrix} 1 & p_1 & p_{1,2} & p_{1,3} \\ 0 & 1 & p_2 & p_{2,3} \\ 0 & 0 & 1 & p_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and}$$

$$\psi_3(w_2) = \begin{pmatrix} 1 & q_1 & q_{1,2} & q_{1,3} \\ 0 & 1 & q_2 & q_{2,3} \\ 0 & 0 & 1 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ satisfy the condition:}$$

$p_i = s.q_i (i = 1,2,3)$, $p_{i,i+1} = s.q_{i,i+1}, (i = 1,2)$, where s is a constant.

Now we recall the new notion of Anti-diagonal matrix of a word.

Anti-diagonal matrix

The notion of anti-diagonal matrix was introduced in [8]. All definitions and results presented in this subsection can be found in [8]. The definition of the anti-diagonal matrix mapping presented below uses a special type of matrix product, called *Anti-diagonal matrix product.*

The notion of an anti-diagonal matrix of a word consider ternary alphabets only. For integers a_i and b_i $(1 \le i \le 3)$, we define the product

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} = a_3 b_1 + a_2 b_2 + a_1 b_3.$$ For two matrices

$$M_1 = \begin{pmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{pmatrix}, \ M_2 = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix}$$

we define $M_1 * M_2 = \begin{pmatrix} c_1 & c_4 & c_7 \\ c_2 & c_5 & c_8 \\ c_3 & c_6 & c_9 \end{pmatrix}$ where $c_i = \begin{pmatrix} a_i \\ a_{i+1} \\ a_{i+2} \end{pmatrix} \begin{pmatrix} b_i & b_{i+1} & b_{i+2} \end{pmatrix}$ for $i = 1,4,7$

where the product defining c_i is as defined above.

Let $\Sigma = \{a, b, c\}$ with $a < b < c$. Then we define

$$M_4(a) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$M_4(b) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ and

$M_4(c) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$.

This can be formulated as an anti-diagonal matrix δ_3 for a ternary word w as

$$\delta_3(w) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & |w|_a \\ 0 & 1 & |w|_b & |w|_{ab} \\ 1 & |w|_c & |w|_{bc} & |w|_{abc} \end{pmatrix}$$

where $|w|_a$ is the number of a in w, $|w|_b$ is the number of b in w,$|w|_c$ is the number of c in w,$|w|_{ab}$ is the number of ab in w, $|w|_{bc}$ is the number of bc in w and $|w|_{abc}$ is the number of abc in w.

For example,

$M_4(abc) = \delta_3(a) * \delta_3(b) * \delta_3(c)$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Using anti-diagonal matrix product, we get

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Ratio property in an anti-diagonal matrix

Definition 2.3. Two words w_1, w_2 over $\Sigma = \{a < b < c\}$ are said to satisfy the ratio property, written as

$w_1 \sim_r w_2$, if $\delta_3(w_1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & p_1 \\ 0 & 1 & p_2 & p_{1,2} \\ 1 & p_3 & p_{2,3} & p_{1,3} \end{pmatrix}$ and $\delta_3(w_2) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & q_1 \\ 0 & 1 & q_2 & q_{1,2} \\ 1 & q_3 & q_{2,3} & q_{1,3} \end{pmatrix}$

satisfy the condition $p_i = s.q_i (i = 1,2,3)$, $p_{i,i+1} = s.q_{i,i+1}$,$(i = 1,2)$, where s is a constant.

3. Existence of commutativity in Parikh matrices

In this section, we examine commutative property in Parikh matrix of a word.

Lemma 3.1. Let $\Sigma = \{a, b\}$ with $a < b$ and $w_1, w_2 \in \Sigma$. If $w_2$was got by replacing the word $\boldsymbol{ab}$ to $\boldsymbol{ba}$ **(**or) $\boldsymbol{ba}$ to $\boldsymbol{ab}$ in any one place ofw_1, then $\psi_2(w_1 w_2) = \psi_2(w_2 w_1)$.

Proof: Let $\Sigma = \{a, b\}$ with $a < b$ and $w_1, w_2 \epsilon \Sigma$. Let $\psi_2(w_1) = \begin{pmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix}$.

In $\psi_2(w_1)$**,** $|w_1|_a = p$, $|w_1|_b = q$ and $|w_1|_{ab}$ = r.

Given that w_2 is replacing the word $\boldsymbol{ab}$ to $\boldsymbol{ba}$ **(**or) $\boldsymbol{ba}$ to $\boldsymbol{ab}$ in any one place ofw_1.

We get$\psi_2(w_2) = \begin{pmatrix} 1 & p & s \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix}$.

In $\psi_2(w_2)$**,** $|w_2|_a = p$, $|w_2|_b = q$ and $|w_2|_{ab}$ = s. $\psi_2(w_1 w_2) = \psi_2(w_1)\psi_2(w_2)$

$$= \begin{pmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & p & s \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix}$$

Using matrix multiplication, we get

$$= \begin{pmatrix} 1 & 2p & s + pq + r \\ 0 & 1 & 2q \\ 0 & 0 & 1 \end{pmatrix}$$

$\psi_2(w_2 w_1) = \psi_2(w_2)\psi_2(w_1)$

$$= \begin{pmatrix} 1 & p & s \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2p & r + pq + s \\ 0 & 1 & 2q \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, $\psi_2(w_1 w_2) = \psi_2(w_2 w_1)$.

Now we illustrate the above result.

Remark 3.1. Note that we consider binary alphabets in the above results in Parikh matrix of words. But this result is not possible for three letter words in Parikh matrix.

Lemma 3.2. Let $\Sigma_3 = \{a, b, c\}$ with $a < b < c$. Letw_1, w_2, w_3 and $w_4 \in \Sigma_3^*$ such that w_1, w_2, w_3 and w_4 satisfy the ratio property $w_1 \sim_r w_2$, $w_3 \sim_r w_1$ and $w_4 \sim_r w_2$. Then $w_1 \sim_r w_4$, $w_3 \sim_r w_4$ and $w_3 \sim_r w_2$.

Proof: Let w_1, w_2, w_3 and w_4 over $\Sigma_3 = \{a < b < c\}$ are said to satisfy the **ratio property**, written $w_1 \sim_r w_2$, $w_3 \sim_r w_1$ and $w_4 \sim_r w_2$.

If $\psi_3(w_1) = \begin{pmatrix} 1 & p_1 & p_{1,2} & p_{1,3} \\ 0 & 1 & p_2 & p_{2,3} \\ 0 & 0 & 1 & p_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

$\psi_3(w_2) = \begin{pmatrix} 1 & q_1 & q_{1,2} & q_{1,3} \\ 0 & 1 & q_2 & q_{2,3} \\ 0 & 0 & 1 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

$\psi_3(w_3) = \begin{pmatrix} 1 & u_1 & u_{1,2} & u_{1,3} \\ 0 & 1 & u_2 & u_{2,3} \\ 0 & 0 & 1 & u_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and

$\psi_3(w_4) = \begin{pmatrix} 1 & v_1 & v_{1,2} & v_{1,3} \\ 0 & 1 & v_2 & v_{2,3} \\ 0 & 0 & 1 & v_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

If $w_1 \sim_r w_2$, we have $\frac{p_1}{q_1} = \frac{p_2}{q_2} = \frac{p_3}{q_3} = \frac{p_{1,2}}{q_{1,2}} = \frac{p_{2,3}}{q_{2,3}} = s$ (1)

If $w_3 \sim_r w_1$, we have $\frac{u_1}{p_1} = \frac{u_2}{p_2} = \frac{u_3}{p_3} = \frac{u_{1,2}}{p_{1,2}} = \frac{u_{2,3}}{p_{2,3}} = t$ (2)

If $w_4 \sim_r w_2$, we have $\frac{v_1}{q_1} = \frac{v_2}{q_2} = \frac{v_3}{q_3} = \frac{v_{1,2}}{q_{1,2}} = \frac{v_{2,3}}{q_{2,3}} = l$ (3)

By $eqn.,$ (1)

$\frac{p_1}{q_1}$ = s

Using $eqn.,$ (3), we get

$\frac{p_1}{\frac{v_1}{l}} = s => \frac{lp_1}{v_1} = s => \frac{p_1}{v_1} = \frac{s}{l} = k \qquad [\because \frac{s}{l} = k]$

$$\frac{p_1}{v_1} = k$$

Likewise, $\frac{p_2}{v_2} = \frac{p_3}{v_3} = \frac{p_{1,2}}{v_{1,2}} = \frac{p_{2,3}}{v_{2,3}} = k$, which means $w_1 \sim_r w_4$

By $eqn.,$ (2)

$$\frac{u_1}{p_1} = t$$

Using $eqn.,$ (1), we get

$$\frac{u_1}{sq_1} = t => \frac{u_1}{q_1} = st$$

Using $eqn.,$ (3), we get

$\frac{u_1}{\frac{v_1}{l}} = st => \frac{lu_1}{v_1} = st => \frac{u_1}{v_1} = \frac{st}{l} = m \qquad [\because \frac{st}{l} = k]$

$$\frac{u_1}{v_1} = m$$

Likewise, $\frac{u_2}{v_2} = \frac{u_3}{v_3} = \frac{u_{1,2}}{v_{1,2}} = \frac{u_{2,3}}{v_{2,3}} = m$, which means $w_3 \sim_r w_4$

By $eqn.,$ (2)

$$\frac{u_1}{p_1} = t$$

Using $eqn.,$ (1), we get

$$\frac{u_1}{sq_1} = t => \frac{u_1}{q_1} = st = n$$

Likewise, $\frac{u_2}{q_2} = \frac{u_3}{q_3} = \frac{u_{1,2}}{q_{1,2}} = \frac{u_{2,3}}{q_{2,3}} = n$, which means $w_3 \sim_r w_2$

Hence, $w_1 \sim_r w_4$, $w_3 \sim_r w_4$ and $w_3 \sim_r w_2$.

Theorem 3.1. Let $\Sigma_3 = \{a, b, c\}$ with $a < b < c$. Let w_1, w_2, w_3 and $w_4 \in \Sigma_3^*$ such that w_1, w_2, w_3 and w_4 satisfy the ratio property $w_1 \sim_r w_2$, $w_3 \sim_r w_1$ and $w_4 \sim_r w_2$.
Then (i) $\psi_3(w_1 w_4) = \psi_3(w_4 w_1)$,
(ii) $\psi_3(w_3 w_4) = \psi_3(w_4 w_3)$ and
(iii) $\psi_3(w_2 w_3) = \psi_3(w_3 w_2)$
Proof: Let $\Sigma_3 = \{a, b, c\}$ with $a < b < c$. Let $w_1, w_2, w_3, w_4 \in \Sigma_3^*$ and w_1, w_2, w_3 and w_4 satisfy the ratio property such that $w_1 \sim_r w_2$, $w_3 \sim_r w_1$ and $w_4 \sim_r w_2$.
From Lemma 3.2, since $w_1 \sim_r w_4$, $w_3 \sim_r w_4$ and $w_3 \sim_r w_2$.
If $w_1 \sim_r w_4$, From i) of Lemma 1 in [9]
Hence $\psi_3(w_1 w_4) = \psi_3(w_4 w_1)$.
Likewise, ii) and iii) are proved.

4. Existence of commutativity in an anti-diagonal matrices

In this section we examine transpose property in an Anti-diagonal matrix of a word.

Proposition 4.1. The transpose of an Anti-diagonal matrix is also an anti-diagonal matrix.

Remark 4.1. Here the Anti-diagonal matrix differs from Parikh matrix as its transpose is not at all a Parikh matrix.

Corollary 4.1. Transpose of an Anti-diagonal matrix is not the same as the reverse matrix as given in Parikh matrices [2].

Example 4.1. Let $\Sigma = \{a < b\}$ and $w \in \Sigma$. Let $w = abaabab$,

$$\delta_2(w) = \delta_2(abaabab) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 3 & 8 \end{pmatrix}$$

$$\delta_2(w^t) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 3 & 8 \end{pmatrix}^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 4 & 8 \end{pmatrix}$$

Since $(w)^{rev} = (abaabab)^{rev} = babaaba$, we get $\delta_2(w^{rev}) = \delta_2(babaaba) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 3 & 4 \end{pmatrix}$

Therefore, $\delta_2(w^t) \neq \delta_2(w^{rev})$

Theorem 4.1. Let $\Sigma = \{a, b\}$ with $a < b$. The word sw_1, w_2 over Σ satisfy transpose property, then $[\delta_2(w_1 * w_2)]^t = [\delta_2(w_2)]^t * [\delta_2(w_1)]^t$. Here we mean $\delta_2(w_1 * w_2)$ as $\delta_2(w_1) * \delta_2(w_2)$.
Proof: Let $\Sigma = \{a, b\}$ with $a < b$. Let $w_1, w_2 \in \Sigma$.

$$\delta_2(w_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & p \\ 1 & q & r \end{pmatrix}$$ and

$\delta_2(w_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & l \\ 1 & m & n \end{pmatrix}$

$\delta_2(w_1 * w_2) = \delta_2(w_1) * \delta_2(w_2)$

$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & p \\ 1 & q & r \end{pmatrix} * \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & l \\ 1 & m & n \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & p+l \\ 1 & q+m & r+pm+n \end{pmatrix}$ as anti-diagonal matrix product

$[\delta_2(w_1 * w_2)]^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & p+l \\ 1 & q+m & r+pm+n \end{pmatrix}^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & q+m \\ 1 & p+l & r+pm+n \end{pmatrix}$

$[\delta_2(w_1)]^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & p \\ 1 & q & r \end{pmatrix}^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & q \\ 1 & p & r \end{pmatrix}$ and

$[\delta_2(w_2)]^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & l \\ 1 & m & n \end{pmatrix}^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & m \\ 1 & l & n \end{pmatrix}$

$[\delta_2(w_2)]^t * [\delta_2(w_1)]^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & m \\ 1 & l & n \end{pmatrix} * \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & q \\ 1 & p & r \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & m+q \\ 1 & l+p & n+mp+r \end{pmatrix}$ as anti-diagonal matrix product.

Hence $[\delta_2(w_1 * w_2)]^t = [\delta_2(w_2)]^t * [\delta_2(w_1)]^t$**.**

Example 4.2. Let $w_1 = abaabab$, $\delta_2(w_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 3 & 8 \end{pmatrix}$ and $w_2 = ababa$ and

$\delta_2(w_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix}$.

$\delta_2(w_1 * w_2) = \delta_2(w_1) * \delta_2(w_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 3 & 8 \end{pmatrix} * \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 7 \\ 1 & 5 & 19 \end{pmatrix}$ as Anti-diagonal matrix product

$[\delta_2(w_1 * w_2)]^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 5 \\ 1 & 7 & 19 \end{pmatrix}$

$[\delta_2(w_1)]^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 3 & 8 \end{pmatrix}^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 4 & 8 \end{pmatrix}$ and

$$[\delta_2(w_2)]^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix}^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix}$$

$$[\delta_2(w_2)]^t * [\delta_2(w_1)]^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix} * \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 4 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 5 \\ 1 & 7 & 19 \end{pmatrix}$$

Hence, $[\delta_2(w_1 * w_2)]^t = [\delta_2(w_2)]^t * [\delta_2(w_1)]^t$.

Theorem 4.2. Let $\Sigma = \{a, b\}$ with $a < b$. Let $w \epsilon \Sigma$ satisfying transpose property, then $[\delta_2(w * mi(w))]^t = [\delta_2(mi(w))]^t * [\delta_2(w)]^t$.

Proof: Let $\Sigma = \{a, b\}$ with $a < b$. Let $w \epsilon \Sigma$.

$\delta_2(w) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & p \\ 1 & q & r \end{pmatrix}$ and

$\delta_2(mi(w)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & p \\ 1 & q & s \end{pmatrix}$

The proof is similar to the proof of Theorem 4.1

5. Conclusion

The topic of Parikh matrices is a promising area of research related to combinatorics on words. In this paper we have obtained certain criteria for words so that their Parikh matrices and Anti-diagonal matrices are commute because of the non-commutativity of words. Most of problems are difficulties to handle mathematically in formal languages.

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