Journal of Mathematics and Informatics
Vol. 10, 2017, 41-48
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 11 December 2017
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/jmi.v10a6

Journal of
Mathematics and Informatics

On the Non-Homogeneous Quadratic Equation with Five Unknowns $x^2+xy-y^2-(z+w)=10p^2$

K.Dhivya[1] **and** ***T.R.Usha Rani***[2]

[1]Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620002
Tamilnadu, India; e-mail:kdhivya10495@gmail.com
[2]Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620002
Tamilnadu, India; e-mail:usharanisigc@gmail.com

Received 2 November 2017; accepted 5 December 2017

Abstract. The non-homogeneous quadratic equation with five unknowns represented by the Diophantine equation $x^2+xy-y^2-(z+w)=10p^2$ is analysed for its non-zero distinct integral solutions. Various interesting relations between the solutions and special numbers are exhibited.

Keywords: non-homogeneous quadratic, quadratic with five unknowns, integral solutions.

AMS Mathematics Subject Classification (2010): 11D09

1. Introduction

The non-homogenous quadratic Diophantine equation offers an unlimited field for research because of their variety [1-3]. For an extensive review of various problems one may refer [4-17]. This communication concerns with yet another interesting equation $x^2+xy-y^2-(z+w)=10p^2$ representing non-homogeneous quadratic equation with five unknowns determining its infinitely many non-zero integral solutions. Also a few interesting properties among the solutions are presented.

2. Notation

$t_{m,n}=n\left(1+\frac{(n-1)(m-2)}{2}\right)$ - Polygonal number of rank *n* with sides*m*

$Ct_{m,n}=\frac{mn(n-1)+2}{2}$ - Centered Polygonal number of rank *n* with sides *m*

$S_n=6n(n-1)+1$ - Star number of rank *n*

$PR_n=n(n+1)$ - Pronic number of rank *n*

$j_n = 2^n + (-1)^n$ - Jacobsthal-Lucas number of rank *n*

$G_n = 2n-1$ - Gnomonic number of rank *n*

$So_n = n(2n^2-1)$ - Stella Octangular number of rank *n*

3. Method of analysis

The non-homogeneous quadratic equation with five unknowns under consideration is,

$$x^2 + xy - y^2 - (z+w) = 10p^2 \quad (1)$$

We have four patterns of solutions of (1) which are presented below.

The substitution of linear transformation

$$x = u+v,\ y = u-v,\ z = 2uv+1,\ w = 2uv-1 \quad (2)$$

in (1) leads to

$$u^2 = 10p^2 + v^2 \quad (3)$$

3.1. PATTERN I

Assume that $p = 2mn$, $u = 10m^2 + n^2$, $v = 10m^2 - n^2$

Substituting the values of *u* and *v* in (1) and simplifying, we obtain the non-zero distinct integer solutions of (1) as follows.

$$x(m) = 20m^2$$
$$y(n) = 2n^2$$
$$z(m,n) = 200m^4 - 2n^4 + 1$$
$$w(m,n) = 200m^4 - 2n^4 - 1$$
$$p(m,n) = 2mn$$

PROPERTIES

1. Each of the following expressions is a Nasty number:
   (i) $3[z(n,n) - w(n,n)]$
   (ii) $3y(n)$
   (iii) $30[x(m)]$
   (iv) $30[z(m,n)] - 30x(3m^2) + 30y(n^2) - 30$
2. $20[z(m,n) + w(m,n)] = x(20m^2) - 10y(2n^2)$
3. $p(n, n(n+1)) - SO_n - nG_n \equiv 0 \pmod 2$
4. $p(n,n) + y(n) - 4PR_n - G_n - S_n + 6t_{4,n} = 0$
5. $x(n) - p(n,n) - 2PR_n + G_n - 16t_{4,n} + 1 = 0$

6. $p(n,n)+y(n)-4t_{4,n}=0$

7. $x(2^n)+y(2^n)-22j_{2n}+22=0$

8. $p(2^n,2^n)-2j_{2n}+2=0$

9. $w(2^n,1)-200j_{4n}+203=0$

10. Each of the following expressions is a biquadratic integer:

(1) $\frac{8[z(m,1)+1]}{100}$, (2) $\frac{8[w(m,1)+3]}{100}$

3.2. PATTERN II

(3) can be written as,

$$u^2-v^2=10p^2 \tag{4}$$

(4) can be written in the form of ratio as

$$u^2-v^2=10p.p \tag{5}$$

$$\frac{u+v}{p}=\frac{10p}{u-v}=\frac{\alpha}{\beta},\beta\neq 0 \tag{6}$$

which is equivalent to the system of double equations

$$p\alpha-u\beta-v\beta=0 \tag{7}$$

$$10p\beta-u\alpha+v\alpha=0 \tag{8}$$

Solving (7) and (8) by the method of cross multiplication, we obtain,

$$p=-2\alpha\beta,\; u=-(\alpha^2+10\beta^2),\; v=10\beta^2-\alpha^2$$

Substituting the values of u and v in (2) and simplifying, we obtain the non-zero distinct integer solutions of (1) as follows.

$$x(\alpha)=-2\alpha^2$$
$$y(\beta)=-20\beta^2$$
$$z(\alpha,\beta)=2\alpha^4-200\beta^4+1$$
$$w(\alpha,\beta)=2\alpha^4-200\beta^4-1$$
$$p(\alpha,\beta)=-2\alpha\beta$$

PROPERTIES

1. Each of the following expressions is a Nasty number:

(i) $3[z(\alpha,\beta)-w(\alpha,\beta)]$

(ii) $3[x(\alpha)-y(\alpha)]$

2. $20[z(\alpha,\beta)+w(\alpha,\beta)]=y(20\beta^2)-10x(2\alpha^2)$

3. $p(n, n(n+1)) + SO_n + nG_n + G_n - 1 \equiv 0 \pmod 2$
4. $p(n, n) - y(n) - 2PR_n + G_n - 16t_{4,n} + 1 = 0$
5. $x(\alpha) + p(\alpha, \alpha) + 4PR_\alpha - S_\alpha + 6t_{4,\alpha} + 1 \equiv 0 \pmod{10}$
6. $x(\alpha) + p(\alpha, \alpha) + 4t_{4,\alpha} = 0$
7. $z(2^n, 2^n) + w(2^n, 2^n) + 396\, j_{4n} - 396 = 0$
8. $z(2^n, 1) - 2\, j_{4n} + 201 = 0$
9. $x(2^n) - y(2^n) - 18\, j_{2n} + 18 = 0$
10. Each of the following expressions is a biquadratic integer:

(1) $\dfrac{8[z(1,\beta) - 3]}{100}$

(2) $\dfrac{8[w(1,\beta) - 1]}{100}$

NOTE:

(4) can also be written as system of double equations

$$u + v = 10p \tag{9}$$

$$u - v = p \tag{10}$$

Solving (9) and (10) we obtain,

$$\left.\begin{aligned} u &= \frac{11p}{2} \\ v &= \frac{9p}{2} \end{aligned}\right\} \tag{11}$$

To obtain the integer solutions, take $p = 2k$ in (11) we get,

$$\left.\begin{aligned} u &= 11k \\ v &= 9k \end{aligned}\right\} \tag{12}$$

Substituting (12) in (2) and simplifying, we obtain the non-zero integer solutions of (1) are given by,

$$x(k) = 20k$$
$$y(k) = 2k$$
$$z(k) = 198k + 1$$
$$w(k) = 198k - 1$$
$$p(k) = 2k$$

3.3. PATTERN III

(4) can be written as,

$$(u + v)(u - v) = 2p.5p \tag{13}$$

(13) can be written in the form of ratio as,

$$\frac{u+v}{2p}=\frac{5p}{u-v}=\frac{\alpha}{\beta},\beta\neq 0 \tag{14}$$

which is equivalent to the system of double equations.

$$2p\alpha-u\beta-v\beta=0 \tag{15}$$

$$5p\beta-u\alpha+v\beta=0 \tag{16}$$

Solving (15) and (16) by the method of cross multiplication, we obtain,

$$p=-2\alpha\beta,\ u=-\left(5\alpha^2+2\beta^2\right),\ v=-5\alpha^2+2\beta^2$$

Substituting the values of u and v in (2) and simplifying, we obtain the non-zero distinct integer solutions of (1) as follows.

$$x(\alpha)=-10\alpha^2$$
$$y(\beta)=-4\beta^2$$
$$z(\alpha,\beta)=50\alpha^4-8\beta^4+1$$
$$w(\alpha,\beta)=50\alpha^4-8\beta^4-1$$
$$p(\alpha,\beta)=-2\alpha\beta$$

PROPERTIES

1. Each of the following expressions is a Nasty number:
   (i) $y(n)-x(n)$ (ii) $3[p(n,n)-y(n)]$
2. $20[z(\alpha,\beta)+w(\alpha,\beta)]=5y\left(4\beta^2\right)-2x\left(10\alpha^2\right)$
3. $y(n)+p(n,n)-x(n)-4PR_n-G_n-S_n+6t_{4,n}=0$
4. $x(n)-p(n,n)+2Ct_{8,n}+16PR_n-2t_{18,n}-2\equiv 0(\mathrm{mod}\ 22)$
5. $p(n,n)-y(n)-2PR_n+G_n+1=0$
6. $p(n,n)+y(n)-x(n)-4t_{4,n}=0$
7. $z\left(2^n,1\right)-50j_{4n}+57=0$
8. $z\left(2^n,1\right)-w\left(2^n,1\right)-100j_{4n}+98=0$
9. $y\left(2^n\right)-x\left(2^n\right)-6j_{4n}+6=0$
10. Each of the following expressions is a biquadratic integer:
    (i) $2[z(1,\beta)-51]$
    (ii) $2[w(1,\beta)-49]$

NOTE:

(13) can also be written as system of double equations

$$u+v=5p \tag{17}$$

$$u-v=2p \tag{18}$$

Solving (17) and (18) we obtain

$$\left.\begin{array}{l} u=\frac{7p}{2} \\ v=\frac{3p}{2} \end{array}\right\} \quad (19)$$

To obtain the integer solutions, take $p = 2k$ in (19) we get,

$$\left.\begin{array}{l} u=7k \\ v=3k \end{array}\right\} \quad (20)$$

Substituting (20) in (2) and simplifying, we obtain the non-zero distinct integer solution are given by,

$$x(k)=10k$$
$$y(k)=4k$$
$$z(k)=42k^2+1$$
$$w(k)=42k^2-1$$
$$p(k)=2k$$

3.4. PATTERN IV

(4) can be written as,

$$(u+v)(u-v)=2.5p^2 \quad (21)$$

(21) can be written as system of double equations

$$u+v=5p^2 \quad (22)$$

$$u-v=2 \quad (23)$$

Solving (22) and (23) we obtain

$$\left.\begin{array}{l} u=\frac{5p^2+2}{2} \\ v=\frac{5p^2-2}{2} \end{array}\right\} \quad (24)$$

To obtain the integer solutions, take $p = 2k$ in (24) we get,

$$\left.\begin{array}{l} u=10k^2+1 \\ v=10k^2-1 \end{array}\right\} \quad (25)$$

Substituting (25) in (2) and simplifying, we obtain the non-zero distinct integer solution of (1) as follows.

$$x(k)=20k^2, \quad y=2, \quad z(k)=200k^4-3$$
$$w(k)=200k^4-1, \quad p(k)=2k$$

PROPERTIES

1. Each of the following expressions is a Nasty number:

(i) $30[x(k)]$, (ii) $6y(k^2)+6p(k^2)$

2. $20[z(k)+w(k)]=x(20k^2)+20y$
3. $x(k)+y(k^2)-p(k^2)-2PR_k+G_k-18t_{4,k}+1=0$
4. $x(k)-p(k^2)-2PR_k+G_k-16t_{4,k}+1=0$
5. $y(k^2)+p(k^2)-4PR_k-G_k-S_k+6t_{4,k}=0$
6. $x(k)+y(k^2)+p(k^2)-24t_{4,k}=0$
7. $z(2^n)+w(2^n)-400j_{4n}+404=0$
8. $x(2^n)+p(2^n)-20j_{2n}+20-2j_n+2(-1)^n=0$
9. $x(2^n)+z(2^n)+w(2^n)-20j_{2n}-400j_{4n}+423=0$
10. Each of the following expression is a biquadratic integer:

(i) $\dfrac{8[z(k)+3]}{100}$, (ii) $\dfrac{8[w(k)+3]}{100}$

4. Conclusion

In this paper we have made an attempt to obtain all integer solutions to the non-homogeneous quadratic equation. One may search for other patterns of solutions and their corresponding properties.

REFERENCES

1. L.E.Dickson, *History of Theory of Numbers and Diophantine Analysis*, Dove Publications, New York, Vol 2 (2005).
2. L.J.Mordell, *Diophantine Equations*, Academic press, New York, (1970).
3. R.D.Carmicheal, *The Theory of Numbers and Diophantine analysis*, Dover Publications, New York, (1959).
4. M.A.Gopalan and D.Geetha, Lattice points on the Hyperboloid of two sheets $x^2-6xy+y^2+6x-2y+5=z^2+4$, *Impact J. Sci. Tech*., 4 (2010) 23-32.
5. M.A.Gopalan, S.Vidhyalakshmi and S.Kavitha, Integral points on the Homogeneous Cone $z^2=2x\ \ -7y^2$, *Diophantus J. Math.*, 1(2) (2012) 127-136.
6. M.A.Gopalan, S.Vidhyalakshmi and G.Sumathi, Lattice points on the Hyperboloid of one sheet $4z^2=2x^2+3y^2-4$, *Diophantuts J. Math.*, 1(2) (2012) 109-115.
7. M.A.Gopalan, S.Vidhyalakshmi and K.Lakshmi, Integral Points on the Hyperboliodof two sheets $3y^2=7x^2-z^2+21$, *Diophantus J Math*., 1(2) (2012) 99-107.

8. M.A.Gopalan, S.Vidhyalakshmi and S.Mallika, Observations on Hyperboloid of one sheet $x^2 + 2y^2 - z^2 = 2$, *Bessel J Math*., 2(3) (2012) 221-226.
9. M.A.Gopalan, S.Vidhyalakshmi, T.R.Usha Rani and S.Mallika, Integral Points on the Homogeneous Cone $6z^2 + 3y^2 - 2x^2 = 0$, *The Impact J. Sci. Tech.*, 6(1) (2012) 7-13.
10. M.A.Gopalan,S.Vidhyalakshmi and T.R.UshaRani, Integral points on the non-homogeneous Cone $2z^2 + 4xy + 8x - 4z = 0$, *Global Journal of Mathematics, Mathematical Sciences*, 2(1) (2010) 61-67.
11. M.A.Gopalan, S.Vidhyalakshmi and J.Uma Rani, Intergal points on the Homogeneous Cone $4y^2 + x^2 = 37z^2$, *Cayley J. of Math*., 2(2) (2013) 101-107.
12. M.A.Gopalan, S.Vidhyalakshmi and A.Kavitha, Observations on the Hyperboloid of two sheet $7x^2 - 3y^2 = z^2 + z(y - x) + 4$, *International Journal of the Latest Research in Scienceand Technology*, 2(2) (2013) 84-86.
13. M.A.Gopalan and B.Sivagami, Integral points on the Homogeneous Cone $z^2 = 3x^2 + 6y^2$, *IOSR Journal of Mathematics*, 8(4) (2013) 24-29.
14. M.A.Gopalan and V.Geetha, Lattice points on the Homogeneous Cone $z^2 = 2x^2 + 8y^2 - 6xy$, *Indian Journal of Science*, 2 (2013) 93-96.
15. M.A.Gopalan, S.Vidhyalakshmi and J.Uma Rani, On the ternary quadratic diophantine equation $6(x^2 + y^2) - 8xy = 21z^2$. ,*Sch. Eng. Tech.*, 2(2A) (2014) 108-112.
16. M.A.Gopalan, S.Vidhyalakshmi and J.Shanthi, Lattice Points on the Homogeneous Cone $8(x^2 + y^2) - 15xy = 56z^2$, *Sch. Journal of Phy. Math. Star.*, 1(1) (2014) 29-32.
17. K.Meena, M.A.Gopalan, E.Bhuvaneshwari and R.Presenna, On quadratic diophantine equation with five unknowns $4w^2 - x^2 - y^2 + z^2 = t^2$, *International Journal of Advanced Scientific Research*, 1(2) (2016) 62-64.