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Some Solutions of the Diophantine Equation $a^x + (3a+4)^y = z^2$ where $a \equiv 15 \pmod{48}$ from the Lucas and Fibonacci Numbers

Tasanai Rangpung

Department of Mathematics, Faculty of Science Ubon Ratchatani Rajabhat University
Ubon Ratchathani, 3400, Thailand. E-mail: Tasanai.r@ubru.ac.th

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Abstract. For any a positive integers of the Diophantine Equation $a^x + (3a+4)^y = z^2$ where $a \equiv 15$ (mod 48) there are only two infinite solutions $(x, y, z) = (1, 0, (a+1)^{1/2})$ where $a = (12t \pm 4)^2-1$ and $(x, y, z) = (1, 1, (4a+4)^{1/2})$ where $a = (12t \pm 4)^2-1$ with x, y and z are non-negative integers. In addition, at the point (x, y) = (3, 2) has non-negative integer solutions and our answers are also applicable to the Fibonacci and Lucas numbers.

Keywords: Congruence, Non-negative integer solutions, Fibonacci and Lucas numbers.

AMS Mathematics Subject Classification (2010): 11D61, 11D72

1. Introduction

A Diophantine equation is an equation that involves many unknown variables and seeks to find integer solutions. Most mathematicians have studied the renowned Diophantine equation in the given form $a^x + b^y = z^2$ when a and b are positive integers, have been researched (refer to, as an example [3, 5, 6, 8, 9, 11, 13]). In 1844, Catalan [1] The Diophantine equation $a^x - b^y = 1$ has $(a,b,x,y) = (3,2,2,3)$ is the unique solution for $\min\{a,b,x,y\} > 1$ where a, x, y, z are positive integers. In 2017, Priya and Vidhyalakshmi [2], studied that on the Non-Homogeneous Ternary Quadratic Equation $2(x^2+y^2) - 3xy + (x+y) + 1 = z^2$ has non-zero distinct integer solutions four different sets and interesting relations between the solutions and special polygonal numbers. In 2022, Pakapongpun and Chattea [4] proved that $a^x + (a+2)^y = z^2$ where $a \equiv 3 (\bmod 20)$ and $a \in \mathbb{N}$ has solution for $(x,y,z) = (1,0,\sqrt{a+1})$ where $a = ((10k-2)^2 - 1)$ and $k \in \mathbb{Z}$. In 2024, Hashim [10] studied the all solutions of the equation in the Fibonacci and Lucas Numbers where the indices i, j, k which are positive integers are defined by the following: $F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$ and

$L_0 = 2,\ L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ of the Diophantine equation $2^x + 2^y = z^2$. Moreover, In 2024, Tadee [12] showed that the Diophantine equation $p^x + q^y = z^2$ when $p = 3$ such that $(x, y, z) = (F_5, L_3, L_6)$ is the unique solution. Inspiration in this paper, focused on finding all solutions of the Diophantine equation $a^x + (3a+4)^y = z^2$ where $a \equiv 15 \pmod{48}$ for all $a \in \mathbb{N}$ when x, y and z are non-negative integers.

2. Some mathematical tools

(Catalan's Conjecture) [1] The unique solution for the Diophantine equation $a^x - b^y = 1$ where $a, b, x, y \in \mathbb{Z}^+$ with $\min\{a, b, x, y\} > 1$ is $(3, 2, 2, 3)$.

Lemma 2.1 [7] If x is an integer, then $x^2 \equiv 0 \bmod 4$ or $x^2 \equiv 1 \bmod 4$.

Lemma 2.2 If the Diophantine equation $a^x + (3a+4)^y = z^2$ where x, y and z are non-negative integers and $a \in \mathbb{N}$ has unique solution at point $(x, y) = (3, 2)$ when $a \equiv 15 \pmod{48}$ then $(x, y, z, a) = (3, 2, 76, 15)$.

Proof Suppose that $a^x + (3a+4)^y = z^2$; $x, y, z \in \mathbb{Z}^+ \cup \{0\}$, let $x = 3$ and $y = 2$.

We get $a^3 + 9a^2 + 24a + 16 = z^2$ such that $\sqrt{a^3 + 9a^2 + 24a + 16} = z$.

Since $a \equiv 15 \pmod{48}$ thus $a = 48m + 15; m \in \mathbb{Z}^+ \cup \{0\}$. Let $m = 0$ then $a = 15$. It implies that $z = 76$. Therefore, $(x, y, z, a) = (3, 2, 76, 15)$.

3. Option pricing

Theorem 2.3 For all a is a positive integer of the Diophantine equation $a^x + (3a+4)^y = z^2$ where $a \equiv 15 \pmod{48}$ and x, y z are non-negative integers has exactly two non-negative integer infinite solution are as follows.

1. $x = 1$ and $y = 0$ have non-negative integer infinite solutions $(x, y, z) = (1, 0, \sqrt{a+1})$ where $a = (12t \pm 4)^2 - 1$

2. $x = 1$ and $y = 1$ have non-negative integer infinite solutions $(x, y, z) = (1, 1, \sqrt{4a+4})$ where $a = (12t \pm 4)^2 - 1$ **.**

In addition, at the point $(x, y) = (3, 2)$ has non-negative integer solutions**.**

Proof Let $x, y, z \in \mathbb{Z}^+ \cup \{0\}$and $a^x + (3a+4)^y = z^2$

Case 1: let $x = 0$ and $y = 0$ obviously, has no solution because $z^2 = 2$ is impossible.

Case 2: let $x > 1$ and $y = 0$, we obtain the Diophantine equation $z^2 - a^x = 1$ has no non-negative integer solution by Catalan's Conjecture.

Case 3: let $x=0$ and $y=1$, we obtain the Diophantine equation $z^2 = 3a+5; a \in \mathbb{N}$. Since $a \equiv 15 \pmod{48}$ and $4|48$, we get $a \equiv 15 \pmod{4}$, we have $3a+5 \equiv 50 \pmod{4}$. It implies that $z^2 \equiv 2 \pmod{4}$. By Lemma 2.1, which contradicts.

Case 4: let $x=0$ and $y>1$, we obtain the Diophantine equation $z^2-(3a+4)^y=1$ have no non-negative integer solution by Catalan's Conjecture.

Case 5: let $x=1$ and $y=0$ becomes $z=\sqrt{a+1}; a \in \mathbb{N}$. Since $a \equiv 15 \pmod{48}$ and $a=48m+15$. It implies that $z=4\sqrt{3m+1}$, consider $k^2=3m+1$ such that $k^2 \equiv 1 \pmod{3}$. It implies that $3|(k-1)$ or $3|(k+1)$. Consider $3|(k-1)$ becomes $m=3t_1^2+2t_1; t_1 \in \mathbb{Z}^+ \cup \{0\}$. Therefore, $a=(12t_1+4)^2-1$. Consider $3|(k+1)$ becomes $m=3t_1^2-2t_1; t_1 \in \mathbb{Z}^+ \cup \{0\}$. Therefore, $a=(12t_2-4)^2-1$

Case 6: Let $x=1$ and $y=1$ becomes $z=2\sqrt{a+1}; a \in \mathbb{N}$. Since $a \equiv 15 \pmod{48}$ and $a=48m+15$. It implies that $8\sqrt{3m+1}$ Consider $k^2=3m+1$ such that $k^2 \equiv 1 \pmod{3}$. It implies that $3|(k-1)$ or $3|(k+1)$. Consider $3|(k-1)$ becomes. It implies that $m=3t_1^2+2t_1; t_1 \in \mathbb{Z}^+ \cup \{0\}$. Therefore, $a=(12t_1+4)^2-1$. Consider $3|(k+1)$ becomes $m=3t_1^2-2t_1; t_1 \in \mathbb{Z}^+ \cup \{0\}$. It implies that $a=(12t_2+4)^2-1$

Case7: $x \geq 1$ and $y \geq 1$,

Subcase 7.1 Let $x=1$ and $y>1$, consider y is even number such that $y=2k; k \in \mathbb{Z}^+$.

It implies that $\left[(3a+4)^k+\left(\frac{a}{2(3a+4)^k}\right)\right]^2-\left(\frac{a}{2(3a+4)^k}\right)^2=z^2$

Since $\frac{a}{2(3a+4)^k}$ is rational number which contradicts.

Consider y is odd number such that $y=2k+1; k \in \mathbb{Z}^+$.

It implies that $(3a+4)\left[\frac{a}{3a+4}+(3a+4)^{2k}\right]=z^2$

Since $\frac{a}{3a+4}$ is rational number which contradicts.

Subcase 7.2. let $x>1$ and $y=1$

Consider y is even number such that $y=2k; k \in \mathbb{Z}^+$.

It implies that $(a)^{2k}+3a+4=z^2$ then $\left(a^k+\frac{3}{2a^{k-1}}\right)^2-\left(\frac{3}{2a^{k-1}}\right)^2+4=z^2$

Since $\frac{3}{2a^{k-1}}$ is rational number which contradicts.

Consider y is odd number such that $y = 2k+1;\ k \in \mathbb{Z}^+$.

It implies that $(a)^{2k+1} + 3a + 4 = z^2$ then $a\left[a^{2k} + 3 + \frac{4}{a}\right] = z^2$

Since $\frac{4}{a}$ is rational number which contradicts.

Subcase 7.3. let $x > 1$ and $y > 1$

Let $(x, y) = (3, 2)$, by lemma 2.2, therefore $(x, y, z, a) = (3, 2, 76, 15)$.

Corollary 2.4. The Diophantine equation $a^x + (3a+4)^y = z^2$ where $a \equiv 15 \pmod{48}$ when a is a positive integer and $(x, y) = (1, 0)$ has a unique $(x, y, z, a, t) = (1,\ 0,\ 4,\ 15,\ 0)$ for all x, y and z are non-negative integers**.**

Proof: Suppose that x, y and z are non-negative integers and a is a positive integer. By theorem 2.3.5, $(x, y, z) = \left(1, 0, \sqrt{a+1}\right)$ where $a = (12t \pm 4)^2 - 1;\ t \in \mathbb{Z}^+ \cup \{0\}$.

Let $t = 0$ then $a = 15$. Since $z = \sqrt{a+1} = \sqrt{15+1} = \sqrt{16} = 4$. Therefore, $(x, y, z, a, t) = (1,\ 0,\ 4,\ 15,\ 0)$

Corollary 2.5. The Diophantine equation $a^x + (3a+4)^y = z^2$ where $a \equiv 15 \pmod{48}$ when a is a positive integer and $(x, y) = (1, 1)$ has a unique $(x, y, z, a, t) = (1,\ 1,\ 8,\ 15,\ 0)$ for all x, y and z are non-negative integers**.**

Proof: Suppose that x, y and z are non-negative integers and a is a positive integer. By theorem 2.3.6, $(x, y, z) = \left(1, 1, \sqrt{4a+4}\right)$ where $a = (12t \pm 4)^2 - 1;\ t \in \mathbb{Z}^+ \cup \{0\}$.

Let $t = 0$ then $a = 15$. Since $z = \sqrt{4a+4} = \sqrt{(4)15+4} = \sqrt{64} = 8$. Therefore, $(x, y, z, a, t) = (1,\ 1,\ 8,\ 15,\ 0)$

4. Conclusion and discussion

For all x, y and z are non-negative integers and a is a positive integer at the point $(x, y) \in \{(1,0), (1,1)(3,2)\}$ of the Diophantine equation $a^x + (3a+4)^y = z^2$ where $a \equiv 15 \pmod{48}$ have only five suitable the written solutions in the Fibonacci and Lucas numbers as follows. $(x, y, z) = (1, 0, 4) = (L_1, F_0, L_3)$, $(x, y, z) = (1, 0, 8) = (L_1, F_0, F_6)$, $(x, y, z) = (1, 0, 76) = (L_1, F_0, L_9)$, $(x, y, z) = (1, 1, 8) = (L_1, L_1, F_6)$ and $(x, y, z) = (3, 2, 76) = (L_2, L_0, L_9)$.

Moreover, we found that $(x,y)=(3,2)$, between 1 to ∞ have all solutions are given table below.

$a^x+(3a+4)^y=z^2$	**Solution of equation**
$(15)^x+(49)^Y=z^2$	$(x,y,z)=(3,2,76)$
$(63)^x+(193)^Y=z^2$	$(x,y,z)=(3,2,536)$
$(255)^x+(769)^Y=z^2$	$(x,y,z)=(3,2,4{,}144)$
$(399)^x+(1{,}201)^Y=z^2$	$(x,y,z)=(3,2,8{,}060)$
$(783)^x+(2{,}353)^Y=z^2$	$(x,y,z)=(3,2,22{,}036)$
$(1{,}023)^x+(3{,}073)^Y=z^2$	$(x,y,z)=(3,2,32{,}864)$
$(1{,}599)^x+(4{,}808)^Y=z^2$	$(x,y,z)=(3,2,64{,}120)$
$(1{,}935)^x+(5{,}809)^Y=z^2$	$(x,y,z)=(3,2,85{,}316)$
$(2{,}703)^x+(8{,}113)^Y=z^2$	$(x,y,z)=(3,2,140{,}764)$
$(3{,}135)^x+(9{,}409)^Y=z^2$	$(x,y,z)=(3,2,175{,}784)$
$(4{,}095)^x+(12{,}289)^Y=z^2$	$(x,y,z)=(3,2,262{,}336)$
$(4{,}623)^x+(13{,}873)^Y=z^2$	$(x,y,z)=(3,2,314{,}636)$

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Authors' contributions. This is author's sole contribution.

REFERENCES

1. E.Catalan, Note extraite d'une lettre adressee a l'editeur par Mr. E. Catalan, Repetiteur a l'ecole polytechnique de Paris, *Journal fur die reine und angewandte Mathematik*, 27 (1844) 192-192.
2. A.Priya and S.Vidhyalakshmi, On the non-homogeneous ternary quadratic equation $2(x^2+y^2)-3xy+(x+y)+1=z^2$, *Journal of Mathematics and Informatics*, 10 (2017) 49-55.

3. N.Burshtein, A short note on solutions of the Diophantine equations $6^x + 11^y = z^2$ and $6^x - 11^y = z^2$ in positive integers x, y, z, *Annals of Pure and Applied Mathematics*, 19(2) (2019) 55-56.
4. A.Pakapongpun, B. Chattae, On the diophantine equation $a^x + (a + 2)^y = z^2$ where $a \equiv 3 (mod\ 20)$, *International Journal of Mathematics and Computer Science*, 17(2) (2022) 711-716.
5. S.Tadee and N.Laomalaw, On the Diophantine equation $n^x - n^y = z^2$ and $2^x - p^y = z^2$, *Phranakhon Rajabhat Research Journal* (*Science and Technology*), 17(1) (2022) 10-16
6. S.Thongnak, W. Chuayjan and T. Kaewong, On the exponential diophantine equation $7^x - 2^y = z^2$ where x, y and z are non-negative integers, *Annals of Pure and Applied Mathematics*, 25(2) (2022) 63-66.
7. S.Thongnak, W. Chuayjan and T. Kaewong, On the exponential diophantine equation $5^x - 2\cdot 3^y = z^2$, *Annals of Pure and Applied Mathematics*, 25(2) (2022) 109-112.
8. S.Tadee, On the diophantine equation $(p+6)^x - p^x = z^2$ where p is a prime number with $p \equiv 1 (\mathrm{mod}\, 28)$, *Journal of Mathematics and Informatics*, 23 (2022) 51-54.
9. K.Srimud, S. Tadee, On the diophantine equation $3^x + b^y = z^2$, *International Journal of Mathematics and Computer Science*, 18(1) (2023) 137-142.
10. H. R.Hashim, On the Solutions of $2^x + 2^y = z^2$ in the Fibonacci and Lucas number, *Journal of Prime Research in Mathematics*, 19(1) (2023) 27-33.
11. S.Tadee, A short note on two diophantine equation $9^x - 3^y = z^2$ and $13^x - 7^y = z^2$, *Journal of Mathematics and Informatics*, 24 (2023) 23-25.
12. S.Tadee, On the positive integer solutions of $p^x + p^y = z^2$ in the Fibonacci and Lucas number, where p is prime, *Journal of Prime Research in Mathematics*, 19(2) (2024) 377-380.
13. N.Malavika and R.Venkatraman, On the exponential diophantine equation $3^x + 121^y = z^2$, *International Journal of Mathematics and Computer Science*, 19(3) (2024) 917-920.