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On The Positive Pell Equation y^2=90 x^2+31

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Abstract. The binary quadratic Diophantine equation represented by the positive pellian $y^2 = 90x^2 + 31$ is analysed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and Pythagorean triangle.

***Keywords*:** Binary quadratic, hyperbola, parabola, integral solutions, pell equation.

AMS Mathematics Subject Classification (2010): 11D09.

1. Introduction

The binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been selected by various mathematicians for its non-trival integer solutions when D takes different integral values[1-4]. For an extensive review of various problems, one may refer [5-14]. In this communication, yet another an interesting equation given by $y^2 = 90x^2 + 31$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

2. Method of analysis

The positive pell equation representing hyperbola under consideration is,

$$y^2 = 90x^2 + 31 \tag{1}$$

The smallest positive integer solutions of (1) are,

$$x_0 = 1, y_0 = 11$$

The general solution (x_n, y_n) of (1) is given by

$$\tilde{y}_n = \frac{1}{2} f_n, \tilde{x}_n = \frac{1}{2\sqrt{90}} g_n$$

where,

$$f_n = \left(19+2\sqrt{90}\right)^{n+1} + \left(19-2\sqrt{90}\right)^{n+1}$$
$$g_n = \left(19+2\sqrt{90}\right)^{n+1} - \left(19-2\sqrt{90}\right)^{n+1}$$

Applying Brahamagupta Lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$2\sqrt{90}x_{n+1} = \sqrt{90}f_n + 11g_n$$
$$2y_{n+1} = 11f_n + \sqrt{90}g_n$$

The recurrence relations satisfied by the solutions x & y are given by

$$x_{n+3} - 38x_{n+2} + x_{n+1} = 0$$
$$y_{n+3} - 38y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x & y satisfying (1) are given in the Table 1 below:

Table I: Examples

N	x_n	y_n
0	1	11
1	41	389
2	1557	14771
3	59125	560909
4	2245193	21299771

From the above table, we observe some interesting relations among the solutions which are presented below:

- Both x_n *and* y_n values are odd.
- Each of the following expressions is a nasty number
  - $\dfrac{66x_{2n+3} - 228x_{2n+2} + 372}{31}$
  - $\dfrac{66x_{2n+4} - 88626x_{2n+2} + 14136}{1178}$
  - $\dfrac{132y_{2n+3} - 44280x_{2n+2} + 7068}{589}$
  - $\dfrac{132y_{2n+4} - 1681560x_{2n+2} + 268212}{22351}$
  - $\dfrac{2334x_{2n+4} - 88626x_{2n+3} + 372}{31}$
  - $\dfrac{4668y_{2n+2} - 1080x_{2n+3} + 7068}{589}$

- $\frac{4668y_{2n+3}-44280x_{2n+3}+372}{31}$
- $\frac{4668y_{2n+4}-1681560x_{2n+3}+7068}{589}$
- $\frac{177252y_{2n+2}-1080x_{2n+4}+268212}{22351}$
- $\frac{177252y_{2n+3}-44280x_{2n+4}+7068}{589}$
- $\frac{177252y_{2n+4}-1681560x_{2n+4}+11532}{961}$
- $\frac{492y_{2n+2}-12y_{2n+3}+744}{62}$
- $\frac{18684y_{2n+2}-12y_{2n+4}+28272}{2356}$
- $\frac{18684y_{2n+3}-492y_{2n+4}+744}{62}$

- Each of the following expressions is a cubical integer
  - $961(11x_{3n+4}-389x_{3n+3}+33x_{n+2}-1167x_{n+1})$
  - $1387684(11x_{3n+5}-14771x_{3n+3}+33x_{n+3}-44313x_{n+1})$
  - $346921(22y_{3n+4}-7380x_{3n+3}+66y_{n+2}-22140x_{n+1})$
  - $499567201(22y_{3n+5}-280260x_{3n+3}+66y_{3n+5}-840780x_{3n+3})$
  - $961(389x_{3n+5}-14771x_{3n+4}+1167x_{n+3}-44313x_{n+2})$
  - $346921(778y_{3n+3}-180x_{3n+4}+2334y_{n+1}-540x_{n+2})$
  - $961(778y_{3n+4}-7380x_{3n+4}+2334y_{n+2}-22140x_{n+2})$
  - $346921(778y_{3n+5}-280260x_{3n+4}+2334y_{n+3}-840780x_{n+2})$
  - $499567201(29542y_{3n+3}-180x_{3n+5}+88626y_{n+1}-540x_{n+3})$
  - $346921(29542y_{3n+4}-7380x_{3n+5}+88626y_{n+2}-22140x_{n+3})$
  - $961(29542y_{3n+5}-280260x_{3n+5}+88626y_{n+3}-840780x_{n+3})$
  - $3844(82y_{3n+3}-2y_{3n+4}+246y_{n+1}-6y_{n+2})$
  - $5550736(3114y_{3n+3}-2y_{3n+5}+9342y_{n+1}-6y_{n+3})$
  - $3844(3114y_{3n+4}-82y_{3n+5}+9342y_{n+2}-246y_{n+3})$
- Relations among the solutions
  - $x_{n+3}=38x_{n+2}-x_{n+1}$
  - $2y_{n+1}=x_{n+2}-19x_{n+1}$
  - $2y_{n+2}=19x_{n+2}-x_{n+1}$
  - $62y_{n+3}=22351x_{n+2}-589x_{n+1}$

- $76y_{n+1} = x_{n+3} - 721x_{n+1}$
- $4y_{n+2} = x_{n+3} - x_{n+1}$
- $76y_{n+3} = 721x_{n+3} - x_{n+1}$
- $19y_{n+1} = y_{n+2} - 180x_{n+1}$
- $19y_{n+3} = 721y_{n+2} + 180x_{n+1}$
- $721x_{n+2} = 2y_{n+3} + 19x_{n+1}$
- $721y_{n+1} = y_{n+3} - 6840x_{n+1}$
- $62y_{n+1} = 589x_{n+3} - 22351x_{n+2}$
- $2y_{n+2} = x_{n+3} - 19x_{n+2}$
- $2y_{n+2} = 19x_{n+3} - x_{n+2}$
- $19y_{n+2} = y_{n+1} + 180x_{n+2}$
- $y_{n+3} = 19y_{n+2} + 180x_{n+2}$
- $721x_{n+2} = 19x_{n+3} - 2y_{n+1}$
- $721y_{n+2} = 19y_{n+1} + 180x_{n+3}$
- $721y_{n+3} = y_{n+1} + 6840x_{n+3}$
- $y_{n+2} = 19y_{n+3} - 180x_{n+3}$

3. Remarkable observations

3.1. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the TableII below.

Table II: Hyperbolas

S.NO	(X, Y)	Hyperbola
1	$(41x_{n+1} - x_{n+2}, 11x_{n+2} - 389x_{n+1})$	$Y^2 - 90X^2 = 3844$
2	$(1557x_{n+1} - x_{n+3}, 11x_{n+3} - 14771x_{n+1})$	$Y^2 - 90X^2 = 5550736$
3	$(389x_{n+1} - y_{n+2}, 11y_{n+2} - 3690x_{n+1})$	$4Y^2 - 360X^2 = 1387684$
4	$(29542x_{n+1} - 2y_{n+3}, 22y_{n+3} - 280260x_{n+1})$	$Y^2 - 90X^2 = 1998268804$
5	$(1557x_{n+2} - 41x_{n+3}, 389x_{n+3} - 14771x_{n+2})$	$Y^2 - 90X^2 = 3844$
6	$(11x_{n+2} - 4y_{n+1}, 778y_{n+1} - 180x_{n+2})$	$Y^2 - 360X^2 = 1387684$
7	$(389x_{n+2} - 41y_{n+2}, 778y_{n+2} - 7380x_{n+2})$	$Y^2 - 360X^2 = 3844$
8	$(29542x_{n+2} - 82y_{n+3}, 778y_{n+3} - 280260x_{n+2})$	$Y^2 - 90X^2 = 1387684$

9	$\left(11x_{n+3}-1557y_{n+1},29542y_{n+1}-180x_{n+3}\right)$	$Y^2-360X^2=1998268804$
10	$389\left(3x_{n+3}-1557y_{n+2},29542y_{n+2}-7380x_{n+3}\right)$	$Y^2-360X^2=1387684$
11	$\left(14771x_{n+3}-1557y_{n+3},29542y_{n+3}-280260x_{n+3}\right)$	$Y^2-360X^2=3844$
12	$\left(82y_{n+1}-2y_{n+2},82y_{n+1}-2y_{n+2}\right)$	$90Y^2-X^2=138340$
13	$\left(22y_{n+3}-29542y_{n+1},3114y_{n+1}-2y_{n+3}\right)$	$90Y^2-X^2=1998264960$
14	$\left(778y_{n+3}-29542y_{n+2},3114y_{n+2}-82y_{n+3}\right)$	$90Y^2-X^2=1383840$

3.2: Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the table III below.

Table III: Parabolas

S.No	(X,Y)	Parabola
1	$\left(41x_{n+1}-x_{n+2},11x_{2n+3}-389x_{2n+2}\right)$	$90X^2=31Y-1922$
2	$\left(1557x_{n+1}-x_{n+3},11x_{2n+4}-14771x_{2n+2}\right)$	$90X^2=1178Y-2775368$
3	$\left(389x_{n+1}-y_{n+2},11y_{2n+3}-3690x_{2n+2}\right)$	$180X^2=Y-589$
4	$\left(29542x_{n+1}-2y_{n+3},22y_{2n+4}-280260x_{2n+2}\right)$	$90X^2=Y-4470$
5	$\left(3247x_{n+2}-58x_{n+3},541x_{2n+4}-30286x_{2n+3}\right)$	$90X^2=Y-1922$
6	$\left(11x_{n+2}-4y_{n+1},778y_{2n+2}-180x_{2n+3}\right)$	$360X^2=589Y-693842$
7	$\left(389x_{n+2}-41y_{n+2},778y_{2n+3}-7380x_{2n+3}\right)$	$360X^2=31Y-1922$
8	$\left(29542x_{n+2}-82y_{n+3},778y_{2n+4}-280260x_{2n+3}\right)$	$90X^2=589Y-693842$
9	$\left(11x_{n+3}-15571y_{n+1},29542y_{2n+2}-180x_{2n+4}\right)$	$360X^2=22351Y-99913440$
10	$\left(389x_{n+3}-1557y_{n+2},29542y_{2n+3}\_7380x_{2n+4}\right)$	$360X^2=589Y-693842$
11	$\left(14771x_{n+3}-1557y_{n+3},29542y_{2n+4}-280260x_{2n+\right.$	$360X^2=Y-1922$

12	$\left(82y_{n+1} - 2y_{n+2}, 82y_{2n+2} - 2y_{2n+3}\right)$	$X^2 = 5580Y - 691920$
13	$\left(22y_{n+3} - 29542y_{n+1}, 3114y_{2n+2} - 2y_{2n+4}\right)$	$X^2 = 212040Y - 999132480$
14	$\left(778y_{n+3} - 29542y_{n+1}, 3114y_{2n+3} - 82y_{2n+4}\right)$	$X^2 = 5580Y - 691920$

3.3. Consider $m = x_{n+1} + y_{n+1}, n = x_{n+1}$. Observe that $m > n > 0$. Treat m,n as the generators of the Pythagorean triangle $T(\alpha, \beta, \gamma)$, where

$$\alpha = 2mn, \beta = m^2 - n^2, \gamma = m^2 + n^2$$

Then the following interesting relations are observed.

a) $\alpha - 45\beta + 44\gamma = -31$

b) $\gamma - 46\alpha + 180\dfrac{A}{P} = 31$

c) $\dfrac{2A}{P} = x_{n+1}y_{n+1}$

$$47\alpha - 45\beta + 43\gamma - 180\frac{A}{P} = -62$$

4. Conclusion

In this paper, we have presented infinitely many integer solutions for the positive pell equation $y^2 = 90x^2 + 31$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of pell equations and determine their integer solutions along with suitable properties.

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