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On the Negative Pellian Equation y^2=110 x^2-29

***R.Suganya*[1] and *D.Maheswari*[2]**

Department of Mathematics, Shrimati Indira Gandhi College
Trichy-620002, Tamilnadu, India.
[1]e-mail: suganyaccc@yahoo.in; [2]e-mail: matmahes@gmail.com

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Abstract. The hyperbola represented by the binary quadratic equation y^2=110 x^2-29 is analyzed for finding its non-zero distinct integer solutions. A few interesting relations among its solutions are presented. Also knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. Also, employing the solutions of the given equation, special pythagorean triangle is constructed.

Keywords: binary quadratic, hyperbola, negative pell equation, integral solution.

AMS Mathematics Subject Classification (2010): 11D09

1. Introduction

Diophantine equation of the form $y^2 = Dx^2 + 1$, where D is a given positive square-free integer is known as pell equation and is one of the oldest Diophantine equation that has interested mathematicians all over all the world, since antiquity, J.L Lagrange proved that all positive pell equation $y^2 = Dx^2 + 1$ has infinitely many distinct integer solutions where as the negative pell equation $y^2 = Dx^2 - 1$does not always have a solution. In [1], an elementary proof of a ceriterium for the solvability of the pell equation $x^2 - Dy^2 = -1$ where D is any positive non-square integer has been presented. For examples the equations $y^2 = 3x^2 - 1, y^2 = 7x^2 - 4$ have no integer solutions, whereas $y^2 = 65x^2 - 1$, $y^2 = 202x^2 - 1$ have integer solutions. In this context, one may refer [2-12]. More specifically, one may refer “The online encyclopedia of Integer sequences” (A031396, A130226, A031398) for values of D for which the negative pell equation $y^2 = Dx^2 - 1$ is solvable or not. In this communication, the negative pell equation given by $y^2=110x^2-29$ is considered and infinitely many integer solutions are obtained. A few interesting relations among the solution are presented. Also knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. Also employing the solutions of the given equation, special pythagorean triangle is constructed.

2. Method of analysis

The negative Pell equation representing hyperbola under consideration is

$$y^2 = 110x^2 - 29 \tag{1}$$

whose smallest positive integer solution is

$$x_0 = 1, y_0 = 9$$

To obtain, the other solutions of (1), consider the pell equation

$$y^2 = 110x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{g_n}{2\sqrt{110}}, \tilde{y}_n = \frac{f_n}{2}$$

where

$$f_n = \left(21 + 2\sqrt{110}\right)^{n+1} + \left(21 - 2\sqrt{110}\right)^{n+1},$$

$$g_n = \left(21 - 2\sqrt{110}\right)^{n+1} - \left(21 - 2\sqrt{110}\right)^{n+1}, \quad n = -1, 0, 1, 2\ldots\ldots$$

Applying Brahmagupta lemma between the solutions of (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{9}{2\sqrt{110}} g_n \tag{2}$$

$$y_{n+1} = \frac{9}{2} f_n + \frac{110}{2\sqrt{110}} g_n \tag{3}$$

Thus (2) and (3) represent the non-zero distinct integer solutions of (1)

The recurrence relations satisfied by the values of x_{n+1} and y_{n+1} are respectively.

$$x_{n+3} - 42x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 42y_{n+2} + y_{n+1} = 0$$

A few numerical examples are given in the following table 1:

Table 1: Examples

n	x_{n+1}	y_{n+1}
-1	1	9
0	39	409
1	1637	17169
2	68715	720689

2.1. A few interesting relations among the solutions are given below

- $2y_{n+2} - 21x_{n+2} + x_{n+1} = 0$
- $638y_{n+3} - 281039x_{n+2} + 6699x_{n+1} = 0$
- $42x_{n+2} - x_{n+1} - x_{n+3} = 0$
- $84y_{n+1} - x_{n+3} + 881x_{n+1} = 0$

- $4y_{n+2} - x_{n+3} + x_{n+1} = 0$
- $84y_{n+3} - 881x_{n+3} + x_{n+1} = 0$
- $x_{n+2} - 21x_{n+1} - 2y_{n+1} = 0$
- $y_{n+2} - 220x_{n+1} - 21y_{n+1} = 0$
- $y_{n+3} - 9240x_{n+1} - 881y_{n+1} = 0$
- $66990x_{n+2} - 18236819x_{n+1} - 38201y_{n+2} = 0$
- $66990x_{n+3} - 765478179x_{n+1} - 1603461y_{n+2} = 0$
- $33495y_{n+1} - 2101055x_{n+1} + 4400y_{n+2} = 0$
- $66990y_{n+3} - 8028402470x_{n+1} + 16817240y_{n+2} = 0$
- $281039y_{n+2} - 6699y_{n+3} + 70180x_{n+1} = 0$
- $6380x_{n+1} - 1888590x_{n+2} + 44990x_{n+3} = 0$
- $2y_{n+2} - x_{n+3} + 21x_{n+2} = 0$
- $6699x_{n+3} - 281039x_{n+2} - 638y_{n+1} = 0$
- $21y_{n+2} - 220x_{n+2} - y_{n+1} = 0$
- $y_{n+1} - 21y_{n+2} - 2y_{n+2} = 0$
- $y_{n+3} - 220x_{n+2} - 21y_{n+2} = 0$
- $6699x_{n+1} - 281039x_{n+2} + 638y_{n+3} = 0$
- $13398y_{n+1} - 562078y_{n+2} + 140360x_{n+3} = 0$
- $y_{n+1} - 881y_{n+3} + 9240x_{n+3} = 0$
- $y_{n+2} - 21y_{n+3} + 220x_{n+3} = 0$
- $70180x_{n+3} - 281039y_{n+2} + 6699y_{n+1} = 0$
- $140360x_{n+1} - 13398y_{n+3} + 562078y_{n+2} = 0$
- $y_{n+1} - 42y_{n+2} + y_{n+3} = 0$

2.2. Each of the following expressions represents a Nasty number

- $\frac{6}{58}\left[116 + 818x_{2n+2} - 18x_{2n+3}\right]$
- $\frac{6}{2436}\left[4872 + 34338x_{2n+2} - 18x_{2n+4}\right]$
- $\frac{6}{29}\left[58 + 220x_{2n+2} - 18y_{2n+2}\right]$
- $\frac{6}{609}\left[1218 + 8580x_{2n+2} - 18y_{2n+3}\right]$

- $\frac{6}{25549}\left[51098+360140x_{2n+2}-18y_{2n+4}\right]$
- $\frac{6}{58}\left[116+34338x_{2n+3}-818x_{2n+4}\right]$
- $\frac{6}{609}\left[1218+220x_{2n+3}-818y_{2n+2}\right]$
- $\frac{6}{29}\left[58+8580x_{2n+3}-818y_{2n+3}\right]$
- $\frac{6}{609}\left[1218+360140x_{2n+3}-818y_{2n+4}\right]$
- $\frac{6}{25549}\left[51098+220x_{2n+4}-34338y_{2n+2}\right]$
- $\frac{6}{609}\left[1218+8580x_{2n+4}-34338y_{2n+3}\right]$
- $\frac{6}{29}\left[58+360140x_{2n+4}-34338y_{2n+4}\right]$
- $\frac{6}{6380}\left[12760+220y_{2n+3}-8580y_{2n+2}\right]$
- $\frac{6}{267960}\left[535920+220y_{2n+4}-360140y_{2n+2}\right]$
- $\frac{6}{6380}\left[12760+8580y_{2n+4}-360140y_{2n+3}\right]$

2.3. Each of the following expressions represents a cubical integer

- $\frac{1}{58}\left[\left(818x_{3n+3}-18x_{3n+4}\right)+3\left(818x_{n+1}-18x_{n+2}\right)\right]$
- $\frac{1}{2436}\left[\left(34338x_{3n+3}-18x_{3n+5}\right)+3\left(34338x_{n+1}-18x_{n+3}\right)\right]$
- $\frac{1}{29}\left[\left(220x_{3n+3}-18y_{3n+3}\right)+3\left(220x_{n+1}-18y_{n+1}\right)\right]$
- $\frac{1}{609}\left[\left(8580x_{3n+3}-18y_{3n+4}\right)+3\left(8580x_{n+1}-18y_{n+2}\right)\right]$
- $\frac{1}{25549}\left[\left(360140x_{3n+3}-18y_{3n+5}\right)+3\left(360140x_{n+1}-18y_{n+3}\right)\right]$
- $\frac{1}{58}\left[\left(34338x_{3n+4}-818x_{3n+5}\right)+3\left(34338x_{n+2}-818x_{n+3}\right)\right]$
- $\frac{1}{609}\left[\left(220x_{3n+4}-818y_{3n+3}\right)+3\left(220x_{n+2}-818y_{n+1}\right)\right]$

- $\frac{1}{29}[(8580x_{3n+4}-818y_{3n+4})+3(8580x_{n+2}-818y_{n+2})]$
- $\frac{1}{609}[(360140x_{3n+4}-818y_{3n+5})+3(360140x_{n+2}-818y_{n+3})]$
- $\frac{1}{25549}[(220x_{3n+5}-34338y_{3n+3})+3(220x_{n+3}-34338y_{n+1})]$
- $\frac{1}{609}[(8580x_{3n+5}-34338y_{3n+4})+3(8580x_{n+3}-34338y_{n+2})]$
- $\frac{1}{29}[(360140x_{3n+5}-34338y_{3n+5})+3(360140x_{n+3}-34338y_{n+3})]$
- $\frac{1}{6380}[(220y_{3n+4}-8580y_{3n+3})+3(220y_{n+2}-8580y_{n+1})]$
- $\frac{1}{267960}[(220y_{3n+5}-360140y_{3n+3})+3(220y_{n+3}-360140y_{n+1})]$
- $\frac{1}{6380}[(8580y_{3n+5}-360140y_{3n+4})+3(8580y_{n+3}-360140y_{n+2})]$

2.4. Each of the following expressions represents a bi-quadratic integer

$\frac{1}{58^2}\left[(47444x_{4n+4}-1044x_{4n+5})+4(818x_{n+1}-18x_{n+2})^2-6728\right],$

$\frac{1}{2436^2}\left[(83647368x_{4n+4}-438948x_{4n+6})+4(34338x_{n+1}-18x_{n+3})^2-11868192\right],$

$\frac{1}{29^2}\left[(6380x_{4n+4}-522y_{4n+4})+4(220x_{n+1}-18y_{n+1})^2-1682\right],$

$\frac{1}{609^2}\left[(5225220x_{4n+4}-10962y_{4n+5})+4(8580x_{n+1}-18y_{n+2})^2-741762\right],$

$\frac{1}{25549^2}\left[(9201216860x_{4n+4}-459882y_{4n+6})+4(360140x_{n+1}-18y_{n+3})^2-1305298441\right],$

$\frac{1}{58^2}\left[(1991604x_{4n+5}-47444x_{4n+6})+4(34338x_{n+2}-818x_{n+3})^2-6782\right],$

$\frac{1}{609^2}\left[(133980x_{4n+5}-498162y_{4n+4})+4(220x_{n+2}-818y_{n+1})^2-741762\right],$

$\frac{1}{29^2}\left[(248820x_{4n+5}-23722y_{4n+5})+4(8580x_{n+2}-818y_{n+2})^2-1682\right],$

$\frac{1}{609^2}\left[(219325260x_{4n+5}-498162y_{4n+6})+4(360140x_{n+2}-818y_{n+3})^2-741762\right],$

$$\frac{1}{25549^2}\begin{bmatrix}(5620780x_{4n+6}-877301562y_{4n+4})+4(220x_{n+3}-34338y_{n+1})^2\\ -1305502802\end{bmatrix},$$

$$\frac{1}{609^2}\begin{bmatrix}(5225220x_{4n+6}-20911842y_{4n+5})+4(8580x_{n+3}-34338y_{n+2})^2\\ -741762\end{bmatrix},$$

$$\frac{1}{29^2}\left[(10444060x_{4n+6}-995802y_{4n+6})+4(360140x_{n+3}-34338y_{n+3})^2-1682\right],$$

$$\frac{1}{6380^2}\left[(1403600y_{4n+5}-54740400y_{4n+4})+4(220y_{n+2}-8580y_{n+1})^2-81408800\right],$$

$$\frac{1}{267960^2}\begin{bmatrix}(58951200y_{4n+6}-9650311440y_{4n+4})+4(220y_{n+3}-360140y_{n+1})^2\\ -14360512320\end{bmatrix},$$

$$\frac{1}{6380^2}\begin{bmatrix}(54740400y_{4n+6}-2297693200y_{4n+5})+4(8580y_{n+3}-360140y_{n+2})^2\\ -81408800\end{bmatrix}.$$

3. Remarkable observations

3.1. Employing the linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the table2 below:

Table 2: Hyperbola

S. No	Hyperbola	(X_n, Y_n)
1	$110X_n^2 - Y_n^2 = 1480160$	$X_n = 818x_{n+1} - 18x_{n+2}$ $Y_n = 220x_{n+2} - 8580x_{n+1}$
2	$110X_n^2 - Y_n^2 = 2611002240$	$X_n = 34338x_{n+1} - 18x_{n+3}$ $Y_n = 220x_{n+3} - 360140x_{n+1}$
3	$110X_n^2 - Y_n^2 = 370040$	$X_n = 220x_{n+1} - 18y_{n+1}$ $Y_n = 220y_{n+1} - 1980x_{n+1}$
4	$110X_n^2 - Y_n^2 = 163187640$	$X_n = 8580x_{n+1} - 18y_{n+2}$ $Y_n = 220y_{n+2} - 89980x_{n+1}$
5	$110X_n^2 - Y_n^2 = 287210616440$	$X_n = 360140x_{n+1} - 18y_{n+3}$ $Y_n = 220y_{n+3} - 3777180x_{n+1}$
6	$110X_n^2 - Y_n^2 = 1480160$	$X_n = 34338x_{n+2} - 818x_{n+3}$ $Y_n = 8580x_{n+3} - 360140x_{n+2}$

7	$110X_n^2 - Y_n^2 = 163187640$	$X_n = 220x_{n+2} - 818y_{n+1}$ $Y_n = 8580y_{n+1} - 1980x_{n+2}$
8	$110X_n^2 - Y_n^2 = 370040$	$X_n = 8580x_{n+2} - 818y_{n+2}$ $Y_n = 8580y_{n+2} - 89980x_{n+2}$
9	$110X_n^2 - Y_n^2 = 163187640$	$X_n = 360140x_{n+2} - 818y_{n+3}$ $Y_n = 8580y_{n+3} - 3777180x_{n+2}$
10	$110X_n^2 - Y_n^2 = 287210616440$	$X_n = 220x_{n+3} - 34338y_{n+1}$ $Y_n = 360140y_{n+1} - 1980x_{n+3}$
11	$110X_n^2 - Y_n^2 = 163187640$	$X_n = 8580x_{n+3} - 34338y_{n+2}$ $Y_n = 360140y_{n+2} - 89980x_{n+3}$
12	$110X_n^2 - Y_n^2 = 370040$	$X_n = 360140x_{n+3} - 34338y_{n+3}$ $Y_n = 360140y_{n+3} - 3777180x_{n+3}$
13	$110X_n^2 - Y_n^2 = 17909936000$	$X_n = 220y_{n+2} - 8580y_{n+1}$ $Y_n = 89980y_{n+1} - 1980y_{n+2}$
14	$110X_n^2 - Y_n^2 = 31593127104000$	$X_n = 220y_{n+3} - 360140y_{n+1}$ $Y_n = 3777180y_{n+1} - 1980y_{n+3}$
15	$110X_n^2 - Y_n^2 = 17909936000$	$X_n = 8580y_{n+3} - 360140y_{n+2}$ $Y_n = 3777180y_{n+2} - 89980y_{n+3}$

3.2. Employing the linear combination among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the table 3 below:

Table 3: Parabola

S. No	Parabola	(X_n, Y_n)
1	$6380X_n - Y_n^2 = 1480160$	$X_n = 818x_{2n+2} - 18x_{2n+3} + 116$ $Y_n = 220x_{n+2} - 8580x_{n+1}$
2	$267960X_n - Y_n^2 = 2611002240$	$X_n = 34338x_{2n+2} - 18x_{2n+4} + 4872$ $Y_n = 220x_{n+3} - 360140x_{n+1}$
3	$3190X_n - Y_n^2 = 370040$	$X_n = 220x_{2n+2} - 18y_{2n+2} + 58$ $Y_n = 220y_{n+1} - 1980x_{n+1}$
4	$66990X_n - Y_n^2 = 163187640$	$X_n = 8580x_{2n+2} - 18y_{2n+3} + 1218$ $Y_n = 220y_{n+2} - 89980x_{n+1}$

5	$2810390X_n - Y_n^2 = 28721061640$	$X_n = 360140x_{2n+1} - 18y_{2n+4} + 51098$ $Y_n = 220y_{n+3} - 3777180x_{n+1}$
6	$6380X_n - Y_n^2 = 1480160$	$X_n = 34338x_{2n+3} - 818x_{2n+4} + 116$ $Y_n = 8580x_{n+3} - 360140x_{n+2}$
7	$66990X_n - Y_n^2 = 163187640$	$X_n = 220x_{2n+3} - 818y_{2n+2} + 1218$ $Y_n = 8580y_{n+1} - 1980x_{n+2}$
8	$3190X_n - Y_n^2 = 370040$	$X_n = 8580x_{2n+3} - 818y_{2n+3} + 58$ $Y_n = 8580y_{n+2} - 89980x_{n+2}$
9	$66990X_n - Y_n^2 = 163187640$	$X_n = 360140x_{2n+3} - 818y_{2n+4} + 1218$ $Y_n = 8580y_{n+3} - 3777180x_{n+2}$
10	$281039X_n - Y_n^2 = 28721061640$	$X_n = 220x_{2n+4} - 34338y_{2n+2} + 51098$ $Y_n = 360140y_{n+1} - 1980x_{n+3}$
11	$66990X_n - Y_n^2 = 163187640$	$X_n = 8580x_{2n+4} - 34338y_{2n+3} + 1218$ $Y_n = 360140y_{n+2} - 89980x_{n+3}$
12	$3190X_n - Y_n^2 = 370040$	$X_n = 360140x_{2n+4} - 34338y_{2n+4} + 58$ $Y_n = 360140y_{n+3} - 3777180x_{n+3}$
13	$701800X_n - Y_n^2 = 17909936000$	$X_n = 220y_{2n+3} - 8580y_{2n+2} + 12760$ $Y_n = 89980y_{n+1} - 1980y_{n+2}$
14	$29475600X_n - Y_n^2 = 315931271040$	$X_n = 220y_{2n+4} - 360140y_{2n+2} + 535920$ $Y_n = 3777180y_{n+1} - 1980y_{n+3}$
15	$701800X_n - Y_n^2 = 17909936000$	$X_n = 8580y_{2n+4} - 360140y_{2n+3} + 12760$ $Y_n = 3777180y_{n+2} - 89980y_{n+3}$

4. Conclusion

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the negative pell equation $y^2 = 110x^2 - 29$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of negative pell equations and determine their integer solutions along with suitable properties.

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