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Integral Solutions of Homogeneous Biquardratic Equations with Five Unknowns $2(x^4-y^4)=(z^2-w^2)p^2$

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Abstract. An attempt has been made to obtain pattern of non-zero distinct integral solutions to the homogeneous biquadratic equation with five unknowns represented by $2\left(x^4-y^4\right)=\left(z^2-w^2\right)p^2$ is analyzed and various interesting relations between the solutions and special numbers namely polygonal numbers , pyramidal numbers are exhibited .

Keywords: Homogeneous biquadratic, biquadratic with five unknowns, integral solutions.

AMS Mathematics Subject Classification (2010): 11D25

1. Introduction

The Theory of Diophantine Equations offer a rich variety of fascinating problems . In particular biquadratic Diophantine homogeneous and non-homogeneous have aroused the interest of numerous mathematicians. Since antiquity $[1-3]$. In this context, one may refer $[4-7]$ for various problems on the biquadratic Diophantine equations. However often we come across homogenous biquadratic equations and as such are may require its integral solutions in its required general form this paper concern with the homogenous biquadratic equations with five unknowns equations for determining its infinitely many non-zero integral solutions. Also a few interesting properties among the solutions are presented.

2. Notations

1. Polygonal number of rank n with sides m

$$t_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$$

2. Pronic number of rank n

$$PR_n=n(n+1)$$

3. Centered hexagonal pyramidal number of rank n

$$CP_{n,6} = n^3$$

4. Centered polygonal number of rank n with m sides

$$Ct_{m.n} = \frac{mn(n-1)+2}{2}$$

5. Stellaoctangular number of rank n

$$So_n = n(2n^2 - 1)$$

3. Method of analysis

The equations representing the biquadratic equation to be solved for its non-zero distinct integer solution is

$$2(x^4 - y^4) = (z^2 - w^2)p^2 \qquad (1)$$

The substitution of linear transformation.

$$x = u + v,\ y = u - v\ ,\ z = u + 4v,\ w = u - 4v \qquad (2)$$

in (1) leads to

$$u^2 + v^2 = p^2 \qquad (3)$$

Assume that

$$p = a^2 + b^2 \qquad (4)$$

3.1. Pattern-1

Rewrite (3) as

$$u^2 + v^2 = p^2 * 1 \qquad (5)$$

Write 1 as

$$1 = \frac{(3+4i)(3-4i)}{25} \qquad (6)$$

Substituting (4) and (6) in (5) and using the method of factorization we get ,

$$(u+iv)(u-iv) = (a+ib)^2 (a-ib)^2\ \frac{(3+4i)(3-4i)}{25} \qquad (7)$$

Equating the positive and negative factors , the resulting equation are

$$(u+iv) = (a+ib)^2\ \frac{(3+4i)}{5} \qquad (8)$$

$$(u-iv) = (a-ib)^2\ \frac{(3-4i)}{5} \qquad (9)$$

Equating Real and Imaginary parts in (8)

$$\left.\begin{aligned} u &= \frac{1}{5}\left[3a^2 - 3b^2 - 8ab\right] \\ v &= \frac{1}{5}\left[4a^2 - 4b^2 + 6ab\right] \end{aligned}\right. \qquad (10)$$

Choose $a=5A$, $b=5B$ in (10)

$$u=15A^2-15B^2-30AB$$
$$v=20A^2-20B^2+30AB \qquad (11)$$

The non-zero distinct integer solutions of (1) are presented by

$$x=35A^2-35B^2-10AB$$
$$y=-5A^2+5B^2-70AB$$
$$z=95A^2-95B^2+80AB$$
$$w=-95A\ \ +95B^2-160AB$$
$$p=25A^2+25B^2$$

Properties:

1. $x(A,A)+y(A,A)+80PR_A\equiv 0(\bmod 5)$
2. $z(1,B)-w(1,B)-2Ct_{240,B}-240PR_A\equiv 0(\bmod 2)$
3. $x(A,A+1)-2Ct_{35,A}+10PR_n\equiv -2(\bmod 35)$

3.2. Pattern-2

The substitution of the linear transformation

$x=u+v$, $y=u-v$, $z=2u+2v$, $w=2u-2v$

in (1) leads to

$$u^2=p^2-v^2 \qquad (12)$$

which is equivalent to

$$u^2=(p+v)(p-v) \qquad (13)$$

which is expressed in the form of solution as

$$\frac{p+v}{u}=\frac{u}{p-v}=\frac{m}{n};n\succ 0 \qquad (14)$$

Now (14) is equivalent to that two equations

$$mu-np-nv=0$$
$$nu+mv-mp=0$$

Applying the method of cross ratio, we get

$$u=2mn$$
$$v=m^2-n^2$$
$$p=m^2+n^2$$

Thus in view of (2) the non-zero integral solutions of (1) are given by

$$x=2mn+m^2-n^2$$
$$y=2mn-m^2+n^2$$
$$z=4mn+2m^2-2n^2$$
$$w=4mn-2m^2+2n^2$$
$$p=m^2+n^2$$

Properties:

1.$z(1,1)+w(1,1)-CP_{2,r}=0$

2.$x(m,m)+y(m,m)-4t_{2,m}=0$

3.$x(m,m+1)+y(m,m+1)-4PR_{m}=0$

3.3. Pattern-3

Instead of (2), one may consider the following transformation

$x=u+v$, $y=u-v$, $z=2uv+2$, $w=2uv-2$

Following the procedure presented in pattern 2, the other choices of integral solutions of (1) are obtained by

$$x=2mn+m^2-n^2$$
$$y=2mn-m^2+n^2$$
$$z=4mn(m^2-n^2)+2$$
$$w=4mn(m^2-n^2)-2$$
$$p=m^2+n^2$$

Properties:

1.$x(m,m)+y(m,m)-4t_{4,m}=0$

2.$z(m,1)+w(m,1)-8Cp_{m,6}\equiv 0(\bmod 8)$

3.$x(m,1)+w(m,1)-2S0_{m}-t_{4,m}\equiv 0(\bmod 3)$

3.4. Pattern-4

Instead of (2), one may consider the following transformation

$x=u+v$, $y=u-v$, $z=4uv+1$, $w=4uv-1$

Following the procedure presented in pattern 2, the other choices of integral solution of (1) are presented by

$$x=2mn+m^2-n^2$$
$$y=2mn-m^2+n^2$$
$$z=8mn(m^2-n^2)+1$$
$$w=8mn(m^2-n^2)-1$$
$$p=m^2+n^2$$

Properties:

1.$p(m,1)-t_{4,m}-1=0$

2.$x(m,1)-2y(m,1)-t_{8,m}\equiv 0(\bmod 3)$

3.$w(m,1)-8CP_{m,6}\equiv -1(\bmod 8)$

3.5. Pattern-5

(3) is also satisfied by

$$u=2rs$$

$v = r^2 - s^2$
$p = r^2 + s^2$

Substituting the values of u, v in (2), the non-zero integer solutions of (1) are given below

$x = 2rs + r^2 - s^2$
$y = 2rs - r^2 + s^2$
$z = 2rs + 4r^2 - 4s^2$
$w = 2rs - 4r^2 + 4s^2$
$p = r^2 + s^2$

Properties:

1. $6(x(1,1) + y(1,1))$ is a nasty number
2. $x(r,1) + z(r,1) - 4PR_r - t_{4,r} + 5 = 0$
3. $y(1,s) - w(1,s) - 5t_{4,s} \equiv 0 \pmod 5$

4. Remarkable observations

Triple 1:

Let u_0, v_0, p_0 be the initial solution of (3)

$u_1 = nu_0 + h,\ v_1 = nv_0 + h,\ p_n = np_0$ (15)

be the 2^{nd} solution of (3) where h is a non-zero integer to be determined

Then, from (3), we get

$h = -nu_0 - nv_0$

$u_1 = -nv_0, v_1 = -nu_0, p_1 = np_0$

Hence the matrix representation of the above solution is

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 & -n \\ -n & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 0 & -n \\ -n & 0 \end{bmatrix}$$

Repeating the above process the general values for u and v are given by

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = A^n \begin{bmatrix} u_0 \\ v_o \end{bmatrix}$$

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left[(n)^n + (-n)^n\right] & -\frac{1}{2}\left[(n)^n - (-n)^n\right] \\ -\frac{1}{2}\left[(n)^n - (-n)^n\right] & \frac{1}{2}\left[(n)^n + (-n)^n\right] \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

Thus, the n^{th} solution as by

$$u_n = \frac{1}{2}\left[(n)^n + (-n)^n\right]u_0 - \frac{1}{2}\left[(n)^n - (-n)^n\right]v_0$$

$$v_n = -\frac{1}{2}\left[(n)^n - (-n)^n\right]u_0 + \frac{1}{2}\left[(n)^n + (-n)^n\right]v_0$$

In view of (2), the general solution of (1) is given by

$$x_n = u_n + v_n$$

$$x_n = u_0(-n)^n + v_0(-n)^n$$

$$y_n = u_n - v_n$$

$$y_n = u_0(n)^n - v_0(n)^n$$

$$z_n = u_n + 4v_n$$

$$z_n = u_0\left[\frac{1}{2}\left[(n)^n + (-n)^n\right] - 2\left[(n)^n - (-n)^n\right]\right] + v_0\left[-\frac{1}{2}\left[(n)^n - (-n)^n\right] + 2\left[(n)^n + (-n)^n\right]\right]$$

$$w_n = u_n - 4v_n$$

$$w_n = u_0\left[\frac{1}{2}\left[(n)^n + (-n)^n\right] + 2\left[(n)^n - (-n)^n\right]\right] - v_0\left[\frac{1}{2}\left[(n)^n - (-n)^n\right] + 2\left[(n)^n + (-n)^n\right]\right]$$

$$p_n = np_0$$

Triple 2:

Let u_0, v_0, p_0 be the initial solution of (3)

$$u_1 = nu_0 + h,\ v_1 = nv_0 - h,\ p_1 = np_0 \tag{16}$$

Following the procedure as above the corresponding integer solutions of (1) are given by

$$x_n = u_0(n)^n + v_0(n)^n$$

$$y_n = u_0(-n)^n - v_0(-n)^n$$

$$z_n = u_0\left[\frac{1}{2}\left[(n)^n + (-n)^n\right] + 2\left[(n)^n - (-n)^n\right]\right] + v_0\left[\frac{1}{2}\left[(n)^n - (-n)^n\right] + 2\left[(n)^n + (-n)^n\right]\right]$$

$$w_n = u_0\left[\frac{1}{2}\left[(n)^n + (-n)^n\right] - 2\left[(n)^n - (-n)^n\right]\right] + v_0\left[\frac{1}{2}\left[(n)^n - (-n)^n\right] - 2\left[(n)^n + (-n)^n\right]\right]$$

$$p_n = np_0$$

Triple 3:

Let u_0, v_0, p_0 be the initial solution of (3)

$$u_1 = nu_0 - h,\ v_1 = nv_0 - h,\ p_1 = np_0 \tag{17}$$

In this case, the corresponding integer solution (1) is given by

$$x_n = u_0(n)^n + v_0(n)^n$$

$$y_n = u_0(n)^n - v_0(n)^n$$

$$z_n = u_0\left[\frac{1}{2}\left[(n)^n + (-n)^n\right] - 2\left[(n)^n - (-n)^n\right]\right] - v_0\left[\frac{1}{2}\left[(n)^n - (-n)^n\right] - 2\left[(n)^n + (-n)^n\right]\right]$$

$$w_n = u_0\left[\frac{1}{2}\left[(n)^n + (-n)^n\right] + 2\left[(n)^n - (-n)^n\right]\right] - v_0\left[\frac{1}{2}\left[(n)^n - (-n)^n\right] + 2\left[(n)^n + (-n)^n\right]\right]$$

$$p_n = np_0$$

5. Conclusion

To conclude one may consider biquadratic equation withmultivariables $(\succ,5)$ and search for their non-zero distinct integer solutions along with their corresponding properties.

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