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Continued Fractions of Ratios of Consecutive Polygonal Numbers

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Abstract. In this paper the patterns of continued fractions of ratios of polygonal numbers for consecutive sizes have been identified.

Keywords: Continued fractions, Simple continued fraction, Euclidean algorithm, Polygonal numbers.

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1. Notations

1. $[a_0, a_1, a_2, a_3, \cdots a_n]$:Continued fraction expansion.
2. $[x_k]$: Integer part of x_k
3. $P_d(n)$: Polygonal number of order d and rank n.

2. Introduction

The ancient Greek mathematician Diophantos was one of the first to study polygonal numbers. A polygonal number can be defined as a sum of equidistant dots used to represent a polygon of a certain size. For example, if you have a square number with rank one it is one, rank two is four because you expand the length and width by one dot each and fill in the outer layer, then rank three would be nine and it continues in this fashion. The rank of a polygonal numbers is the number of dots on a side of the outermost layer of the polygonal number. This holds true for all polygonal numbers.

A polygonal number is denoted by $P_d(n)$ where d is the number of sides to the corresponding polygon and n is the rank, or order, of the polygonal number. For instance $P_5(4)$ would be a pentagonal number with rank four. All polygonal numbers with rank one equals one, and all polygonal numbers of rank two are equal to the number of sides on the corresponding polygon.

$P_3(3)=6$ $\quad P_4(3)=9$ $\quad P_5(3)=12$

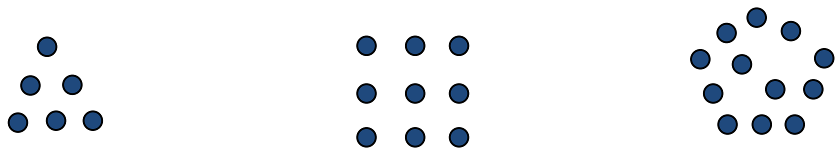

Figure 1:

Furthermore, you can find any polygonal number by using the formula

$P_d(n) = \dfrac{(d-2)n^2 + (4-n)n}{2}$. Note that $d, n \in N$ and, since less than three sides would not form a polygon.

Table 1: Values of some polygonal numbers

Sides	**Rank of polygonal numbers**											
d	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
3	1	3	6	10	15	21	28	36	45	55	66	78
4	1	4	9	16	25	36	49	64	81	100	121	144
5	1	5	12	22	35	51	70	92	117	145	176	210
6	1	6	15	28	45	66	91	120	153	190	231	276
7	1	7	18	34	55	81	112	148	189	235	286	342
8	1	8	21	40	65	96	133	176	225	280	341	408
9	1	9	24	46	75	111	154	204	261	325	396	474
10	1	10	27	52	85	126	175	232	297	370	451	540
11	1	11	30	58	95	141	196	260	333	415	506	606
12	1	12	33	64	105	156	217	288	369	460	561	672
13	1	13	36	70	115	171	238	316	405	505	616	738
14	1	14	39	76	125	186	259	344	441	550	671	804
15	1	15	42	82	135	201	280	372	477	595	726	870
16	1	16	45	88	145	216	301	400	513	640	781	936
17	1	17	48	94	155	231	322	428	549	685	836	1002
18	1	18	51	100	165	246	343	456	585	730	891	1068
19	1	19	54	106	175	261	364	484	621	775	946	1134
20	1	20	57	112	185	276	385	512	657	820	1001	1200
21	1	21	60	118	195	291	406	540	693	865	1056	1266
22	1	22	63	124	205	306	427	568	729	910	1111	1332
23	1	23	66	130	215	321	448	596	765	955	1166	1398
24	1	24	69	136	225	336	469	624	801	1000	1221	1464
25	1	25	72	142	235	351	490	652	837	1045	1276	1530
26	1	26	75	148	245	366	511	680	873	1090	1331	1596
27	1	27	78	154	255	381	532	708	909	1135	1386	1662
28	1	28	81	160	265	396	553	736	945	1180	1441	1728
29	1	29	84	166	275	411	574	764	981	1225	1496	1794
30	1	30	87	172	285	426	595	792	1017	1270	1551	1860

Continued fraction plays an important role in number theory. It is used to represent the rational numbers to an another form by using Euclidean algorithm.

An expression of the form

$$\frac{p}{q} = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{\ddots}}}}$$

where a_i, b_i are real or complex numbers is called a continued fraction.

An expression of the form

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

where $b_i = 1\ \forall i,$ and $a_0, a_1, a_2, \cdots$ are each positive integers is called a simple continued fraction.

The continued fraction is commonly expressed as

$$\frac{p}{q} = a_0 + \frac{1}{a_1 +}\frac{1}{a_2 +}\frac{1}{a_3 +}\cdots \text{or simply as } [a_0, a_1, a_2, a_3, \cdots].$$

The elements $a_0, a_1, a_2, a_3, \cdots$ are called the partial quotients. If there are finite number of partial quotients, we call it finite simple continued fraction, otherwise it is infinite.

The continued fraction expansion of 22/7 is [3; 7]. Here $a_0 = 3$.

3. The continued fraction algorithm

Suppose we wish to find continued fraction expansion of $x \in R$.

Let $x_0 = x$ *and set* $a_0 = [x_0]$.

Define $x_1 = \dfrac{1}{x_0 - [x_0]}$ *and set* $a_1 = [x_1]$.

and $x_2 = \frac{1}{x_1 - [x_1]} \Rightarrow a_2 = [x_2],....., x_k = \frac{1}{x_{k-1} - [x_{k-1}]} \Rightarrow a_k = [x_k],....$

This process is continued infinitely or to some finite stage till an $x_i \in N$ exists such that $a_i = [x_i]$.

Examples

1. Continued fraction expansion of 414/283 = 1.4629 is [1; 2, 6, 4, 5]
2. Continued fraction expansion of $\sqrt{3}$ and $\sqrt{7}$ are [1; 1, 2 , 1, 2, 1, 2,…] and [2; 1, 1, 1, 4, 1, 1, 1, 4,…]. Which are known as periodic continued fractions. The above periodic continued fractions are also denoted by $[1;\overline{1,2}]$ and $[2;\overline{1,1,1,4}]$.

In this paper we try to find some patterns of ratios of consecutive polygonal numbers of different sides and different orders using continued fractions.

Theorem 1. The continued fraction of ratio of consecutive polygonal number of rank 2 is $[0;1,d\,]$ where d is the sides of a polygon and $d \geq 3$.

In other words $\frac{P_d(2)}{P_{d+1}(2)} = [0;1,d\,]$, where $d \geq 3$.

Proof: Take $d = 3$.

Therefore $\frac{P_d(2)}{P_{d+1}(2)} = \frac{3}{4}$.

Using continued fraction algorithm,

Take $x_0 = \frac{3}{4}$, so $a_0 = 0$. Then $x_1 = \frac{1}{x_0 - [x_0]} = \frac{3}{4} = 1 + \frac{1}{3} \Rightarrow a_1 = 1$.

$x_2 = \frac{1}{x_1 - [x_1]} = 3 \Rightarrow a_2 = 3$.

Therefore $\frac{P_3(2)}{P_4(2)} = \frac{3}{4} = [0;1,3\,]$.

The result is true when $d = 3$.

Assume the result is true for $d = k-1, k \geq 4$.

Therefore $\frac{P_{k-1}(2)}{P_k(2)} = [0;1,k-1]$.

We prove the result for $d = k$.

$$\frac{P_k(2)}{P_{k+1}(2)} = \frac{(k-2)4+(k-2)2}{(k-1)4+(3-k)2} = \frac{(k-2)2+(k-2)}{(k-1)2+(3-k)} = \frac{k}{k+1}.$$

Take

$$x_0 = \frac{k}{k+1}$$

so $a_0 = 0.$

Then $x_1 = \dfrac{1}{x_0 - [x_0]} = \dfrac{k+1}{k} = 1 + \dfrac{1}{k} \Rightarrow a_1 = 1$

$x_2 = \dfrac{1}{x_1 - [x_1]} = k \Rightarrow a_2 = k.$

Therefore $\dfrac{P_k(2)}{P_{k+1}(2)} = [0;1,k].$

Hence by induction the result is true for all values of d, where $d \geq 3$.

Theorem 2. The continued fraction of ratio of consecutive polygonal number of rank 3 is $[0;1,d-1]$ where d is the sides of a polygon and $d \geq 3$.

In other words $\dfrac{P_d(3)}{P_{d+1}(3)} = [0;1,d-1]$, where $d \geq 3$.

Proof: Similar to the proof of theorem 1.

Theorem 3. The continued fraction of ratio of consecutive polygonal number of side ≥ 4 and order ≥ 3 is

$\left[0;1,d-2,\dfrac{R-1}{2}\right]$ if R is odd and $\left[0;1,d-2,\dfrac{R-2}{2},2\right]$ if R is even.

In other words $$\frac{P_d(R)}{P_{d+1}(R)} = \begin{cases} \left[0;1,d-2,\dfrac{R-1}{2}\right] & if\ R\,is\,odd \\ \left[0;1,d-2,\dfrac{R-2}{2},2\right] & if\ R\,is\,even \end{cases}$$

Proof:

Case (i): When R is odd ie., $R = 2n+1, n \geq 2.$

Take $d = 3$ and $R = 5.$

Therefore $\dfrac{P_3(5)}{P_4(5)} = \dfrac{15}{25}.$

Using continued fraction algorithm,

Take $x_0 = \dfrac{15}{25}$, so $a_0 = 0$. Then $x_1 = \dfrac{1}{x_0 - [x_0]} = \dfrac{25}{15} = 1 + \dfrac{10}{15} \Rightarrow a_1 = 1.$

$x_2 = \dfrac{1}{x_1 - [x_1]} = \dfrac{15}{10} = 1 + \dfrac{5}{10} \Rightarrow a_2 = 1.$

$$x_3 = \frac{1}{x_2 - [x_2]} = \frac{10}{5} = 2 \Rightarrow a_3 = 2$$

Therefore $\frac{P_3(5)}{P_4(5)} = \frac{15}{25} = [0;1,1,2]$.

Therefore the result is true for $d = 3$ and $R = 5$.

Assume the result is true for $d = k-1$ and $R = 2n-1$.

$$\frac{P_{k-1}(2n-1)}{P_k(2n-1)} = \left[0;1,k-3,\frac{2n-2}{2}\right] = [0;1,k-3,n-1].$$

Prove the result is true for $d = k$ and $R = 2n+1$.

Therefore

$$\frac{P_k(2n+1)}{P_{k+1}(2n+1)} = \frac{(k-2)(2n+1)^2 + (4-k)(2n+1)}{(k-1)(2n+1)^2 + (3-k)(2n+1)} = \frac{(k-2)(2n+1) + (4-k)}{(k-1)(2n+1) + (3-k)}$$

$$= \frac{2kn - 4n + 2}{2kn - 2n + 2}$$

$$= \frac{kn - 2n + 1}{kn - n + 1}$$

Therefore $\frac{P_k(2n+1)}{P_{k+1}(2n+1)} = \frac{kn - 2n + 1}{kn - n + 1}$

Using continued fraction algorithm,

Take $x_0 = \frac{kn - 2n + 1}{kn - n + 1} \Rightarrow a_0 = 0.$ Then

$$x_1 = \frac{1}{x_0 - [x_0]} = \frac{kn - n + 1}{kn - 2n + 1} = 1 + \frac{n}{kn - 2n + 1} \Rightarrow a_1 = 1.$$

$$x_2 = \frac{1}{x_1 - [x_1]} = \frac{kn - 2n + 1}{n} = k - 2 + \frac{1}{n} \Rightarrow a_2 = k - 2.$$

$$x_3 = \frac{1}{x_2 - [x_2]} = n \Rightarrow a_3 = n$$

Therefore $\frac{P_k(2n+1)}{P_{k+1}(2n+1)} = [0;1,k-2,n]$.

Hence by induction the result is true for all value of d and R when R is odd and $d \geq 3$.

Case (ii): When R is even ie., $R = 2n$, $n \geq 2$.

Take $d = 3$ and $R = 4$.

Therefore $\frac{P_3(4)}{P_4(4)} = \frac{10}{16}$.

Using continued fraction algorithm,

Take $x_0 = \frac{10}{16}$, so $a_0 = 0$. Then $x_1 = \frac{1}{x_0 - [x_0]} = \frac{16}{10} = 1 + \frac{6}{10} \Rightarrow a_1 = 1.$

$$x_2 = \frac{1}{x_1 - [x_1]} = \frac{10}{6} = 1 + \frac{4}{6} \Rightarrow a_2 = 1.$$

$$x_3 = \frac{1}{x_2 - [x_2]} = \frac{6}{4} = 1 + \frac{2}{4} \Rightarrow a_3 = 1$$

$$x_4 = \frac{1}{x_3 - [x_3]} = \frac{4}{2} = 2 \Rightarrow a_4 = 2$$

Therefore $\frac{P_3(4)}{P_4(4)} = \frac{10}{16} = [0;1,1,1,\ 2\ ].$

Therefore the result is true for $d = 3$ and $R = 4$.

Assume the result is true for $d = k-1$ and $R = 2n-1$.

$$\frac{P_{k-1}(2n-1)}{P_k(2n-1)} = \left[0;1,k-3,\frac{2n-2}{2}\right] = [0;1,k-3,n-1].$$

Prove the result is true for $d = k$ and $R = 2n$.

Therefore $\frac{P_k(2n)}{P_{k+1}(2n)} = \frac{(k-2)(2n)^2 + (4-k)(2n)}{(k-1)(2n)^2 + (3-k)(2n)} = \frac{(k-2)(2n) + (4-k)}{(k-1)(2n) + (3-k)}$

$$= \frac{2kn - 4n - k + 4}{2kn - 2n - k + 3}$$

Therefore $\frac{P_k(2n)}{P_{k+1}(2n)} = \frac{2kn - 4n - k + 4}{2kn - 2n - k + 3}$

Using continued fraction algorithm,

Take $x_0 = \frac{2kn - 4n - k + 4}{2kn - 2n - k + 3} \Rightarrow a_0 = 0.$ Then

$$x_1 = \frac{1}{x_0 - [x_0]} = \frac{2kn - 2n - k + 3}{2kn - 4n - k + 4} = 1 + \frac{2n-1}{2kn - 4n - k + 4} \Rightarrow a_1 = 1.$$

$$x_2 = \frac{1}{x_1 - [x_1]} = \frac{2kn - 4n - k + 4}{2n-1} = k - 2 + \frac{2}{2n-1} \Rightarrow a_2 = k - 2.$$

$$x_3 = \frac{1}{x_2 - [x_2]} = \frac{2n-1}{2} \Rightarrow n - 1 + \frac{1}{2} a_3 = n - 1$$

$$x_4 = \frac{1}{x_3 - [x_3]} = 2 \Rightarrow a_4 = 2$$

Therefore $\frac{P_k(2n)}{P_{k+1}(2n)} = [0;1,k-2,n-1,2\ ].$

Hence by induction the result is true for all values of d and R when R is even and $d \geq 3$.

$$\text{Hence}\quad \frac{P_d(R)}{P_{d+1}(R)} = \begin{cases} \left[0;1,d-2,\frac{R-1}{2}\right] & \text{if } R \text{ is odd} \\ \left[0;1,d-2,\frac{R-2}{2},2\right] & \text{if } R \text{ is even} \end{cases}$$

4. Illustration

The following table gives the patterns of continued fractions of consecutive polygonal numbers of different orders and ranks.

Consecutive fractions of polygonal numbers	**Continued fraction expansion**
$\frac{P_5(2)}{P_6(2)}$	[0; 1, 5]
$\frac{P_{12}(2)}{P_{13}(2)}$	[0; 1, 12]
$\frac{P_4(3)}{P_5(3)}$	[0; 1, 3]
$\frac{P_{15}(3)}{P_{16}(3)}$	[0; 1, 14]
$\frac{P_9(5)}{P_{10}(5)}$	[0; 1, 7, 2]
$\frac{P_{17}(6)}{P_{18}(6)}$	[0; 1, 15, 2, 2]
$\frac{P_{10}(9)}{P_{11}(9)}$	[0; 1, 8, 4]
$\frac{P_{26}(10)}{P_{271}(10)}$	[0; 1, 24, 4, 2]
$\frac{P_{29}(12)}{P_{30}(12)}$	[0; 1, 27, 5, 2]

5. Conclusion

In this paper, we have identified various patterns of continued fractions of ratios of polygonal numbers of consecutive sizes. This work may be extended to higher order figurate numbers like pyramidal numbers.

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