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Observation on the Non-Homogeneous Ternary Quadratic Equation $x^2-xy+y^2+2(x+y)+4=12z^2$

N. Bharathi[1] **and *S. Vidhyalakshmi***[2]

Department of Mathematics, Shrimathi Indira Gandhi College
Trichy-2, Tamil Nadu, India.
[1]E-mail:bharathiakilla95@gmail.com; [2]E-mail: vidhyasigc@gmail.com

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Abstract. A search is made for obtaining infinitely many non-zero distinct integer solutions to the non-homogeneous quadratic equation given by $x^2-xy+y^2+2(x+y)+4=12z^2$. Different choices of integer solution to the above equation are obtained. A few interesting relations between the solutions and special polygonal numbers are obtained.

Keywords: Non-homogeneous quadratic, ternary quadratic integer solution

AMS Mathematics Subject Classification (2010): 11D09

1. Introduction

The Diophantine equations offer on unlimited field for research due to their variety [1-3]. In particular, one may refer [4-8] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $x^2-xy+y^2+2(x+y)+4=12z^2$ representing non-homogeneous quadratic equation with three unknowns for determining its infinitely many non-zero integral points. Also, few interesting relations among the solutions are presented.

2. Notation

1. Polygonal number of rank n with sides m

$$t_{m,n} = n\left[1+\frac{(n-1)(m-2)}{2}\right]$$

2. Pronic number of rank n

$$pRn = n(n+1)$$

3. Centered hexagonal pyramidal number of rank n

$$cP_{n,6} = n^3$$

4. Square number of rank n

$$t_{4,n} = n^2$$

3. Method of analysis

The ternary quadratic Diophantine equation to be solved for its non-zero solution is

$$x^2 - xy + y^2 + 2(x+y) + 4 = 12z^2 \quad (1)$$

We present below different patterns of integer solutions to (1)

Introducing the linear transformation $(u \neq v \neq 0)$

$$x = u + v, y = u - v \quad (2)$$

in(1), it leads to $U^2 + 3v^2 = 12z^2$ (3)

where $U = u + 2$ (4)

The above equation (3) is solved through different approaches and then, in view of 2 we obtain different patterns of integer solutions to (1)

3.1. Pattern-1

Write 12 as

$12 = (3 + i\sqrt{3})(3 - i\sqrt{3})$(5)

Assume $z = a^2 + 3b^2$ (6)

where $a, b \succ 0$

Using (5) and (6) in (3)

$U + 3v^2 = (3 + i\sqrt{3})(3 - i\sqrt{3})(a^2 + 3b^2)^2$ and

employing the method of factorization define

$$(u + iv\sqrt{3})(u + iv\sqrt{3}) = (3 + i\sqrt{3})(3 - i\sqrt{3})(a + i\sqrt{3}b)^2 (a - i\sqrt{3}b)^2$$

Equating the real and imaginary parts, we get

$$U = U(a,b) = 3a^2 - 9b^2 - 6ab + 2$$

$$v = v(a,b) = 6ab + a^2 - 3b^2$$

In view of (2) we get

$x = x(a,b) = 4a^2 - 12b^2 - 2$ (7)

$y = y(a,b) = 2a^2 - 6b^2 - 12ab - 2$ (8)

Thus (6),(7),(8) represents non-zero distinct integral solution of (1) in two parameters

Properties:

1. $x(a, a+1) - 2y(a, a+1) - 2 = 24pR_a$
2. $x(a,-1) - z(a,-1) - 2t_{3,a} \equiv 0 (\bmod 11)$
3. $x(a,b) - 2y(a,b) - 24cP_{n,6} = 0$

3.2. Pattern-2

Write (3) in form of ratio as

$$\frac{U + 3z}{z + v} = \frac{(z - v)}{U - 3z} = \frac{\alpha}{\beta}, \beta \neq 0 \quad (9)$$

which is equivalent to the following two equations

$\beta U - \alpha v + z(3\beta - \alpha) = 0$ (10)

$-\alpha U-3v\beta+3z(\beta+\alpha)=0$ (11)

Solving (10) and (11) by the method of cross-multiplication, we have

$U=U(\alpha,\beta)=9\beta^2-3\alpha^2-6\alpha\beta$ (12)

$v=v(\alpha,\beta)=\alpha^2-3\beta^2+3\beta-3\alpha\beta$ (13)

$z=z(\alpha,\beta)=-3\beta^2-\alpha^2$ (14)

Substituting U and v values in (4) and (2), we get

$x=x(\alpha,\beta)=6\beta^2-2\alpha^2-3\beta-9\alpha\beta-2$ (15)

$y=y(\alpha,\beta)=12\beta^2-4\alpha^2+3\beta-3\alpha\beta-2$ (16)

Thus (14),(15),(16) represents non-zero distinct integral solution of (1) in two parameters.

Properties:

1. $y(1,\beta)-2x(1,\beta)-t_{44,\beta}\equiv 21 \pmod{29}$
2. $z(\alpha,\alpha)+2t_{4,\alpha}=0$
3. $x(\alpha,\alpha+1)+y(\alpha,\alpha+1)\equiv 14 \pmod{24}$

3.3. Pattern-3

Consider

$z=X+3T$ (17)

$v=X+12T$ (18)

$U=3w$ (19)

Substituting (17),(18),(19) in (3), we get

$X^2=36T^2+w^2$ (20)

which is in the form of Pythagorean equation and is satisfied by

$X=9R^2+S^2$

$T=RS$

$w=9R^2-S^2$

In view of (2), the integer solutions are given by

$x=36R^2-2S^2+12RS-2$

$y=18R^2-4S^2-12RS-2$

$z=9R^2+S^2+3RS$

Properties:

1. $x(R,R)+y(R,R)-48t_{4,R}\equiv 0 \pmod 2$
2. $z(R,1)-t_{20,R}-1\equiv 0 \pmod{11}$
3. $x(1,S)-y(1,S)-2t_{4,S}\equiv 18 \pmod{24}$

Also, note that (20) is satisfied by

$$\mathrm{X} = 36\left(R^2 + S^2\right)$$
$$\mathrm{T} = 6\left(R^2 - S^2\right)$$
$$w = 72RS$$

in this case the corresponding solutions to (1) are given by

$$x = 216RS - 2 + 108R^2 - 36S^2$$
$$y = 126R - 2 - 108R^2 + 36S^2$$
$$z = 54R^2 + 18S^2$$

Properties:

1. $z(R,R) - 72t_{4,R} = 0$
2. $x(S,S) - 2z(S,S) - 144t_{4,S} + 2 = 0$
3. $y(1,S) + y(1,S) - t_{38,S} \equiv 69(\mathrm{mod}\,233)$

3.4. Pattern- 4

Note that (20) is expressed as the system of double equations as follows:

	System1	System2	System3
$X + w$	T^2	$6T^2$	$12T$
$X - w$	36	6	$3T$

Solving each of the above systems, the corresponding solutions to (1) are given below:

Solution for system 1:

Solving the double equations, we have

$$X = 2k^2 + 18$$
$$w = 2k^2 - 18$$
$$T = 2k$$

In view of (2), the integer solutions are given by

$$x = 8k^2 + 24k - 38$$
$$y = 4k^2 - 24k - 74$$
$$z = 2k^2 + 6k + 18$$

Properties:

1. $x(k) + y(k) - 12t_{4,k} + 112 = 0$
2. $z(k) - t_{6,k} \equiv 4(\mathrm{mod}\,7)$
3. $x(k) - 2z(k) - t_{10,k} \equiv 14(\mathrm{mod}\,15)$

Solution for system 2:

Solving the double equations, we have

$$X = 3T^2 + 3$$
$$w = 3T^2 - 3$$
$$U = 9T^2 - 9$$

In view of (2) ,the integer solutions are given by

$$x = 12T^2 + 12k - 8$$
$$y = 6T^2 - 24T - 14$$
$$z = 3T^2 + 3T + 3$$

Properties:

1. $z(T) - 6t_{3,T} \equiv 0 \pmod 3$
2. $x(T) + y(T) - 17t_{4,T} \equiv 0 \pmod 7$
3. $x(T) - 2z(T) - t_{12,T} \equiv 6 \pmod{10}$

Solution for system 3:

Solving the double equations, we have

$$X = 15k$$
$$w = 9k$$
$$T = 2k$$

In view of (2), the integer solutions are given by

$$x = 66k - 2$$
$$y = -12k - 2$$
$$z = 21k$$

4. Conclusion

In this paper, we have made an attempt of find all integer solutions to the ternary quadratic equation given by $x^2 - xy + y^2 + 2(x + y) + 4 = 12z^2$ .As quadratic equations in three unknowns are rich in variety, one way attempt to find integer solutions to other choices of ternary quadratic equations along with suitable properties.

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