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On the Non-Homogeneous Cubic Equation with Four Unknowns $(x\text{-}y)^2 = 2z^3+w^2$

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Abstract. The non-homogeneous cubic equation with four unknowns given by $(x-y)^2 = 2z^3 + w^2$ is analyzed for its distinct integer solutions. Three different patterns of integer solutions to the above equation are obtained. A few interesting relations between the solutions and special polygonal numbers are also obtained.

Keywords: Non-homogeneous cubic equation, cubic with four unknowns, integer solutions.

AMS Mathematics Subject Classification (2010): 11D25

1. Introduction

Integral solutions for the non-homogeneous Diophantine cubic equation is an interesting concept as it can be seen from [1, 2, 3]. In [4-8] a few special cases of cubic Diophantine equation with three and four unknowns are studied. In this communication, we present the integral solutions of an interesting cubic equation with four unknowns $(x\text{-}y)^2 = 2z^3+w^2$. A few remarkable relations between the solutions are presented.

2. Notations

1) Polygonal number of rank ‘n’ with m sides

$$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$$

2) Gnomonic number of rank ‘n’

$$G_n = 2n - 1$$

3) Pronic number of rank ‘n’

$$PR_n = n(n+1)$$

4) Centered Polygonal number of rank ‘n’ with m sides

$$Ct_{m,n} = \frac{mn(n-1)+2}{2}$$

5) Centered hexagonal Pyramidal number of rank 'n' with m sides

$CP_{k,6}= n^3$

6) Star number of rank 'n'

$S_n = 6n(n-1)+1$

3. Method of analysis

The non-homogeneous cubic equation with four unknowns under consideration is

$$(x-y)^2 = 2z^3 + w^2 \tag{1}$$

Introducing the linear transformations

$$x = u+v,\ y = u-v,\ z = 2T \tag{2}$$

in (1), it is written as

$$v^2 - s^2 = 4T^3 \tag{3}$$

The above equation (3) is written in the system of double equations as follows:

System	**1**	**2**	**3**	**4**	**5**
$v+s$	T^3	T^2	$2T$	2	$4T^2$
$v-s$	4	$4T$	$2T^2$	$2T^3$	T

Consider **system (1)** and solving for v and s,

we have $v = \dfrac{4+T^3}{2}$, $s = \dfrac{T^3-4}{2}$, $T = 2k$

Using the above values in (2) we have,

$$x = u+4k^3+2,\ y = u-4k^3-2,\ z = 4k,\ w = 8k^3-4 \tag{4}$$

which represent the integer solutions of (1)

Properties:

1. $3\left[(x-y)^2 + w^2 - 32\right]$ is a nasty number.
2. $x(u,k) - y(u,k) - CP_{2k,6} - 4 = 0$
3. $x(u,k) + y(u,k)$ is always even.

Consider **system (2)** and solving for v and s, we have

$v = \dfrac{T^2+4T}{2}$, $s = \dfrac{T^2-4T}{2}$, $T = 2k$

Using the above values in (2) we have,

$$x = u+2k^2+4k,\ y = u-2k^2-4k,\ z = 4k,\ w = 4k^2-8k \tag{5}$$

which represent the integer solutions of (1)

Properties:

1. $x(u,k) + y(u,k) - G_u - 1 = 0$
2. $z(k) + w(k) - 4PR_k + 4G_k + 4 = 0$
3. $y(k,k) + z(k) + t_{6,n} = 0$

Consider **system (3)** and solving for v and s we have,

$v = T + T^2$, $s = T - T^2$, $T = 2k$

Using the above values in (2) we have,

$$x = u + 2k + 4k^2,\ y = u - 2k - 4k^2,\ z = 4k,\ w = 4k - 8k^2 \tag{6}$$

which represent the integer solutions of (1)

Properties:

1. $x(k,k) - y(k,k) - 8t_{4,k} \equiv 0 \pmod 4$
2. $z(k) - w(k) - 16t_{3,k} + 4G_k - 4 = 0$
3. $x(k,k) + w(k) - 4PR_k + 8t_{4,k} \equiv 0 \pmod 3$

Consider **system (4)** and solving for v and s we have,

$v = 1 + T^3$, $s = 1 - T^3$, $T = 2k$

Using the above values in (2) we have,

$$x = u + 1 + 8k^3,\ y = u - 1 - 8k^3,\ z = 4k,\ w = 2 - 16k^3 \tag{7}$$

which represent the integer solutionsof (1)

Properties:

1. $6[x(u,k) - y(u,k) + w(k)]$ is a nasty number
2. $x(u,k) + y(u,k) - G_k - 1 = 0$
3. $y(1,k) - w(k) - CP_{2k,6} + 2 = 0$

Consider **system** (5) and solving for v and s we have,

$v = \dfrac{4T^2 + T}{2}$, $s = \dfrac{4T^2 - T}{2}$, $T = 2k$

Using the above values in (2) we have,

$$x = u + 8k^2 + k,\ y = u - 8k^2 - k,\ z = 4k,\ w = 16k^2 - 2k \tag{8}$$

which represent the integer solutions of (1)

Properties:

1. $x(k,k) - 8PR_n \equiv 0 \pmod 6$
2. $6[x(2,k) + y(2,k)]$ is a nasty number.
3. $z(k) + w(k) - 12t_{4,n} - 2t_{6,n} \equiv 0 \pmod 4$

Note that equation (3) is solved as follows,

Replacing v by $2V$ and s by $2P$ in equation (3) we have,

$$V^2 - P^2 = T^3 \tag{9}$$

Following the procedure as above we obtain two sets of integer solution to (1) represented as below.

Set 1

$x = u + 2t_{3,T}$, $y = u - 2t_{3,T}$, $z = 2T$, $w = 4t_{3,(T-1)}$

Set 2

$x = u + 8k^3 + 2k$, $y = u - 8k^3 - 2k$, $z = 4k$, $w = 16k^3 - 4k$

In addition to the above sets of solutions we have some more choices of solutions to (1) illustrated below:

Taking $x = y + U$ (10)

in (1), we have

$U^2 - w^2 = 2z^3$ (11)

The above equation (11) is written in the system of double equations as follows:

System	**6**	**7**	**8**
$U+w$	$2z$	2	$2z^2$
$U-w$	z^2	z^3	z

Consider **system (6)** and solving for U and w, we have

$$U = \frac{2z+z^2}{2},\ w = \frac{2z-z^2}{2},\ z = 2k$$

Using the above values in (10) we have,

$x = s + 2k + 2k^2$, $y = s$, $z = 2k$, $w = 2k - 2k^2$ (12)

which represent the integer solutions of (1)

Properties:

1. $x(s,k) - y(s) - 2PR_k = 0$
2. $z(k) + w(k) - 2G_k + 2t_{4,k} - 2 = 0$
3. $z(n) + y(n) + 2Ct_{3,n} - 3t_{4,n} - 2 = 0$

Consider **system (7)** and solving for U and w, we have

$x = s + 1 + 4k^3$, $y = s$, $z = 2k$, $w = 1 - 4k^3$ (13)

which represent the integer solutions of (1)

Properties:

1. $z(n) - w(n) - G_n - 4Ct_{n,6} = 0$
2. $x(n,n) + y(n) - 4CP_{n,5} - 2CP_{n,6} - 1 = 0$
3. $w(n) + x(n,n) + 2Ct_{1,n} - t_{4,n} - 4 = 0$

Consider **system (8)** and solving for U and w, we have

$U = 4k^2 + k$, $w = 4k^2 - k$, $z = 2k$

Using the above values in (10) we have,

$x = k + s + 4k^2$, $y = s$, $z = 2k$, $w = 4k^2 - k$ (14)

which represent the integer solutions of (1).

Properties:

1. $x(k,s) - y(s) - 4PR_k \equiv 0 \pmod 3$

2. $z(k)+w(k)-4t_{4,k}-G_k+k-1=0$

3. $x(k,k)+z(k)-4PR_k=0$

Note that equation (11) is also solved as follows:
Replacing U by $\alpha+\beta$ and w by $\alpha-\beta$ in equation (11) we have,

$$2\alpha\beta=z^3 \tag{15}$$

Choosing $\alpha=2^{3k+2}\beta^2,\ k\geq 0$
we get ,

$z=2^{k+1}\beta,\ w=2^{3k+2}\beta^2-\beta,\ U=2^{3k+2}\beta^2+\beta$

Using the above values in (10) we have,

$$x=s+2^{3k+2}\beta^2+\beta,\ y=s,\ z=2^{k+1}\beta,\ w=2^{3k+2}\beta^2-\beta \tag{16}$$

which represent the integer solutions of (1)

Properties:

1. $6[z(2,1)+y(1)]$ is a nasty number.
2. $z(0,n)+w(0,n)-4PR_n-2Ct_{3,n}-3t_{4,n}-2=0$
3. $x(1,n,n)+y(n)-26t_{4,n}+s_n-1\equiv 0 \pmod 7$

4. Conclusion

In this paper, we have made an attempt to obtain all possible integer solutions to the non-homogeneous cubic equation with four unknowns represented by $(x-y)^2=2z^3+w^2$ Since, by definition, the cubic equation in four unknowns are rich in variety, to conclude one may search for integer solutions to other choices of cubic equations along with suitable properties.

REFERENCES

1. L.E.Dickson, History of Theory of Numbers, Vol 2, Chelsea publishing company, New York, (1952).
2. L.J.Mordell, Diophantine Equations, Academic press, London, (1969).
3. R.D.Carmichael, The theory of numbers and Diophantine analysis, New York, Dover, (1959).
4. M.A.Gopalan and S.Premalatha, Integral solutions of $(x+y)(xy+w^2)=2(k^2+1)z^3$. *Bulletin of Pure and Applied Sciences*, 28E (2) (2009) 197-202.
5. M.A.Gopalan and V.Pandichelvi, Remarkable solutions on the cubic equation with four unknowns $x^3+y^3+z^3=28(x+y+z)w^2$ *Antarctica J. of Maths.*, 4(4) (2010) 393-401.
6. M.A.Gopalan and B.Sivagami, Integral solutions of homogeneous cubic equation with four unknowns $x^3+y^3+z^3=3xyz+2(x+y)w^3$, *Impact. J. Sci. Tec*, 4(3) (2010) 53-60.

7. M.A.Gopalan and S.Premalatha, On the cubic Diophantic equations with four unknowns $(x-y)(xy-w^2)=2(n^2+2n)z^3$, *International Journal of Mathematical Sciences*, 9(1-2) ((2010) 171-175.
8. M.A.Gopalan and J.Kaliga Rani, Integral solutions of $x^3+y^3+(x+y)xy=z^3+w^3+(z+w)zw$, *Bulletin of Pure and Applied Sciences,* 29E (1) (2010) 169-173.