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On the Homogeneous Ternary Quadratic Equation $11x^2-2y^2 = 9z^2$

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Abstract. An attempt has been made to obtain all integer solutions of the homogeneous ternary quadratic Diophantine equation given by $11x^2 - 2y^2 = 9z^2$. Different choices of integer solution to the above equation are obtained. A few interesting relations between the solutions and special polygonal numbers are presented.

Keywords: Homogeneous quadratic, ternary quadratic, integer solution.

AMS Mathematics Subject Classification (2010): 11D09

1. Introduction

The Diophantine equations offer an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-8] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $11x^2 - 2y^2 = 9z^2$ representing homogeneous quadratic equation with three unknowns for determining its infinitely many non-zero integral points. Also, few interesting relations among the solutions are presented.

2. Notations

i) Polygonal number of rank 'n' with size m:

$$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$$

ii) Pronic number of rank 'n':

$$PR_n = n(n+1)$$

iii) Centered hexagonal pyramidal number of rank 'n':

$$CP_{n,6} = n^3$$

iv) Centered triangular pyramidal number of rank 'n':

$$CP_{n,3} = \frac{n^3 + n}{2}$$

3. Method of analysis

The ternary quadratic equation under consideration is

$$11x^2 - 2y^2 = 9z^2 \quad (1)$$

3.1. Introducing the linear transformations

$$x = X + 2T \ , \ y = X + 11T \quad (2)$$

in (1), it is written as

$$X^2 = 22T^2 + z^2 \quad (3)$$

which is satisfied by

$$T = 2rs, \ X = 22r^2 + s^2 \ , \ z = 22r^2 - s^2 \quad (4)$$

Substituting the above values of X and T in (2) , we have

$$x = 22r^2 + s^2 + 4rs \ , \ y = 22r^2 + s^2 + 22rs \quad (5)$$

Thus the integer solutions of (1) are given by (4) and (5).

Properties:

$i) x(r,r) + z(r,r) - 48t_{4,r} = 0$

$ii) y(r,r+1) - x(r,r+1) - 36t_{3,r} = 0$

$iii). x(r,1) - t_{46,r} - 1 \equiv 0 \pmod 5$

3.2. Note that (3) is expressed in the system of double equations as follows:

$X+z$	$11T^2$	$11T$	$22T$
$X-z$	2	$2T$	T

Solving each of the above systems and performing a few calculations, the corresponding three sets of solutions to (1) are given by

Set 1

$$x = 22k^2 + 4k + 1 \ , \ y = 22k^2 + 22k + 1 \ , \ z = 22k^2 - 1$$

Properties:

$i) y(k) + z(k) - t_{90,k} \equiv 0 \pmod{13}$

$ii) y(k) - 44t_{3,k} - 1 = 0$

$iii) y(k(k+1)) - x(k(k+1)) - 36t_{3,k} = 0$

Set 2

$$x = 17k \ , \ y = 35k \ , \ z = 9k$$

Set 3

$$x = 27k \ , y = 45k \ , \ z = 21k$$

3.3. Observe that (3) is expressed in the form of ratio as

$$\frac{(X+z)}{2T}=\frac{11T}{(X-z)}=\frac{\alpha}{\beta},\beta\neq 0 \tag{6}$$

The above equation (6) is equivalent to the system of equations

$$\beta X+\beta z-2\alpha T=0\ ,\ \alpha X-\alpha z-11\beta T=0$$

The above system is solved by applying the method of cross multiplication and after simplification the values of x, y, z are given by

$$x=2\alpha^2+11\beta^2+4\alpha\beta\ ,\ y=2\alpha^2+11\beta^2+22\alpha\beta\ ,\ z=2\alpha^2-11\beta^2$$

Properties:

$i)\, x(\alpha,1)-t_{6,\alpha}\equiv 1(\mathrm{mod}\,5)$

$ii)\, y(\alpha,\alpha+1)-x(\alpha,\alpha+1)-36t_{3,\alpha}=0$

$iii)\, x(\alpha,1)+y(\alpha,1)+z(\alpha,1)-t_{14,\alpha}\equiv 11(\mathrm{mod}\,31)$

Note that equation (3) may also be represented in three ways as follows:

Way 1

$$\frac{(X+z)}{T}=\frac{22T}{(X-z)}=\frac{\alpha}{\beta},\beta\neq 0$$

Way 2

$$\frac{(X+z)}{11T}=\frac{2T}{(X-z)}=\frac{\alpha}{\beta},\beta\neq 0$$

Way 3

$$\frac{(X+z)}{22T}=\frac{T}{(X-z)}=\frac{\alpha}{\beta},\beta\neq 0$$

Following the above procedure, the corresponding solutions to (1) obtained from each of the above representations are presented below.

Integer solutions from way 1

$$x=\alpha^2+22\beta^2+4\alpha\beta\ ,\ y=\alpha^2+22\beta^2+22\alpha\beta\ ,\ z=\alpha^2-22\beta^2$$

Properties:

$i)\, y(\alpha,1)+z(\alpha,1)-t_{6,\alpha}\equiv 0(\mathrm{mod}\ 23)$

$ii)\, y(\alpha,1)-x(\alpha,1)-18PR_{\alpha}-18t_{4,\alpha}=0$

$iii)\, z(\alpha,1)-t_{4,\alpha}\equiv 0(\mathrm{mod}\ 2)$

Integer solutions from way 2

$$x=11\alpha^2+2\beta^2+4\alpha\beta\ ,\ y=11\alpha^2+2\beta^2+22\alpha\beta\ ,\ z=11\alpha^2-2\beta^2$$

Properties:

$i)\, x(1,\beta)-t_{6,\beta}\equiv 1(\mathrm{mod}\,5)$

$ii)\, y(\alpha,1)+z(\alpha,1)-44t_{3,\alpha}=0$

$iii) z(\alpha,\alpha+1) - t_{20,\alpha} \equiv 2 \pmod 4$

Integer solutions from way 3

$$x = 22\alpha^2 + \beta^2 + 4\alpha\beta \text{ , } y = 22\alpha^2 + \beta^2 + 22\alpha\beta \text{ , } z = 22\alpha^2 - \beta^2$$

Properties:

$i) z(\alpha,\alpha+1) - t_{44,\alpha} + 1 \equiv 0 \pmod{18}$

$ii) y(\alpha,\alpha^2) - x(\alpha,\alpha^2) - 18CP_{\alpha,6} = 0$

$iii) y(\alpha,2\alpha-1) - x(\alpha,2\alpha-1) - 18t_{6,\alpha} = 0$

In addition to the above solutions we have other choices of solutions to (1) which are illustrated below.

Choice 1

3.4. Introducing the linear transformations

$$x = X + 9T \text{ , } z = X + 11T \tag{7}$$

in (1), it is written as

$$X^2 = 99T^2 + y^2 \tag{8}$$

which is satisfied by

$$T = 2rs \text{ , } X = 99r^2 + s^2 \text{ , } y = 99r^2 - s^2 \tag{9}$$

Substituting the above values of X and T in (7), we have

$$x = 99r^2 + s^2 + 18rs \text{ , } z = 99r^2 + s^2 + 22rs \tag{10}$$

Thus the integer solutions of (1) are given by (9) & (10)

Properties:

$i) z(2s-1,s) - 4t_{3,s} = 0$

$ii). x(1,s) - y(1,s) - 2t_{4,s} \equiv 0 \pmod{18}$

$iii) z(1,s) - t_{4,s} \equiv 11 \pmod{22}$

3.5. Note that (8) is expressed in the system of double equations as follows:

$X+y$	$9T^2$	$11T$	$33T$	$3T^2$
$X-y$	11	$9T$	$3T$	3

Solving each of the above systems and performing a few calculations, the corresponding four sets of solutions to (1) are given by,

Set 1

$$x = 18k^2 + 36k + 19 \text{ , } y = 18k^2 + 18k - 1 \text{ , } z = 18k^2 + 40k + 21$$

Properties:

$i) z(k^2) - y(k^2) - 22t_{4,k} \equiv 0 \pmod 2$

$ii)\, y(k) - 18PR_k + 1 = 0$

$iii)\, z(k^2) - x(k^2) - 4t_{4,k} \equiv 0 \pmod 2$

Set 2

$$x = 38k \;,\; y = 2k \;,\; z = 42k$$

Set 3

$$x = 54k \;,\; y = 30k \;,\; z = 58k$$

Set 4

$$x = 66k^2 + 84k + 27 \;,\; y = 66k^2 + 66k + 15 \;,\; z = 66k^2 + 88k + 29$$

Properties:

$i)\, z(k(k+1)) - y(k(k+1)) - 22PR_k \equiv 0 \pmod 2$

$ii)\, x(k) + y(k) - t_{268,k} \equiv 42 \pmod{281}$

$iii)\, z(k(2k-1)) - x(k(2k-1)) - 4t_{6,4} \equiv 0 \pmod 2$

3.6. Observe that (8) is expressed in the form of ratio as,

$$\frac{(X+y)}{9T} = \frac{11T}{(X-y)} = \frac{\alpha}{\beta}, \beta \neq 0 \tag{11}$$

The above equation (11) is equivalent to the system of equations

$$\beta X + \beta y - 9\alpha T = 0 \;,\; \alpha X - \alpha y - 11\beta T = 0$$

The above system is solved by applying the method of cross multiplication and after simplification the values of x, y, z are given by

$$x = 9\alpha^2 + 11\beta^2 + 18\alpha\beta \;,\; y = 9\alpha^2 - 11\beta^2 \;,\; z = 9\alpha^2 + 11\beta^2 + 22\alpha\beta$$

Properties:

$i)\, x(\alpha,1) + y(\alpha,1) - 36t_{3,\alpha} = 0$

$ii)\, y(1,\beta) + 11t_{4,\beta} \equiv 0 \pmod 3$

$iii)\, x(\alpha,1) - 9t_{4,\alpha} \equiv 11 \pmod{18}$

Note that (8) may also be represented in three ways as follows:

Way 1

$$\frac{(X+y)}{3T} = \frac{33T}{(X-y)} = \frac{\alpha}{\beta}, \beta \neq 0$$

Way 2

$$\frac{(X+y)}{11T} = \frac{9T}{(X-y)} = \frac{\alpha}{\beta}, \beta \neq 0$$

Way 3

$$\frac{(X+y)}{33T}=\frac{3T}{(X-y)}=\frac{\alpha}{\beta},\beta\neq 0$$

Following the above procedure, the corresponding solutions to (1), obtained from each of the above representations are presented below:

Integer solutions from way 1

$$x=3\alpha^2+33\beta^2+18\alpha\beta\ ,\ y=3\alpha^2-33\beta^2\ ,\ z=3\alpha^2+33\beta^2+22\alpha\beta$$

Properties:

$i)\, y(\alpha,1)+z(\alpha,1)-t_{14,\alpha}\equiv 0(\text{mod } 27)$

$ii)\, 6[z(\alpha,1)-x(\alpha,1)]$ is a nasty number.

$iii)\, z(\alpha,1)-t_{8,\alpha}\equiv 9(\text{mod } 24)$

Integer solutions from way 2

$$x=11\alpha^2+9\beta^2+18\alpha\beta\ ,\ y=11\alpha^2-9\beta^2\ ,\ z=11\alpha^2+9\beta^2+22\alpha\beta$$

Properties:

$i)\, y(\alpha,1)+z(\alpha,1)-44t_{3,\alpha}=0$

$ii)\, y(\alpha,\alpha+1)-t_{6,\alpha}\equiv 8(\text{mod } 17)$

$iii)\, z(\alpha,\alpha+1)-x(\alpha,\alpha+1)-4PR_{\alpha}=0$

Integer solutions from way 3

$$x=33\alpha^2+3\beta^2+18\alpha\beta\ ,\ y=33\alpha^2-3\beta^2,\ z=33\alpha^2+3\beta^2+22\alpha\beta$$

Properties:

$i)\, y(\alpha,\alpha)-30t_{4,\alpha}=0$

$ii)\, 6[z(\beta,\beta)-x(\beta,\beta)]$ is a nasty number.

$iii)\, x(\alpha,1)+y(\alpha,1)-66t_{4,\alpha}\equiv 0(\text{mod } 18)$

Choice 2

3.7. Introducing the linear transformations

$$y=X+9T\ \ ,\ z=X-2T \tag{12}$$

in (1), it is written as

$$x^2=18T^2+X^2 \tag{13}$$

which is satisfied by

$$T=2rs\,,\ X=18r^2-s^2\ ,\ x=18r^2+s^2 \tag{14}$$

Substituting the above values of X and T in (12) we have,

$$y=18r^2-s^2+18rs\ \ z=18r^2-s^2-4rs \tag{15}$$

Thus the integer solutions of (1) are given by (14) and (15)

Properties:

$$i)\, y(r,1)+z(r,1)-t_{74,r} \equiv 47 \pmod{49}$$
$$ii)\, x(r,1)+z(r,1)-36t_{4,r} \equiv 0 \pmod{4}$$
$$iii)\, y(r,r^2+1)-z(r,r^2+1)-44CP_{r,3}=0$$

3.8. Note that (13) is expressed in the system of double equations as follows:

$x+X$	$9T$	$9T^2$	$6T$	$3T^2$
$x-X$	$2T$	2	$3T$	6

Solving each of the above systems and performing few calculations, the corresponding four sets of solutions to (1) are given by

Set 1

$$x=11k \text{ , } y=25k \text{ , } z=3k$$

Set 2

$$x=18k^2+1 \text{ , } y=18k^2+18k-1 \text{ , } z=18k^2-4k-1$$

Properties:

$$i)\, z(k^3)-y(k^3)+22CP_{k,6}=0$$
$$ii)\, x(k)-18t_{4,k}-1=0$$
$$iii)\, y(k^2)-x(k^2)-18t_{4,k} \equiv 0 \pmod{2}$$

Set 3

$$x=9k \text{ , } y=21k \text{ , } z=-k$$

Set 4

$$x=6k^3+3 \text{, } y=6k^2+18k-3 \text{ , } z=6k^2-4k-3$$

Properties:

$$i)\, y(k^2)-z(k^2)-22t_{4,k}=0$$
$$ii)\, x(k)+y(k)-t_{26,k} \equiv 0 \pmod{29}$$
$$iii)\, x(k)-6t_{4,k} \equiv 0 \pmod{3}$$

3.9. Observe that (13) is expressed in the form of ratio as

$$\frac{(x+X)}{2T}=\frac{9T}{(x-X)}=\frac{\alpha}{\beta},\beta\neq 0 \qquad (16)$$

The above equation (16) is equivalent to the system of equations

$$\beta x+\beta X-2\alpha T=0 \text{ , } \alpha x-\alpha X-9\beta T=0$$

The above system is solved by applying the method of cross multiplication and after simplification the values of x, y, z are given by

$x = 2\alpha^2 + 9\beta^2$, $y = 2\alpha^2 - 9\beta^2 + 18\alpha\beta$, $z = 2\alpha^2 - 9\beta^2 - 4\alpha\beta$

Properties:

$i) 6\left[x(\alpha, \alpha+1) + y(\alpha, \alpha+1) - 36t_{6,\beta}\right]$ is a nasty number.

$ii) y(2\beta - 1, \beta) - z(2\beta - 1, \beta) - 22t_{6,\beta} = 0$

$iii) x(\alpha, \alpha) - 11t_{4,\alpha} = 0$

Note that (13) may also be represented in three ways as follows:

Way 1

$$\frac{(x+X)}{3T} = \frac{6T}{(x-X)} = \frac{\alpha}{\beta}, \beta \neq 0$$

Way 2

$$\frac{(x+X)}{T} = \frac{18T}{(x-X)} = \frac{\alpha}{\beta}, \beta \neq 0$$

Way 3

$$\frac{(x+X)}{9T} = \frac{2T}{(x-X)} = \frac{\alpha}{\beta}, \beta \neq 0$$

Following the above procedure, the corresponding solutions to (1) obtained from each of the above representations are presented below:

Integer solutions from way 1

$x = 3\alpha^2 + 6\beta^2$, $y = 3\alpha^2 - 6\beta^2 + 18\alpha\beta$, $z = 3\alpha^2 - 6\beta^2 - 4\alpha\beta$

Properties:

$i) y(\alpha, \alpha^2) - z(\alpha, \alpha^2) - 22CP_{\alpha,6} = 0$

$ii) x(\alpha, \alpha+1) + y(\alpha, \alpha+1) - 6t_{4,\alpha} - 36t_{3,\alpha} = 0$

$iii) z(\alpha, 1) - t_{8,\alpha} \equiv 0 \pmod{2}$

Integer solutions from way 2

$x = \alpha^2 + 18\beta^2$, $y = \alpha^2 - 18\beta^2 + 18\alpha\beta$, $z = \alpha^2 - 18\beta^2 - 4\alpha\beta$

Properties:

$i) x(\alpha, \alpha) - 19t_{4,\alpha} = 0$

$ii) x(\alpha, 1) + y(\alpha, 1) - t_{6,\alpha} \equiv 0 \pmod{19}$

$iii) y(\alpha, \alpha^2) - z(\alpha, \alpha^2) - CP_{\alpha,6} = 0$

Integer solutions from way 3

$x = 9\alpha^2 + 2\beta^2$, $y = 9\alpha^2 - 2\beta^2 + 18\alpha\beta$, $z = 9\alpha^2 - 2\beta^2 - 4\alpha\beta$

Properties:

$i)\, x(\alpha,1)+y(\alpha,1)-18PR_{\alpha}=0$

$ii)\, y(\alpha,2\alpha-1)-z(\alpha,2\alpha-1)-22t_{6,\alpha}=0$

$iii)\, x(\alpha,\alpha)-11t_{4,\alpha}=0$

4. Conclusion

In this paper we have made an attempt to find all integer solutions to the ternary quadratic equation given by $11x^2-2y^2=9z^2$.It is worth to mention that the above equation represents a cone. As quadratic equations in three unknowns are rich in variety, one may attempt to find integer solutions to other choices of ternary quadratic equations along with suitable properties.

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