Journal of Mathematics and Informatics
Vol. 25, 2023, 1-6
ISSN: 2349-0632 (P), 2349-0640 (online)
Published 2 June 2023
www.researchmathsci.org
DOI:http://dx.doi.org/10.22457/jmi.v25a01222

Journal of **Mathematics and Informatics**

Population Projection of Mozambique

Gabriel Jamo

25 Bairro Muntanhane, Marracuene, Maputo,Mozambique
e-mail: jamounitiva@gmail.com

Received 9 March 2023; accepted 20 May 2023

Abstract. In this paper, the evolution of a dynamical system representing the population growth of Mozambique is analyzed. The correspondence between the boundary value problem and the integral equation is leveraged to address the issue of the local existence of solutions to the boundary value problem. The conclusion that the function converges to a function that is the unique solution to the boundary value problem is arrived at by way of constructing a sequence of approximations using Picard's method of successive approximations and contraction mapping. The exponential function is globally Lipschitz, hence uniformly continuous; however, its solution does not converge to a fixed point implying that the population will grow without bounds as t→∞. The logistic model solves T (ϕ) = ϕ, whence T has a unique fixed point ϕ that is a continuous solution to the integral equation and consequently to the boundary value problem. Therefore, population growth is bounded. In addition, this function is locally Lipschitz and, therefore, not uniformly continuous.

Keywords: Population, continuous function, Lipschitz condition, contraction mapping, fixed point

AMS Mathematics Subject Classification (2010): 11J17

1. Introduction

In this article, the evolution of a system described by a dated vector of real numbers $x(t)\epsilon\mathbb{R}^n$ is analyzed. Here the vector-valued function mapping the set XxℝxΩ into $\mathbb{R}^n$, i.e, $\phi x\mathbb{R}x\Omega \subseteq \mathbb{R}^{n+1+p} \to \mathbb{R}^n$ with $x(t) = \phi(x^0, \mathrm{t}, \alpha)$ gives the values of the state vector at any time *t* as a function of the initial condition $x^0 \epsilon \mathbb{R}^n$ and a vector of parameters $\alpha \epsilon \mathbb{R}^p$. By assigning values to *t* given $x^0\epsilon \mathbb{R}^n$and $\alpha \epsilon \mathbb{R}^p$ the time path of the system obtains [1]. In this particular case, the system represents the population growth of Mozambique.

Mathematical modelling, in particular, the exponential function and the logistic model of limited growth and some of the properties governing differential equations, are used in this paper. Specifically, the correspondence between the boundary value problem and the integral equation is considered. The continuity of a function and Lipschitz condition are also reviewed in a prelude to the local existence and uniqueness of solutions as well as contraction mapping concepts. The data used was obtained from the World Bank database [2].

2. Methodology

Consider a parameterized continuous time dynamical system which characterizes population growth, given by

$$\frac{\dot{x}}{x} = f(x, \alpha, t) \tag{1}$$

where the function f maps the set $\mathrm{X x \Omega x I}$ into X, a subset of $\mathbb{R}^{\mathrm{n}}$, that is, $\mathrm{f}: \mathrm{X x \Omega x I} \subseteq \mathbb{R}^{\mathrm{n+p+1}} \to \mathrm{X} \subseteq \mathbb{R}^{\mathrm{n}}$. As described in [3], a differentiable function $\phi(t): J_\varphi \to \mathrm{X}$ defined on the interval $J_\varphi \subseteq I$ with the property that for all t in J_φ, $(\varphi(t), t) \epsilon D$ and $\varphi'(t) = f[\varphi(t), \varphi, t]$ where D is an open and connected set in $\mathbb{R}^{\mathrm{n+1}}$ is a particular solution of equation (1). With $\varphi(t)$ being a solution function and $\mathrm{t}_0 \, \varepsilon \, J_\varphi$, then by setting $x^0 = \varphi(t)$ it is the case that $\varphi(t)$ must be a solution of the boundary value problem

$$\dot{x} = f(x, \alpha, t), \qquad x(\mathrm{t}_0) = x^0 \tag{2}$$

Its orbit through the point (x^0, t_0) induced by the corresponding solution function $\varphi(t): \phi(\mathrm{t}, x^0, \mathrm{t}_0, \alpha)$ is given by $\gamma(x^0, \mathrm{t}_0) = \varphi([\mathrm{t}_0, b)) = \{x \epsilon X; x = \varphi(t)$ for all $t \, \varepsilon \, J_m(x^0, \mathrm{t}_{0,} \alpha) = (a, b)\}$, where $J_m(x^0, \mathrm{t}_{0,} \alpha)$ is the maximal interval of definition. The positive orbit through the point (x^0, t_0) is specified as $\gamma^+(x^0, \mathrm{t}_0) = \varphi([\mathrm{t}_0, b)) = \{x \epsilon X; x = \varphi(t), t \, \varepsilon \, [a, b) \subseteq J_m(x^0, \mathrm{t}_{0,} \alpha)\}$ and the negative orbit given by $\gamma^-(x^0, \mathrm{t}_0) = \varphi(a, \mathrm{t}_0] = \{x \epsilon X; x = \varphi(t), t \, \varepsilon \, (a, \mathrm{t}_0] \subseteq J_m(x^0, \mathrm{t}_{0,} \alpha)\}$. Note that $\gamma(x^0, \mathrm{t}_0) = \gamma^+(x^0, \mathrm{t}_0) \cup \gamma^-(x^0, \mathrm{t}_0)$ [4].

The problem of the existence and uniqueness of solutions to the boundary value problem is akin to determining the existence and uniqueness of continuous solutions of the integral equation, which is a solution to the boundary value problem, there being a correspondence between the boundary value problem and the integral equation. In particular, given a continuous function $f(x, t)$ in some domain $D = X\mathrm{x}I$ with the point (x^0, t_0) in D for $t \, \epsilon \, J_\varphi$, the function $\phi: J \subseteq I \to \mathbb{R}^{\mathrm{n}}$ where $\mathrm{t}_0 \in J$ is a solution of the integral equation

$$\phi(t) = x^0 + \int_{\mathrm{t}_0}^{t} f(s, x) \, ds \tag{3}$$

whose solution function is constructed recursively using Picard's method of successive approximations to yield a sequence of functions $\{\phi_n\}$, $\phi_{n+1}(\mathrm{t}) = x^0 + \int_{t_0}^{t} [\phi_n, s] \, ds$ for each $t \epsilon J$ given $\phi_0(\mathrm{t}) = x^0$ [5,6].

Local existence and uniqueness of solutions are predicated on a function being continuous and satisfying the Lipschitz condition. In particular, a function mapping $f: X^n \to Y^n$ at $x^0 \epsilon R^n$ satisfies the Lipschitz condition if there is a neighbourhood in the open ball $B(x^0, \varepsilon)$ and some constant $M > 0$ such that $\|f(x) - f(y)\| \leq M\|x - y\|$, with M being applicable in the whole interval. If the function is globally Lipschitz, then it is uniformly continuous. Otherwise, it is locally Lipschitz, that is, $\|f(x) - f(y)\| \leq M_0\|x - y\|$ and consequently not uniformly continuous. Here M_0 is not a fixed value [7,8]. By definition a function $f: (X, d) \to (Y, \delta)$ is uniformly continuous on $E \subseteq X$ if $\forall x, y \epsilon E$ given $\varepsilon > 0 \; \exists \; \delta(\varepsilon)$ independent of x such that $\delta[f(x), f(y)] < \varepsilon$ whenever $d(x, y) \leq \delta(\varepsilon)$ [9].

The lemma on local existence and uniqueness of solutions makes it precise that a solution to the boundary value problem in some neighbourhoods of x^0 exists. Specifically, given a function defined on a closed box by $\mathrm{B}(x^0, \mathrm{t}_0) = Bx^0\mathrm{x}I_0 = \{(x, t); |t - \mathrm{t}_0| \leq a\|x - x^0\| \leq b\}$ and some constant $\mathrm{M} > 0$ such that $\|f(x, t) - f(y, t)\| \leq M\|x - y\|$ for all (x,t) and (y,t) in $B(x^0, t_0)$ then there is a number $h \leq a$ such that the boundary value problem $\dot{x} = f(x,t)$ with $x(t_0) = x^0$ has a unique solution $\phi(t)$ defined on the interval $J = [t_0 - h, t_0 + h]$ with $\phi(t)$ in $Bx^0\ \forall\, t\ \in\ J$. Now consider a space of continuous real-valued functions C(J) defined on the interval $J = [t_0 - h, t_0 + h]$ with $\phi(t)$ in $B(x^0, t_{0)}$ for all $t\ \varepsilon\ J$. Let $\varphi(t) = x^0 + \int_{t_0}^{t} f[\varphi(s), s]ds\ \forall\ t\ \varepsilon\ J]$ and define the operator $T\colon C(J) \to C(J)$. This operator is applied to obtain successive Picard's approximation given by the sequence of functions $\{\boldsymbol{\phi_n}\}$ defined for each t in J by $\phi_0 = x^0$ and $\phi_{n+1}(t) = T\phi_n(t)$ for $n = 1,2,3, \dots$. The function $\boldsymbol{\phi}$ is a solution of the integral equation provided that $T(\phi) = \phi$, meaning that it has a fixed point of T [10]. Given a metric space (X, d) and some constant c the distance function giving the metric $d(T_x, T_y) \leq cd(x,y)$ for all $x, y \in X$ where $c \in (0,1)$ ensures the existence and uniqueness of solutions of a function in view of the fact that T is a contraction mapping and therefore a fixed point [11,12,16].

3. Analysis and results

Population growth is a function of time t and population x described by the equation

$$\frac{\dot{x}}{x} = f(x,t) \tag{4}$$

where $f\colon X \text{x}\ I \subseteq \mathbb{R}^{n+1} \to X \subseteq \mathbb{R}^n$. The assumption that migration does not affect the system's trajectory yields a function that depends on birth rate $b(t,p)$ and death rate $d(t,p)$, that is, $f(x,t) = b(x,t) -\ d(x,t)$. Given a function $f(x,t)$ the evolution of the system $x(t)\epsilon\mathbb{R}^n$ satisfies the growth equation $\dot{x} = (x,t)x$ which together with the prescribed initial conditions $x(t_0) = x^0$ yields the boundary value problem

$$\dot{x} = (t,x)x, \qquad x(t_0) = x^0 \tag{5}$$

Assuming a constant rate of growth, that is, $f(x,t) = \alpha\ \forall (x,t)\epsilon\ \mathbb{R}^{n+1}$, with $\alpha \in \mathbb{R}^p$ the following dynamical system obtains

$$\dot{x} = \alpha x \qquad x(t_0) = x^0 \tag{6}$$

where $f\colon X \text{x}\ \Omega \subseteq \mathbb{R}^{n+p} \to X \subseteq \mathbb{R}^n$ and $\alpha > 1$ is the base of the exponential function and $x \neq 0$. Thus the solution depends on the initial condition $x^0 \epsilon\ \mathbb{R}^n$ and a parameter $\alpha \in \mathbb{R}^p$ [13].

Showing that the function in equation (6) is globally Lipschitz and, therefore, uniformly continuous is straightforward. For this purpose, let $\varepsilon > 0$ be given and assume that there exists some constant $M > 0$ in the interval $J_m[t_0, \infty)$ satisfying the Lipschitz condition. Then by definition, $\|\alpha x - \alpha y\| \leq M\|x - y\| \Rightarrow \alpha\ \|x - y\|/\|x - y\| \leq M \Rightarrow \alpha < M$, where the derivative is bounded by a unique value α at every point in the interval $[t_0, \infty)$. The function $f(x) = \alpha x$ is uniformly continuous since for all $x, y\ \varepsilon\ [t_0, \infty)$ and given $\varepsilon > 0$ there is some number $\delta(\varepsilon) > 0$ independent of x such that $\|f(x) - f(y)\| < \varepsilon$ whenever $\|x - y\| < \delta$. Obviously, for $M = 0, \|x - y\|\ \leq 0 < \varepsilon$. Next, let $\varepsilon > 0$ be arbitrary and choose $\delta(\varepsilon) =\ \varepsilon/M$. Then for any $\|\alpha x - \alpha y\|\ < \delta = \varepsilon/M$ it is the case that $\|\alpha x - \alpha y\| \leq\ M\|x - y\| < M\varepsilon/M = \varepsilon$ and we are done.

Picard's method of successive approximations yields a sequence of functions $\phi_n(t) = x^0 + \sum_{n=1}^{\infty} \alpha x^n/n!$ to the unique solution to the boundary value problem (5) as $n \to \infty$ and satisfies this equation. Notice that there does not exist $c \in (0,1)$ satisfying $d(T_x, T_y) \leq cd(x,y)$ for all $x, y \in X$ since $d[f(x) - f(y)] = \|f(x) - f(y)\| = \|\alpha(f(x) - f(y)\|\ \leq \alpha\ \|x - y\| \leq \alpha d(x,y)$ where $\alpha > 1$. Hence no fixed point implying that the model predicts that population growth is unbounded. It can also be said that the system is not exponentially stable [14]. The exponential function has been modified to take into account the assumption that population growth is bounded by some number $\xi > 0$ such that if $x < \xi$, $f(x,t) > 0$ and $f(x,t) < 0\ if\ x > \xi$. It is generally assumed that f is a linear function of x: $f(x,t) = \beta(\xi - x)\ \forall\, x\, \epsilon\, \mathrm{X} \subseteq \mathbb{R}^{\mathrm{n}}$ where $\beta, \xi > 0$ and $\alpha = \beta\xi$. Thus the growth equation now becomes

$$\dot{x} = (\alpha - \beta x)x^2 = \alpha x - \beta x^2 \qquad x(\mathrm{t}_0) = x^0 \tag{7}$$

For small population size the population growth is governed by the exponential function while for large size the term βx^2 dominates with the result that population growth is dumped until it reaches ξ [15].

This function is locally Lipschitz since by definition for any $\varepsilon > 0$ there is some constant $M_0 > 0$ such that $\|(\alpha x - \beta x^2) - (\alpha y - \beta y^2)\| \leq M_0\|x - y\| = \|(\alpha x - \beta x^2)$ $-(\alpha y - \beta y^2)\|/\|x - y\| \leq M_0 \leq (\alpha - 2\beta x) \leq M$. The slope of the function at any given point x^0 is bounded by $(\alpha - 2\beta x)$ where α and β are fixed positive parameters < 0 with $\beta \ll \alpha$, but it is not unique hence δ varies in the interval $[t_0, \infty)$. In order to show that the function is not uniformly continuous it is necessary to negate the definition of uniform continuity. In particular, we must show that given $\varepsilon > 0$ there is $\delta > 0$ such that for $x, y\, \epsilon\, [\mathrm{t}_0, \infty)$, $\|f(x) - f(y)\| \geq \varepsilon$ whenever $\|x - y\| < \delta$. To proceed let $\varepsilon > 0$ and fix some δ independent of x. Then $\|f(x) - f(y)\| = \|(\alpha x - \beta x^2) - (\alpha y - \beta y^2)\| =$ $\alpha\,\|x - y\| - \beta\,\|x^2 - y^2\| = \alpha\,\|x - y\| - \beta\{\|x - y\|\ \|x + y\|\}$. Now set $a = \|x - y\| \Rightarrow a < \delta$ and without loss of generality let $x < y$. Since $\|x - y\|$ one can also write $\|y - x\|$ and so $\|y - x\| = a \Rightarrow y - x = a \Rightarrow y = x + a$. It follows that $\|f(x) - f(y)\| = \alpha a - \beta a(x + y) \Rightarrow \alpha a - \beta a(x + x + a) \geq \varepsilon \Rightarrow -\beta(2x + a) \geq \varepsilon - a\alpha/a \Rightarrow$ $x \geq -\varepsilon + \alpha a - \beta a^2$. Solving for a results in two solutions but the case where $a = \sqrt{\varepsilon}$ is considered. Given $\varepsilon > 0$ fix some arbitrary δ and any number $a = \inf(\delta, \sqrt{\varepsilon})$ and let
$y = x + a = (1/2\beta)\,[-\varepsilon + \alpha a - \beta a^2 + 2\beta a^2] > 0$.
Then $\|y - x\| = ||x - (x + a)|| = \|-a\| = \|a\| < \delta \Rightarrow a < \delta$.
Since $\|f(x) - f(y)\| = \alpha\|y - x\| - \beta\|y - x\|\|y + x\| - \ \beta\{a(x + y)\},\ +y)\}$.
Observe $\varepsilon \geq \varepsilon$, following the substitution of the expressions for *x* and *y* and simplification. This concludes the proof.

Next it is shown that the function is a contraction mapping, to this end let

$$d = [f(x) - f(y)] = \|f(x) - f(y)\| = \|(\alpha x - \beta x^2) - (\alpha y - \beta y^2)\|$$

$\Rightarrow \alpha\|x - y\| - \|x + y\|\} = \alpha\|x - y\|\,(1/2)\beta\|x - y\| = (1/2)\,(\alpha - \beta)\|x - y\| \leq (1/2)\,(\alpha - \beta)d(x, y)$. So there is $c\,\epsilon\,(0,1)$ satisfying the condition

$(T_x, T_y) \leq cd(x, y)\ \forall x, y\, \epsilon\, X \Rightarrow f(x) = \alpha x - \beta x^2$ is a contraction mapping and therefore has a fixed point $T(\alpha/\beta) = (\alpha/\beta)$.

It is immediately clear following a few Picard's iterations in respect of equation of (7) that the solution function is $\phi_n(t) \to x^0 + (\sum_{n=1}^{\infty} \alpha\, t^n/n! - \sum_{n=2}^{\infty}\ \beta^{(2^{n-2})} t^{(2^n-1)}/$ $(3^2)^{n-2}(2^n - 1)$ converging to a function that is the unique solution to the boundary value problem (7) as $n \to \infty$. Notice that this solution is the maximal solution. The solution function of equation (7) can be exhibited explicitly. By method of separation of variables and partial fractions it is shown that the solution function is

$$x(t) = \alpha x^0/\beta x^0 + (\alpha - \beta x^0)e^{-\alpha(t - t_0)} \qquad \forall t \epsilon \mathbb{R} \tag{8}$$

Differentiating equation (7) yields $\ddot{x} = [(\alpha - \beta x)x(\alpha - \beta x)]x$, henceforth $\ddot{x} > 0$ if $x\epsilon(0, (\alpha/2\beta)) \cup (\alpha/\beta), \infty)$ meaning that population rises rapidly and $\ddot{x} < 0$ if $x\, \epsilon((\alpha/2\beta), \alpha/\beta))$ implying that population rises at a decreasing rate [17]. To determine the parameter α, as described in [18] let t_0, t_1 and t_2 be the years with corresponding population size x^0, x^1 and x^2 respectively then from equation (8) the following expression emerges

$$e^{-\alpha} = x^0(x^2 - x^1)/x^2(x^1 - x^0) \quad (9)$$

Considering population values for t_0, t_1 and t_2 corresponding to years 2002, 2003 and 2004 being $x^0 = 19139658$, $x^1 = 22846758$ and $x^2 = 20312705$, respectively, then $\alpha =$0.026802. The procedure for calculating β is as follows [19]. Given that the growth rate of the population per year in 2002 is 2.96% and since $(1/x)\dot{x} = (\alpha - \beta x^0)$, then $\beta = 1.48073\text{E} - 10$ so that $\alpha/\beta = 1.81\text{E} + 08$ is the fixed point which the solution converges to. With the parameters α and β as well as x^0 and t_0 the orbit can now be obtained and is shown in figure 1 for $t \epsilon$ (a, ∞). Notice that this is the maximal interval of the existence of the solution.

4. Conclusion

The objective of this article is to determine the state trajectory of the population of Mozambique. The exponential function is uniformly continuous, being globally Lipschitz, while the logistic function is locally Lipschitz and, therefore, not uniformly continuous. The former predicts unlimited population growth, while the latter indicates that it is bounded and has an S-shaped curve. The population size is expected to increase rapidly until 2082; thereafter, it will increase at a decreasing rate until 2301, when the upper bound is reached.

Acknowledgements. I thank the reviewer for taking the time and effort to review this manuscript and the comments that have improved its quality.

Conflicts of Interest. The authors declare no conflicts of interest.

Authors' contributions. All authors contributed equally to this work.

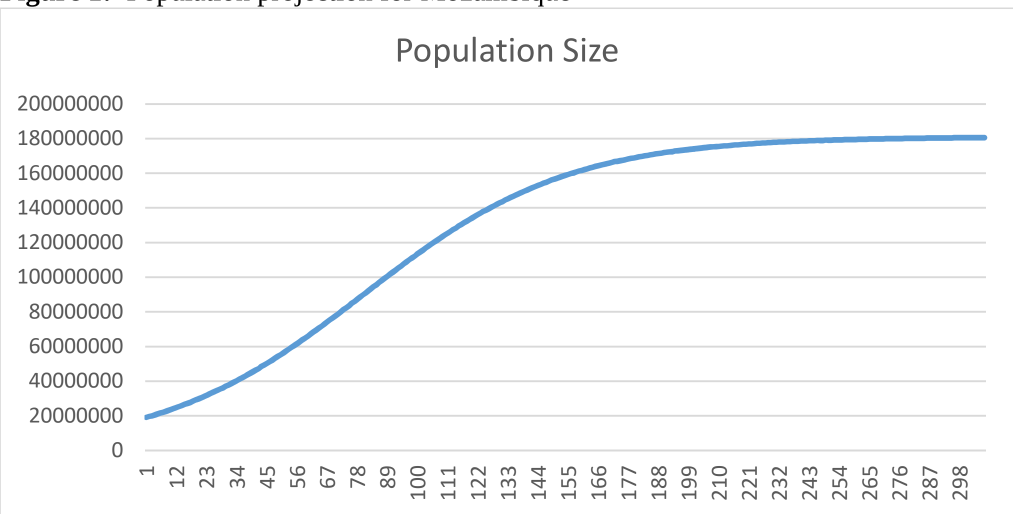
REFERENCES

1. A.De, la Fuente, Mathematical methods and models for economists, *Cambridge University Press, New York*, (2000) 391.
2. World Bank, *https://databank.worldbank.org/source/world-development-indicators*, accessed 10 February, 2023.
3. W.A.Brock and A.G.Malliaris, Differential equations, stability and chaos in dynamic economics, *Elsevier Science Publishers B.V., Amsterdam*, (1989) 2-3.
4. H.Amann, Ordinary differential equations, an introduction to nonlinear analysis, *Walter de Gruyter, New York*, (2011) 126.
5. W.A.Brock and A.G.Malliaris, Differential equations, stability and chaos in dynamic economics, *Elsevier Science Publishers B.V., Amsterdam*, (1989) 18-19.
6. E.A.Coddington and N.Levinson, Theory of differential equations, *Robert E. Krieger Publishing Company* Inc., *Malabar, Florida*, (1983) 2.
7. R.G.Bartle, The elements of real analysis, *John Wiley and Sons*, NJ (1976) 161.
8. I.I.Vrabie, Differential equations, an introduction to basic concepts, results, and applications, *World Scientific Publishing Co. Pte. Ltd*, (2004) 63.
9. W.Rudin, Principles of mathematical analysis, *International Series in Pure and Applied Mathematics*, *McGraw-hill*, *New York* 3 (1976) 85-86.

10. A.De la Fuente, Mathematical methods and models for economists, *Cambridge University Press, New York*, (2000) 433-434.
11. R.M.Brooks and K.Schmitt, The contraction mapping principle and some applications, *Electronic Journal of Differential Equations*, 9 (2009) 17-18.
12. R.Krishnakumar and D.Dhamodharan, Cone c-class function with common fixed point theorems for cone b-metric, *Journal of Mathematics and Informatics*, 8(2017) 83-94.
13. H.Amann, Ordinary differential equations, an introduction to nonlinear analysis, *Walter de Gruyter*, (2011) 3-4.
14. S.M.Banu and S.R.Ramadevi, Stability analysis by Lyapunov function in neural network, *Journal of Mathematics and Informatics*, 11 (2007) 143-146
15. M.Haque, F.Ahmed, S.Anam and R.Kabir, Future population projection of Bangladesh by growth rate modeling using logistic population model, *Annals of Pure and Applied Mathematics*, 1(2) (2012) 192-202.
16. A.Pariya, P.Pathak, V.H.Badshah and N.Gupta, Fixed point theorems for various types of compatible mappings of integral type in modular metric spaces, *Journal of Mathematics and Informatics*, 8 (2017) 7-18.
17. M.Haque, F.Ahmed, S.Anam and R.Kabir, Future population projection of Bangladesh by growth rate modeling using logistic population model, *Annals of Pure and Applied Mathematics*, 1(2) (2012) 192-202.
18. A.Wali, D.Ntubabare and V.Mboniragira, Mathematical Modeling of Rwanda's Population Growth, *Applied Mathematical Sciences*, 5 (53) (2011) 2617-2628.
19. M.Braun, Differential equations and their applications, an introduction to applied mathematics, *Springer-Verlag New York Inc., NJ* (1983) 31.

Appendix

Figure 1: Population projection for Mozambique



Source: World Bank