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Integral Points on the Cone $7x^2$-$3y^2$=$16z^2$

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Abstract. The cone represented by the ternary quadratic Diophantine equation $7x^2 - 3y^2 = 16z^2$ is analyzed for its patterns of non-zero distinct integral solutions. A few interesting properties between the solutions and special polygonal numbers are exhibited.

Keywords: Ternary quadratic, cone, integral solutions.

AMS Mathematics Subject Classification (2010): 11D09

1. Introduction

The ternary homogeneous quadratic Diophantine equation offers an unlimited field for research because of their variety [1-2]. For an extensive review of various problems, one may refer [3-5]. In this context one may also see [6-9] for integer solutions satisfying special three dimensional graphical representation. This communication concerns with yet another interesting ternary quadratic equation $7x^2 - 3y^2 = 16z^2$ representing a cone for determining its infinitely many non zero integer solutions. A few interesting properties among the solution and special numbers are presented. Also, given an integer solution, three different triples of integer generating infinitely many integer solutions are exhibited.

2. Notations used

- Polygonal number of rank **n** with size **m**

$$T_{m,n} = n[1 + \frac{(n-1)(m-2)}{2}]$$

- Pyramidal number of rank **n** with size **m**

$$P_n^m = \frac{1}{6}[n(n+1)][(m-2)n + (5-m)]$$

- Pronic number of rank n

$$\Pr_n = n(n+1)$$

- Centered polygonal number of rank **n** with **m**

$$Ct_{m,n} = \frac{mn(n-1)+2}{2}$$

3. Method of analysis

The ternary quadratic Diophantine equation to be solved for its non-zero distinct integral solution is $7x^2 - 3y^2 = 16z^2$ (1)

To start with it is observed that (1) is satisfied by the following two non-zero integer triples: $(16k-8, 24k-12, 2k-1), (28k-14, 36k-29, 10k-5)$. However, we have the other choices of solutions which are illustrated below.

3.1. PATTERN-1

Introduce the linear transformations

$$x = X + 3T,\ y = X + 7T \tag{2}$$

in (1) leads to,

$$X^2 = 21T^2 + (2Z)^2 \tag{3}$$

which can be written as,

$$(X+2Z)(X-2Z) = 21T^2 \tag{4}$$

The equation (4) is written as the system of two equations as follows

System	1	2	3
$X-2Z$	T^2	$7T^2$	$3T^2$
$X-2Z$	21	3	7

System-1:

Consider,

$$X + 2Z = T^2$$
$$X - 2Z = 21$$

Solving these two equations we get,

$$\left.\begin{aligned} X &= 2k^2 - 2k + 11 \\ Z &= k^2 - k - 5 \\ T &= 2k - 1 \end{aligned}\right\} \tag{5}$$

Substituting (5) in (2), we get the corresponding non-zero distinct integer solutions to (1) as follows:

$$x = 2k^2 + 4k + 8$$
$$y = 2k^2 + 12k + 4$$
$$z = k^2 - k - 5$$

Properties:

- $x(k) + y(k) - 8t_{3,4} + 1 \equiv 0 \pmod 4$
- $6 * y(2)$ is a nasty number
- $x(k) + z(k) - 3\,\mathrm{Pr}_k \equiv 0 \pmod 3$

System-2:

Consider,

$$X + 2Z = 7T^2$$

$$X-2Z=3$$

Solving these two equations we get,

$$\left.\begin{aligned} X &= 14k^2-14k+5 \\ Z &= 7k^2-7k+1 \\ T &= 2k-1 \end{aligned}\right\} \tag{6}$$

Substituting (6) in (2), we get the corresponding non-zero distinct integer solutions to (1) as follows:

$$x=14k^2-8k+2$$
$$y=14k^2-2$$
$$z=7k^2-7k+1$$

Properties:

- $x(k)+y(k)-3T_{12,k}-3Ct_{4,k}\equiv 0(\bmod 3)$
- $y(k)-z(k)-7\,\mathrm{Pr}_k\equiv 0(\bmod 3)$
- $x(1)-CP_{2,6}=0$

System-3:

Consider,

$$X+2Z=3T^2$$
$$X-2Z=7$$

Solving these two equations we get,

$$\left.\begin{aligned} X &= 6k^2-6k+5 \\ Z &= 3k^2-3k-1 \\ T &= 2k-1 \end{aligned}\right\} \tag{7}$$

Substituting (7) in (2), we get the corresponding non-zero distinct integer solutions to (1) as follows,

$$x=6k^2+2$$
$$y=6k^2+8k-2$$
$$z=3k^2-3k-1$$

Properties:

- $x(k)+y(k)-4T_{8,k}\equiv 0(\bmod 4)$
- $x(k)-z(k)-3k\,\mathrm{Pr}_k\equiv 0(\bmod 3)$
- $2y(1)$ is a nasty number.

3.2. Pattern-2

Rewrite (4) in the form of ratio as

$$\frac{X+2Z}{21}=\frac{T^2}{X-2Z}=\frac{\alpha}{\beta},\beta\neq 0 \tag{8}$$

which is equivalent to the following two equations

$$\left.\begin{aligned} \beta X + 2\beta Z - 21\alpha = 0 \\ \alpha X - 2\alpha Z - \beta T^2 = 0 \end{aligned}\right\} \tag{9}$$

Employing the method of cross multiplication we get,

$$X = -2(\beta^2 + 21\alpha^2)$$
$$Z = \beta^2 + 21\alpha^2$$
$$T = -4\alpha\beta$$

Thus in view of (2), we get the non-zero distinct integral solution (1) are obtained by

$$x = -2\beta^2 - 42\alpha^2 - 12\beta\alpha$$
$$y = -2\beta^2 - 42\alpha^2 - 28\beta\alpha$$
$$z = \beta^2 - 21\alpha^2$$

Properties:

- $-x(\alpha,1) - 2T_{4,\alpha} - 42 \equiv 0 \pmod 2$
- $z(1,\beta) - T_{4,\beta} \equiv 0 \pmod 3$
- $6(x(\alpha,\alpha) - y(\alpha,\alpha))$ is a nasty number.

Pattern 2(a)

In addition to (4) is expressed in the form of ratio as

$$\frac{X+2Z}{3} = \frac{7T^2}{X-2Z} = \frac{\alpha}{\beta}, \beta \neq 0 \tag{10}$$

Following the procedure as above the corresponding solutions of (1) are presented by

$$x = -14\beta^2 - 6\alpha^2 - 12\beta\alpha$$
$$y = -14\beta^2 - 6\alpha^2 - 28\beta\alpha$$
$$z = 7\beta^2 - 3\alpha^2$$

Properties:

- $6 * (-2z(\alpha^2,1) - y(\alpha^2,1) - 24T_{3,\alpha})$ is a nasty number.
- $-x(1,\beta) - y(1,\beta) - 28\,\mathrm{Pr}_{\beta} - 12 \equiv 0 \pmod 2$
- $z(\alpha,\alpha+1) - T_{10,\alpha} - 7 \equiv 1 \pmod 2$

Pattern 2(b)

In addition to equation (4) is expressed in the form of ratio as

$$\frac{X+2Z}{7} = \frac{3T^2}{X-2Z} = \frac{\alpha}{\beta}, \beta \neq 0 \tag{11}$$

Following the procedure as above the corresponding solutions are

$$x = -6\beta^2 - 14\alpha^2 - 12\beta\alpha$$
$$y = -6\beta^2 - 14\alpha^2 - 28\beta\alpha$$
$$z = 3\beta^2 - 7\alpha^2$$

Properties:

- $z(1,\beta) - x(1,\beta) - 3T_{8,\beta} - 7 \equiv 0 \pmod 6$

- $-y(\alpha,\alpha+2)-128t_{3,\alpha}+1-12\equiv 0(\bmod 2)$
- $x(\alpha,1)-+z(\alpha,1)+3St_{3,\alpha}+1+9T_{4,n}=0$

3.3. Pattern-3

Equation (3) is also satisfied by

$$\left.\begin{aligned} T &= 2mn \\ 2Z &= 21m^2-n^2 \\ X &= 21m^2+n^2 \end{aligned}\right\} \tag{12}$$

as our interest centres an integer solutions,we choose $m=2M,n=2N$ in (12), we get

$$T=8MN$$
$$Z=42M^2-2N^2$$
$$X=84M^2+4N^2$$

Thus in view of (2), corresponding the non- zero distinct integer solutions of equation (1) are presented by

$$x=84M^2+4N^2+24MN$$
$$y=84M^2+4N^2+56MN$$
$$z=42M^2-2N^2$$

Properties:

- $60z(M,M)$ is a nasty number.
- $24[x(M,M)]-12[y(M,M)]=1408T_{4,M}$
- $28x(M,M)-12y(M,M)+z(M,M)-2896T_{3,M}\equiv 0(\bmod 4)$

3.4. Remarkable obsevations-4

I. If the non-zero integer triple (x_0,y_0,z_0) is any solutions of (1) then each of the following of non-zero distinct integer solution on also satisfies (1)

Triple 1: (X_n,Y_n,Z_0)

Let x_0,y_0,z_0 be the initial solution of (1)

Let

$$\left.\begin{aligned} x_1 &= x_0+2h \\ y_1 &= y_0+3h \\ z_1 &= z_0 \end{aligned}\right\} \tag{13}$$

be the second solution of (1), where h is a non-zero integer to be determined.

Then, from(1), we get

$$h=18y_0-28x_0$$

$$\therefore x_1=-55x_0+36y_0$$
$$y_1=-84x_0-55y_0$$

Hence the matrix representation of above solution is,

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{bmatrix} -55 & 36 \\ -84 & 55 \end{bmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

where A= $\begin{bmatrix} -55 & 36 \\ -84 & 55 \end{bmatrix}$

Repeating the above process the general value for x and y are given by

$$A^n = \begin{bmatrix} -27 & 18 \\ -42 & 28 \end{bmatrix} + (-1)^n \begin{bmatrix} 28 & -18 \\ 42 & -27 \end{bmatrix}$$

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{pmatrix} -27+28(-1)^n & 18(1-(-1)^n) \\ 42+(-1+(-1)^n) & 28-27(-1)^n \end{pmatrix} \begin{pmatrix} x_0 \\ y_o \end{pmatrix}$$

Thus the n^{th} solution as

$$x_n = (-27+28(-1)^n)x_o + 18(1-(-1)^n)y_o$$
$$y_n = 42(-1+(-1)^n)x_o + 28 - 27(-1)^n y_0$$
$$z_n = z_o$$

Triple 2: (X_n, Y_0, Z_n)

Let

$$\left.\begin{aligned} x_1 &= x_0 + 3h \\ y_1 &= y_0 \\ z_1 &= z_o + 2h \end{aligned}\right\} \qquad (14)$$

be the second solution of (1).

Following the procedure as above, the corresponding integer solutions to (1) is given by

$$x_n = (64+63(-1)^n)x_o - 96(1+(-1)^n)z_o$$
$$y_n = y_0$$
$$z_n = 42(1+(-1)^n)x_o - (63+64(-1)^n)z_0$$

Triple 3: (X_0, Y_n, Z_n)

Let

$$\left.\begin{aligned} x_1 &= 24x_0 \\ y_1 &= 24y_0 - 4h \\ z_1 &= 24z_o + h \end{aligned}\right\} \qquad (15)$$

In this case the following procedure as above the corresponding integer solutions to (1) is given by,

$$x_n = 24^n x_0$$

$$y_n = \left[\frac{(24)^n}{48}(12) + \frac{(-24)^n}{(-48)}(-36)\right]y_0 + \left[\frac{(24)^n}{48}(48) + \frac{(-24)^n}{(-48)}(48)\right]z_0$$

$$z_n = \left[\frac{(24)^n}{48}(9) + \frac{(-24)^n}{(-48)}(9)\right]y_0 + \left[\frac{(24)^n}{48}(36) + \frac{(-24)^n}{(-48)}(-12)\right]z_0$$

II. Employing the solutions (x, y, z) of (1) each of following expressions among the special polygonal, pyramidal, central polygonal and pronic numbers is a perfect square

- $7\left[\frac{18P_{x-2}^3}{Ct_{6,x-2} - 1}\right]^2 - 3\left[\frac{3P_y^3}{t_{3,y+1}}\right]^2$
- $7\left[\frac{3P_x^3}{t_{3,x}}\right]^2 - 3\left[\frac{6P_y^5}{Ct_{6,y} - 1}\right]^2$
- $7\left[\frac{6P_x^4}{t_{3,2x}}\right]^2 - 3\left[\frac{3(P_y^4 - P_y^3)}{t_{3,y}}\right]^2$
- $7\left[\frac{P^5 x}{t_{3,x}}\right]^2 - 3\left[\frac{2P^5 y}{t_{4,y}}\right]^2$
- $7\left[\frac{6P_x^3}{\Pr_x}\right]^2 - 3\left[\frac{P_y^3}{t_{3,y}}\right]^2$

6. Conclusion

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic Diophantine equation represented by $7x^2 - 3y^2 = 16z^2$. As quadratic equations are rich in variety, one may search for their choices of quadratic equation with variables greater than or equal to 3 and determine their properties through special numbers.

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