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On the Cubic Diophantine Equation with Four Unknowns $x^2+y^2=z^3-w^3$

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Abstract. The sequences of integral solutions to the cubic equation with four variables $x^2+y^2=z^3-w^3$ are obtained. A few properties among the solutions are presented. Also, Employing the integer solutions of the considered equation, integer solutions for different choices of hyperbolas and parabolas are obtained.

Keywords: Cubic equation with four unknowns, non-homogeneous cubic, cubic with four unknowns, integral solutions.

AMS Mathematics Subject Classification (2010): 11D25

1. Introduction

The Diophantine equation offers an unlimited field for research due to their variety [1-4]. In particular, one may refer [5-13] for cubic equation with three unknowns. In [14,15] cubic equations with four unknowns are studied for its non-trivial integral solutions and in [16,17] cubic equations with five unknowns and one may refer [18,19] for cubic equation with six unknowns are analyzed for its distinct integer solutions.

This communication concerns with the problem of obtaining infinitely many non-zero distinct integral solutions of cubic equation with four variables given by $x^2+y^2=z^3-w^3$. A few interesting properties among the solutions are presented.

2. Method of analysis

The cubic Diophantine equation with four unknowns to be solved

for getting non-zero integral solution is

$$x^2+y^2=z^3-w^3 \tag{1}$$

To start with, it is observed by trial and error that the following quadruples

(x,y,z,w): $(5,1,3,1)$, $(6,1,4,3)$, $(5,6,5,4)$, $(6,9,5,2)$, $(10,14,8,6)$, $\left(-(\alpha^2+1),\ \alpha(\alpha^2+1), 0, -(\alpha^2+1)\right)$ satisfy (1).

To obtain an infinite set of non-zero distinct integer solutions to (1), we proceed as follows:
On substituting the linear transformations

$$z = x + h\,,\; w = x + k\,,\; h \neq k \neq 0 \tag{2}$$

in (1) it leads to

$$x^2 + y^2 = (3h - 3k)x^2 + (3h^2 - 3k^2)x + h^3 - k^3\,,\; h \neq k \tag{3}$$

To Solve (3), we have to go in for particular values for h and k. for simplicity and brevity, we present below a few illustrations when h and k take special values.

2.1. Illustration 1

Choose $h = k + 1$ in the above equation We have

$$y^2 = 2x^2 + 3x(2k+1) + (3k^2 + 3k + 1) \tag{4}$$

for which the solutions are presented below when $k = 1, 2$

Case (1):

By taking $k = 1$ in (4), we obtain

$$y^2 = 2x^2 + 9x + 7$$

Performing some algebraic simplifications, the above equation is written as

$$X^2 = 8y^2 + 25 \tag{5}$$

where $X = 4x + 9$ (6)

The smallest positive integer solution of (5) is

$$X_0 = 15, y_0 = 5 \tag{7}$$

To obtain the other solutions of (5), consider the pell equation

$$X^2 = 8y^2 + 1 \tag{8}$$

whose general solution is given by

$$\tilde{y}_n = \frac{g_n}{4\sqrt{2}}$$

$$\tilde{X}_n = \frac{f_n}{2}$$

where

$$f_n = (3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1}$$

$$g_n = (3 + 2\sqrt{2})^{n+1} - (3 - 2\sqrt{2})^{n+1}\;, n = 0,2,4,...$$

Applying Brahmagupta lemma between (X_0, y_0) and $(\tilde{X}_n, \tilde{y}_n)$, the other integer solutions of (5) are given by

$$X_{n+1} = \frac{5}{2}\left[3f_n + 2\sqrt{2}\,g_n\right]$$

$$y_{n+1} = \frac{5}{4\sqrt{2}}\left[2\sqrt{2}\,f_n + 3g_n\right] \tag{9}$$

By using (9) in (6) and using (2), we obtain the non-zero distinct integral solutions to (1) are given by

$$x_{n+1} = \frac{5\left[3f_n + 2\sqrt{2}\,g_n\right] - 18}{8}$$

$$y_{n+1} = \frac{5}{4\sqrt{2}}\left[2\sqrt{2}f_n + 3g_n\right]$$

$$z_{n+1} = \frac{5\left[3f_n + 2\sqrt{2}\,g_n\right] - 2}{8}$$

$$w_{n+1} = \frac{5\left[3f_n + 2\sqrt{2}\,g_n\right] - 10}{8}\ ,\ n = 0,2,4...$$

The recurrence relations satisfied by x, y, z and w are given by

$$x_{n+5} - 34x_{n+3} + x_{n+1} = 72.$$

$$y_{n+5} - 34y_{n+3} + y_{n+1} = 0.$$

$$z_{n+5} - 34z_{n+3} + z_{n+1} = 8.$$

$$w_{n+5} - 34w_{n+3} + w_{n+1} = 40\,, n = 0,2,4,...$$

A few numerical examples are given below in Table 1:

Table 1 : Numerical examples

n	x_{n+1}	y_{n+1}	z_{n+1}	w_{n+1}
0	19	30	21	20
2	719	1020	721	720
4	24499	34650	24501	24500
6	832319	1177080	832321	832320
8	28274419	39986070	28274421	28274420

From the above table , we observe some interesting relations among the solutions which are presented below:

1. x_{n+1} and z_{n+1} are always odd
2. y_{n+1} and w_{n+1} are always even
3. Relations among the solutions:
    - ❖ $x_{n+5} = 34x_{n+3} - x_{n+1} + 72$
    - ❖ $12y_{n+1} = x_{n+3} - 17x_{n+1} - 36$
    - ❖ $12y_{n+3} = 17x_{n+3} - x_{n+1} + 36$
    - ❖ $12y_{n+5} = 577x_{n+3} - 17x_{n+1} + 1260$

- $12y_{n+1} = 17x_{n+5} - 577x_{n+3} - 1260$
- $60y_{n+3} = 1331663x_{n+3} - 39199x_{n+5} + 2908044$
- $12y_{n+5} = 17x_{n+5} - x_{n+3} + 36$
- $577y_{n+3} = 24x_{n+5} + 17y_{n+1} + 54$
- $577y_{n+5} = 816x_{n+5} + y_{n+1} + 1836$
- $y_{n+1} = 34y_{n+3} - y_{n+5}$
- Each of the following expressions represents a nasty number:
  - $\frac{4}{5}\{70x_{2n+2} - 2x_{2n+4} + 168\}$
  - $\frac{4}{5}\{2378x_{2n+4} - 70x_{2n+6} + 5208\}$
  - $\frac{12}{2885}\{12x_{2n+6} - 9512y_{2n+2} + 2912\}$
  - $\frac{2}{5}\{99y_{2n+6} - 3363y_{2n+4} + 30\}$
- Each of the following expressions in Table 2 represents a hyperbola:

Table 2: Hyperbola

Hyperbola	$(p_{n,} q_n)$
$2p_n^2 - 9q_n^2 = 450$	$(70x_{n+1} - 2x_{n+3} + 153, 33x_{n+1} - x_{n+3} + 72)$
$2p_n^2 - q_n^2 = 450$	$(2378x_{n+3} - 70x_{n+5} + 5193, 3363x_{n+3} - 99x_{n+5} + 7344)$
$p_n^2 - 8q_n^2 = 8323225$	$(12x_{n+5} - 9512y_{n+1} + 27, 4x_{n+5} - 3363y_{n+1} + 9)$
$p_n^2 - 2q_n^2 = 900$	$(3363y_{n+3} - 99y_{n+5}, 2378y_{n+3} - 70y_{n+5})$

- Each of the following expressions in Table 3 represents a parabola:

Table 3: Parabola

Parabola	$(p_{n,}q_n)$
$3q_n^2 = 5p_n - 150$	$(70x_{2n+2} - 2x_{2n+4} + 168, 33x_{n+1} - x_{n+3} + 72)$
$q_n^2 = 15p_n - 450$	$(2378x_{2n+4} - 70x_{2n+6} + 5208, 3363x_{n+3} - 99x_{n+5} +$
$16q_n^2 = 2885p_n - 16646450$	$(12x_{2n+6} - 9512y_{2n+2} + 2912, 4x_{n+5} - 3363y_{n+1} + 9$
$2q_n^2 = 15p_n - 900$	$(99y_{2n+6} - 3363y_{2n+4} + 30, 2378y_{n+3} - 70y_{n+5})$

Case (2):

By taking $k = 2$ in (4), we obtain

$$y^2 = 2x^2 + 15x + 19$$

Performing some algebraic simplifications, the above equation is written as

$$\alpha^2 = 8y^2 + 73 \tag{10}$$

where $\alpha = 4x + 15$ (11)

The smallest positive integer solution of (10) is

$$\alpha_0 = 9, y_0 = 1 \tag{12}$$

To obtain the other solutions of (10), consider the pell equation

$$\alpha^2 = 8y^2 + 1 \tag{13}$$

whose general solution is given by

$$\tilde{y}_n = \frac{g_n}{4\sqrt{2}}$$

$$\tilde{\alpha}_n = \frac{f_n}{2}$$

where

$$f_n = (3+2\sqrt{2})^{n+1} + (3-2\sqrt{2})^{n+1}$$

$$g_n = (3+2\sqrt{2})^{n+1} - (3-2\sqrt{2})^{n+1} \quad , n = 0,2,4,...$$

Applying Brahmagupta lemma between (X_0, y_0) and $(\tilde{X}_n, \tilde{y}_n)$, the other integer solutions of (10) are given by

$$\begin{aligned} \alpha_{n+1} &= \frac{1}{2}\left[9 f_n + 2\sqrt{2}\, g_n\right] \\ y_{n+1} &= \frac{1}{4\sqrt{2}}\left[2\sqrt{2} f_n + 9 g_n\right] \end{aligned} \tag{14}$$

By using (14) in (11) and using (2), we obtain the non-zero distinct integral solutions to (1) are given by

$$x_{n+1}=\frac{9f_n+2\sqrt{2}\,g_n-30}{8}$$

$$y_{n+1}=\frac{1}{4\sqrt{2}}\left[2\sqrt{2}f_n+9g_n\right]$$

$$z_{n+1}=\frac{9f_n+2\sqrt{2}\,g_n-6}{8}$$

$$w_{n+1}=\frac{9f_n+2\sqrt{2}\,g_n-14}{8}\ ,\ n=0,2,4...$$

The recurrence relations satisfied by x,y,z and w are given by

$$x_{n+5}-34x_{n+3}+x_{n+1}=72.$$

$$y_{n+5}-34y_{n+3}+y_{n+1}=0.$$

$$z_{n+5}-34z_{n+3}+z_{n+1}=24.$$

$$w_{n+5}-34w_{n+3}+w_{n+1}=56, n=0,2,4,...$$

A few numerical examples are given below in Table 4:

Table 4: Numerical examples

n	x_{n+1}	y_{n+1}	z_{n+1}	w_{n+1}
0	5	12	8	7
2	289	414	292	291
4	9941	14064	9944	9943
6	337825	477762	337828	337827
8	11476229	16229844	11476232	11476231

From the above table, we observe some interesting relations among the solutions which are presented below:

1. x_{n+1} and z_{n+1} are always odd
2. y_{n+1} and w_{n+1} are always even
3. Relations among the solutions:
    - ❖ $x_{n+5}=34x_{n+3}-x_{n+1}+120$
    - ❖ $12y_{n+1}=x_{n+3}-17x_{n+1}-60$
    - ❖ $12y_{n+3}=17x_{n+3}-x_{n+1}+60$
    - ❖ $1752y_{n+5}=84242x_{n+3}-2482x_{n+1}+306600$

- $1752y_{n+1} = 2482x_{n+5} - 84242x_{n+3} - 306600$
- $12y_{n+3} = x_{n+5} - 17x_{n+3} - 60$
- $12y_{n+5} = 17x_{n+5} - x_{n+3} + 60$
- $42121y_{n+3} = 1752x_{n+5} + 1241y_{n+1} + 6570$
- $577y_{n+5} = 816x_{n+5} + y_{n+1} + 3060$
- $y_{n+1} = 34y_{n+3} - y_{n+5}$
- Each of the following expressions represents a nasty number:
  - $\frac{1}{73}\{568x_{2n+2} - 8x_{2n+4} + 2976\}$
  - $\frac{4}{73}\{4826x_{2n+4} - 142x_{2n+6} + 17784\}$
  - $\frac{2}{42121}\{36x_{2n+6} - 19304y_{2n+2} + 42256\}$
  - $\frac{2}{73}\{9y_{2n+4} - 201y_{2n+2} + 438\}$

Each of the following expressions in Table 5 represents a hyperbola:

Table 5: Hyperbola

Hyperbola	$(p_{n,} q_n)$
$p_n^2 - 8q_n^2 = 767376$	$(568x_{n+1} - 8x_{n+3} + 2100, 201x_{n+1} - 9x_{n+3} + 720)$
$2p_n^2 - q_n^2 = 95922$	$(4826x_{n+3} - 142x_{n+5} + 17565, 6825x_{n+3} - 201x_{n+5} + 2484$
$p_n^2 - 8q_n^2 = 1$	$(36x_{n+5} - 19304y_{n+1} + 15, 4x_{n+5} - 6825y_{n+1} + 15)$
$p_n^2 - 8q_n^2 = 191844$	$(201y_{n+1} - 9y_{n+3}, 71y_{n+1} - y_{n+3})$

- Each of the following expressions in Table 6 represents a parabola:

Table 6: Parabola

Parabola	$(p_{n,}q_n)$
$4q_n^2 = 219p_n - 383688$	$(568x_{2n+2} - 8x_{2n+4} + 2976, 201x_{n+1} - 9x_{n+3} + 720$
$4q_n^2 = 876p_n - 383688$	$(4826x_{2n+4} - 142x_{2n+6} + 17784, 6825x_{n+3} - 201x_n$
$16q_n^2 = 42121p_n - 35483572$	$(36x_{2n+6} - 19304y_{2n+2} + 42256, 4x_{n+5} - 6825y_{n+1}$
$8q_n^2 = 219p_n - 191844$	$(9y_{2n+4} - 201y_{2n+2} + 438, 71y_{n+1} - y_{n+3})$

2.2. Illustration 2

By taking $k = 0, h = 1$ in (4), we obtain

$$y^2 = 2x^2 + 3x + 1$$

Performing some algebraic simplifications, the above equation is written as

$$\beta^2 = 8y^2 + 1 \tag{15}$$

where $\beta = 4x + 3$ (16)

The smallest positive integer solution of the above equation is

$$\beta_0 = 3, y_0 = 1 \tag{17}$$

and the general solution of equation (15) is given by

$$y_n = \frac{g_n}{4\sqrt{2}}$$
$$\beta_n = \frac{f_n}{2} \tag{18}$$

where $f_n = (3+2\sqrt{2})^{n+1} + (3-2\sqrt{2})^{n+1}$

$g_n = (3+2\sqrt{2})^{n+1} - (3-2\sqrt{2})^{n+1}$, $n = 0,2,4,...$

By using (18) in (16) and using (2), we obtain the non-zero distinct integral solutions to (1) are given by

$$x_n = \frac{f_n - 6}{8}$$

$$y_n = \frac{g_n}{4\sqrt{2}}$$

$$z_n = \frac{f_n + 2}{8}$$

$$w_n = x_n = \frac{f_n - 6}{8} \text{ , } n = 0,2,4...$$

2.3. Illustration 3

Choosing $k = -h$ in (3), we obtain

$$x^2 + y^2 = 6hx^2 + 2h^3 \tag{19}$$

To solve (19), we have to take particular values of h.

For illustrations, taking h=4, in (19), we have

$$y^2 = 23x^2 + 128$$

By taking $y = 2Y$ and $x = 2X$, we get (20)

$$Y^2 = 23X^2 + 32 \tag{21}$$

The smallest positive integer solution of (21) is

$$X_0 = 4, Y_0 = 20 \tag{22}$$

To obtain the other solutions of (21), consider the pell equation

$$Y^2 = 23X^2 + 1 \tag{23}$$

whose general solution is given by

$$\tilde{Y}_n = \frac{f_n}{2}$$

$$\tilde{X}_n = \frac{g_n}{2\sqrt{23}}$$

where
$$f_n = (24 + 5\sqrt{23})^{n+1} + (24 - 5\sqrt{23})^{n+1}$$
$$g_n = (24 + 5\sqrt{23})^{n+1} - (24 - 5\sqrt{23})^{n+1} \quad , n = -1,0,1,2,...$$

Applying Brahmagupta lemma between (X_0, Y_0) and $(\tilde{X}_n, \tilde{Y}_n)$, the other integer solutions of (21) are given by

$$\begin{aligned} X_{n+1} &= 2 f_n + \frac{10}{\sqrt{23}} g_n \\ Y_{n+1} &= 10 f_n + 2\sqrt{23}\, g_n \end{aligned} \tag{24}$$

By applying (23) in (20) ,we obtain the non-zero distinct integral solutions to (1) are given by

$$x_{n+1} = 4 f_n + \frac{20}{\sqrt{23}} g_n$$

$$y_{n+1} = 20 f_n + 4\sqrt{23}\, g_n$$

$$z_{n+1} = 4 f_n + \frac{20}{\sqrt{23}} g_n + 4$$

$$w_{n+1} = 4 f_n + \frac{20}{\sqrt{23}} g_n - 4, \; n = -1,0,1,2,...$$

Following the analysis presented in Illustration 1, one may obtain relations among the solutions for Illustrations 2 and 3.

4. Conclusion

In this paper, we have presented sets of infinitely many non-zero distinct integer solutions to the cubic equation with four unknowns given by $x^2 + y^2 = z^3 - w^3$. In other words, we have obtained quadruples such that, in each quadruple, the sum of the squares of any two members equals the difference of cubes of its other two members. As Diophantine equations are rich in variety due to their definition, one may attempt to find integer solutions to higher degree Diophantine equations with multiple variables along with their suitable properties.

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