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(n,m) -Power Quasi Normal Operators in Semi-Hilbertian Spaces

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Abstract. Let H be a Hilbert space and let A be a positive bounded operator on H. The Semi Inner Product $\langle \xi | \eta \rangle_A := \langle \xi | \eta \rangle$, ξ ,η∈ H induces a Semi norm $\|.\|_A$on H. This makes H into a Semi- Hilbertian Space. In this Manuscript we introduce the Generalization of Normal Operator named as (n,m) - power quasi normal operators [6] in Semi - Hilbertian Space $(H, \|.\|_A)$ with the help of the papers like positive normal, A-normal, A-quasi normal, (A,n)-power quasi normal and also isometries in the same space. Now we generalize the above papers and named it as (n,m) - power quasi normal operators in semi - hilbertian spaces.

Keywords: Semi–Hilbertian space, A- Self Adjoint, A- normal, (A,n)- power quasi normal operators.

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1. Introduction

A bounded linear operator T on a complex Hilbert space is quasi-normal if T and T^*T commute. The class of quasi-normal operators was first introduced and studied by A.Brown [5] in 1953. From the definition, it is easily seen that this class contains normal operators and isometries. In [6] the author introduced some generalization of normal powers operators in Hilbert Space and studied some of the properties. The Purpose of the present paper is to study the class of (n,m)- power quasi normal Operators in Semi-Hilbertian Spaces. Through out this Manuscript, H denotes the complex Hilbert Space with Inner Product $\langle \cdot | \cdot \rangle$ $B(H)$ is the algebra of all bounded linear Operators on H, $B(H)^+$ is the cone of Positive (semi definite) Operators of $B(H)$ i.e., $B(H)^+ := \{A \in B(H): \langle A\xi | \xi \rangle \geq 0, \forall \xi \in H\}$.

2. Preliminaries

For every $T \in B(H), N(T), R(T)$ and $\overline{R(T)}$ denotes the null space, the range space and the closure of the range of T and its adjoint operator byT^*. Given the positive operator $A \in B(H)^+$, we consider the Semi Inner Product $\langle . | . \rangle_A : H \times H \to C$ defined by

$\langle\xi \mid \eta\rangle_A = \langle A\xi|\eta\rangle \forall\xi,\eta \in H$. In general this semi inner product induces a semi norm, $\|.\|_A$, defined by $\|\xi\|_A = (\langle A\xi \mid \xi\rangle)^{\frac{1}{2}} = \left\|A^{\frac{1}{2}}\xi\right\|$.

It is easily seen that $\|.\|_A$ is a norm on H if and only if A is injective, and the semi – normed space $(B(H), \|.\|_A$ is complete if and only if $R(A)$ is closed.

The above semi norm induces a semi norm on the subspace $B^A(H)$ of $B(H)$
$B^A(H) := \{T \in B(H)/\exists c > 0: \|T\xi\|_A \leq c\|\xi\|_A \forall\xi \in H\}$.
Operators in $B^A(H)$ are called A- bounded operators.

3. Main results

The set of all A- bounded operators which admit an A-adjoint is denoted by $B_A(H)$. By Douglas theorem [2,8] we have that

$$B_A(H) = \{T \in B(H)/R(T^*A)cR(A)\}$$

If $T \in B_A(H)$, then there exists a distinguished A- adjoint operator of T, namely the reduced solution of equation $AX = T^*A$, that is A^lT^*A. We denote this operator by $T^\#$. There fore $T^\# = A^lT^*A$ and
$AT^\# = T^*A$, $R(T^\#)c\overline{R(A)}$ and$N(T^\#) = N(T^*A)$.

Here A^l denotes the Moore - penrose inverse of A. It is defined as the unique linear extension of $\check{A}^{-1}$ where $\check{A}$ is the isomorphism. We are having four Moore – penrose equations such that

$$AXA = A, XAX = X, XA = P_{N(A)^\perp} AX = P_{\overline{R(A)}}\ |_{D(A^l)}$$

1. Let $T \in B_A(H)$. Then the following statements hold.
   a) $T^\# \in B_A(H)$, $(T^\#)^\# = P_{\overline{R(A)}}TP_{\overline{R(A)}}$and $(T^\#)^\#)^\# = T^\#$.
   b) If $S \in B_A(H)$ then $TS \in B_A(H)$and $(TS)^\# = S^\#T^\#$.
   c) $T^\#T$and$TT^\#$ are A-self adjoint.
   d) $\|T\|_A = \left\|T^\#\right\|_A = \left\|T^\#T\right\|_A^{\frac{1}{2}} = \left\|TT^\#\right\|_A^{\frac{1}{2}}$
   e) $\|S\|_A = \left\|T^\#\right\|_A$for every $S \in B(H)$ which is an A- adjoint of T.
   f) If $S \in B_A(H)$then $\|TS\|_A = \|ST\|_A$.
      2. $(A^t)^\# = A^t$for every $t > 0$, if $A \in B(H)^+$ and if $AT = TA$ then $T^\# = PT^*$, here $P = P_{\overline{R(A)}}$ [8].

The classes of normal, quasi normal, n- quasi normal, isometries, partial isometries, etc., on Hilbert Spaces have been generalized to semi-Hilbertain spaces by many authors in many papers.

In the following definition we collect the notions of some classes of A- Operators.

2. For $T \in B(H)$, an operator $S \in B(H)$ is called an A-adjoint of T if for every $\xi,\eta \in H$ we have,$\langle T\xi \mid \eta\rangle_A = \langle\xi \mid S\eta\rangle_A$ i.e.,$AS = T^*A$.

If T is an adjoint of itself, then T is called an A- self adjoint operator($AT = T^*A$).

Definitions 3.1.

1. A- isometry if $T^*AT = A \Leftrightarrow \|T\xi\|_A = \|\xi\|_A, \forall\xi \in H$
2. A- normal if $T^*AT = TAT^* \Leftrightarrow \|T\xi\|_A = \|T^*\xi\|_A, \forall\xi \in H$
3. A- partial isometry if $\|T\xi\|_A = \|\xi\|_A \forall\xi \in N(AT)^{\perp A}$
4. A- unitary if for any $\xi \in H$,then$T^*AT = TAT^* = A \Leftrightarrow \|T^*\xi\|_A = \|T\xi\|_A = \|\xi\|_A$

5. A- hyponormal if $TAT^* \leq T^*AT \Leftrightarrow \|T^*\xi\|_A \leq \|T\xi\|_A, \forall \xi \in H$

4. $A-(n,m)$- power quasi normal operator in semi-hilbertian spaces $(H, \langle . | . \rangle_A)$

In this section we introduce the concept of $A-(n,m)$- power quasi normal operators in semi-Hilbertian spaces.

Definition 4.1. An Operator$T \in B_A(H)$ is said to be (n,m)- quasi normal operators in for a positive integer n,m if

$T^n(T^\#)^m T = (T^\#)^m T T^n$ (1)

Remark: Every n-quasi normal is $(n,1)$- quasi normal but the converse, every $(n,1)$-quasi normal is not n-quasi normal by [6].

The equality (1) can be written as $[T^n(T^\#)^m - (T^\#)^m T^n]T = 0$. We called it as $A-(n,m)$-power quasi normal operators in Semi-Hilbertian Space.

Definition 4.2. We say that $T \in B(H)$ is an A- positive if $AT \in B(H)^+$ (or) equivalently $\langle T\xi | \xi \rangle_A \geq 0 \ \forall \xi \in H$.

Remark. The Inequality Cauchy-Schwartz for A- positive operator. If $T \in B(H)$ is A- positive, then

$$|\langle T\xi|\eta\rangle_A|^2 \leq \langle T\xi|\eta\rangle_A \langle T\xi|\eta\rangle_A \forall \xi, \eta \in H$$

Result 4.3. $T \in B_A(H)$is said to be A- normal then

$$R(TT^\#) \subset \overline{R(A)} \Leftrightarrow \|T^\# T\xi\|_A = \|TT^\#\xi\|_A \forall \xi \in H$$

Lemma 4.4. Let $T \in B_A(H)$, then T is $A-(n,m)$- quasi normal operator if and only if $\|T^{m\#}\xi\|_A = \|T^{n+1}\xi\|_A$.

Proof: Let us choose the (n,m)- quasi normal in the following way to prove the result.

$$[T^n(T^\#)^m - (T^\#)^m T^n]T = 0$$

$$\Leftrightarrow \langle A[T^n(T^\#)^m - (T^\#)^m T^n]T\xi | \xi \rangle = 0$$

$$\Leftrightarrow \langle AT^n(T^\#)^m T\xi | \xi \rangle = \langle A(T^\#)^m (T^n T)\xi | \xi \rangle$$

$$\Leftrightarrow \langle AT^n(T^{m\#})T\xi | \xi \rangle = \langle (T^\#)^m A(T^n T)\xi | \xi \rangle$$

$$\Leftrightarrow \langle AT^{\#m} | T^{\#m}\xi \rangle = \langle T^{n+1} | AT^{n+1}\xi \rangle$$

$$\Leftrightarrow \|T^{m\#}\xi\|_A^2 = \|T^{n+1}\|_A^2$$

$$\Leftrightarrow \|T^{m\#}\xi\|_A = \|T^{n+1}\xi\|_A.$$

Proposition 4.5. Let $T, S \in B_A(H)$ are $A-(n,m)$-quasi normal operators. Then the following property hold.

If $S(T^{m\#}T) = (T^{m\#}T)S$and $(ST)^n = S^n T^n$, then TS is $A-(n,m)$-quasi normal operator.

Proof: For all $\xi \in H$, we have that

$$\left\|(TS)^{m\#}TS\xi\right\|_A = \left\|S^{m\#}T^{m\#}TS\xi\right\|_A$$
$$\leq \left\|S^{m\#}ST^{m\#}T\xi\right\|_A$$
$$\leq \left\|S^{m\#}ST^{m\#}T\xi\right\|_A$$
$$\left\|(TS)^{m\#}TS\xi\right\|_A \leq \left\|((TS)^{m+1})^{\#}\xi\right\|_A$$

Proposition 4.6. Let $T \in B_A(H), B = T^n + T^{m\#}T$and $C = T^n - T^{m\#}T$. Then we have $T \in A - (n, m)$-quasi normal if and only if B commutes with .

Proof: We have $BC = CB$

$\Leftrightarrow(T^n + T^{m\#}T)\big(T^n - T^{m\#}T\big)$ =$(T^n - T^{m\#}T)(\,T^n + T^{m\#}T)$

$\Leftrightarrow(T^{2n} - T^{n+1}T^{m\#} + T^{m\#}T^{n+1} - (T^{m\#}T)^2$

= $(T^{2n} + T^{n+1}T^{m\#} - T^{m\#}T^{n+1} - (T^{m\#}T)^2$

$\Leftrightarrow(T^nT^{m\#}T = T^{m\#}TT^n)$

Theorem 4.7. Let S, T be two $A - (n, m)$-quasi normal such that $ST = TS = S^{m\#}T = T^{\#m}S = 0$. Then $S + T$ is $A - (n, m)$-quasi normal.

Proof:

$$(S+T)^n(S+T)^{m\#}(S+T) = (S^n + T^n)(S^{m\#} + T^{m\#})(S+T)$$
$$= \big(S^nS^{m\#} + S^nT^{m\#} + T^nS^{m\#} + T^nT^{m\#}\big)(S+T)$$
$$= \big(S^{m\#}S^n + T^nT^{m\#}\big)(S+T)$$
$$= \big(S^{m\#}S^{n+1}\big) + \big(T^{n+1}T^{m\#}\big)$$
$$= (S+T)^{m\#}(S+T)^{n+1}$$

Hence proved.

5. Conclusion

Here by concluded that the Generalization of Normal Operators can also be worked in the Semi-Hilbertian Spaces. In further studies this work will be extended and serves as a tool for other works.

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