Intern. J. Fuzzy Mathematical Archive
Vol. 1, 2013, 92-98
ISSN: 2320 –3242 (P), 2320 –3250 (online)
Published on 24 April 2013
www.researchmathsci.org

International Journal of
Fuzzy Mathematical Archive

New Algorithms for Solving Interval Valued Fuzzy Relational Equations

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Received 3 April 2013; accepted 17April 2013

Abstract. In this paper, new algorithm is proposed to solve the Interval valued *(I – V)* fuzzy relational equation $P \cdot Q = R$ with *max-min* composition and max product composition. The algorithm operates systematically and graphically on a matrix pattern to get all the interval solution of P. An example is given to illustrate its effectiveness.

Keywords: Fuzzy relational equations, Interval valued fuzzy relational equations, Interval maximum solution, Algorithms.

AMS Mathematics Subject Classifications (2012): 15B15, 15A09, 94D05

1. Introduction

It is well known that the fuzzy relational equation xA = b, where A is a fuzzy matrix and b is a fuzzy vector is consistent, then it has a unique maximum solution and a finite number of minimum solutions [12]. A study on fuzzy relational equations was introduced by Sanchez [9] and later developed by many researchers, for more details one may refer [6]. In [4], Kim and Roush have developed a theory of fuzzy matrices analogous to that for Boolean Matrices [3]. In [1], Cho has proved that a fuzzy relational equation xA = b is consistent whenever A is regular and bX is a solution for some g–inverse X of A. Recently the concept of Interval valued fuzzy matrices as a generalization of fuzzy matrix was introduced and developed by Shyamal and Pal [10], by extending the max-min operations on fuzzy algebra F = [0, 1], for elements a, b∈F determined as a + b = max{a, b} and a.b = min{a, b} and the standard order '≥' of real numbers over F. A matrix $A \in F_{mn}$ is said to be regular if there exist $X \in F_{mn}$ such that AXA = A. X is called a generalized inverse of A and is denoted by A^-. A new algorithm is proposed to solve the fuzzy relation equation $P \circ Q = R$ with max–min composition and max–product composition in [5]. In our earlier work [7], we have represented an IVFM $A = (a_{ij}) = ([a_{ijL}, a_{ijU}])$ where each a_{ij} is a subinterval of the interval [0, 1], as the Interval matrix $A = [A_L, A_U]$ whose ij th entry is the interval $[a_{ijL}, a_{ijU}]$. Hence the lower limit A_L

= (a_{ijL}) and upper limit A_U = (a_{ijU}) are fuzzy matrices such that $A_L \leq A_U$. By using the IVFM Matrix operation introduced and developed in [10], we have discussed the regularity of IVFM in terms of the regularity of the lower and upper limit Matrices A_L and A_U [7]. In [11], consistency of system of Interval fuzzy relational equation x A^I = b^I where A^I is the set of fuzzy matrices { A' / $A_L \leq A' \leq A_U$} and b^I = {b' /$b_L \leq b' \leq b_U$} is discussed and complete set of solutions for $b_L \leq xA' \leq b_U$ is determined. Recently, it was restructured by G Li and SC Fang in [2]. In [8], consistency of the Interval valued fuzzy relational equations and the complete set of solutions of xA = b where A is an Interval valued fuzzy matrix and b is an interval valued vector is determined and Equivalent condition for the existence of Interval maximum solution is obtained

In this paper, a new algorithm is proposed to solve the Interval valued fuzzy relational equations as a generalization of that of fuzzy relational equations. In section 2, we present the basic definitions and notations, required results on fuzzy relational equations. In section 3, we have proposed a new algorithm to solve the Interval valued *(I – V)* fuzzy relational equation $P \cdot Q = R$ with *max-min* composition and max product composition. The algorithm operates systematically and graphically on a matrix pattern to get all the interval solution of *P*. An example is given to illustrate its effectiveness.

2. Preliminaries

In this section, some basic definitions and results needed and notations are given. Let $(IVFM)_{mn}$, F_{mn}, F_m, N_n denotes the set of all m×n interval valued fuzzy matrices, set of all fuzzy matrices, set of all fuzzy vectors and set of all natural numbers 1 to n respectively.

Let I = {1,2,...,m} and J = { 1,2, …,n} be index set. Also define X = { x∈ F_m /0 ≤ x_i ≤ 1, for all i∈ I} and Ω (A,b) = {x∈ X / xA = b} which represents the solution set of system xA = b where A ∈F_{mn}.

According to [6, 9] when Ω(A,b) ≠ ϕ for some fuzzy matrix A, then it can be completely determined by the unique maximum solution and a finite number of minimum solutions. Moreover, $\hat{x}$ ∈ Ω(A,b) is called the maximum solution, if x ≤ $\hat{x}$ for all x∈ Ω(A,b) and $\check{x}$∈ Ω(A,b) is called the minimum solution, if $\check{x}$ ≤ x for all x∈ Ω(A,b). If Ω(A,b) ≠ ϕ, then the unique maximum solution of the equation xA = b is determined by the following formula [9]

$$\hat{x} = A \circ b = \left(\min \left(a_{ij} \circ b_j \right) \right), j \in J, i \in I \quad \text{where } a_{ij} \circ b_j = \begin{cases} 1, & if\ a_{ij} \leq b_j \\ b_j, & otherwise \end{cases}$$

Definition 2.1. For a pair of fuzzy matrices E = (e_{ij}) and F = (f_{ij}) in F_{mn} such that E ≤ F, let us define the interval matrix denoted as [E,F], whose ij^{th} entry is the interval with lower limit e_{ij} and upper limit f_{ij}, that is [e_{ij}, f_{ij}].

In particular for E = F, IVFM [E,E] reduces to E∈ F_{mn}

For A = (a_{ij}) = ([a_{ijL} , a_{ijU}]) ∈ $(IVFM)_{mn}$ let us define A_L = (a_{ijL}) and A_U = (a_{ijU}) clearly A_L and A_U belongs to F_{mn} such that $A_L \leq A_U$. Therefore an IVFM A can be written as A = [A_L, A_U]. (2.1)

Here we shall follow the basic operations on IVFM as given in [7].

For $A = (a_{ij})_{mn} = ([a_{ijL}, a_{ijU}])$ and $B = (b_{ij})_{np} = ([b_{ijL}, b_{ijU}])$ their product denoted as
$AB = (C_{ij}) = [\sum_{k=1}^{n} a_{ik} b_{kj}]$ $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,p$
$= [\sum_{k=1}^{n} (a_{ikL}, b_{kjL}), \sum_{k=1}^{n} (a_{ikU}, b_{kjU})]$
If $A = [A_L, A_U]$ and $B = [B_L, B_U]$ then $AB = [A_LB_L, A_UB_U]$ (2.2)
For $A = (a_{ij}), B = (b_{ij}) \in (IVFM)_{mn}$

$A \leq B$ if and only if $a_{ijL} \leq b_{ijL}$ and $a_{ijU} \leq b_{ijU}$ for all $i = 1,...m$ and $j = 1,2,\ldots n$ (2.3)

In particular if $a_{ijL} = a_{ijU}$ and $b_{ijL} = b_{ijU}$ then (2.2) reduces to the standard max-min composition of fuzzy matrices [4, 6]

Lemma 2.2. (Theorem 3.2 of [7]) For $A = [A_L, A_U] \in (IVFM)_{mn}$ and $B = [B_L, B_U] \in (IVFM)_{np}$, the following hold.

(i) $A^T = [A_L^T, A_U^T]$

(ii) $AB = [A_LB_L, A_UB_U]$

Lemma 2.3. (Theorem (3.3) of [7]) Let $A = [A_L, A_U] \in (IVFM)_{mn}$, then A is regular IVFM $\Leftrightarrow$ A_L and $A_U \in F_{mn}$ are regular.

Remark 2.4. If A is regular IVFM then there exist an IVFM X such that AXA = A, X is called g–inverse of A and $X \in A\{1\}$ the set of all g–inverses of A. By (2.2), AXA = A reduces to $A_LX_LA_L = A_L$ and $A_UX_UA_U = A_U$ for some $X_L \in A_L\{1\}$ and $X_U \in A_U\{1\}$.
Therefore $X = [X_L, X_U]$ is a g – inverse of A.

Lemma 2.5. (Lemma 2.5.1 of [6]) Let x A = b where $A = (a_{ij}) \in F_{mn}$, $b = (b_{ij}) \in F_{1n}$. If $\max_j (a_{ij}) < b_k$ for some $k \in N_n$, then $\Omega(A,b) = \phi$.

In [2], the set of solution for $xA' = b'$
where $A' \in A^I = \{ A' / A' \in [A_L, A_U]\}$ and $b' \in b^I = \{ b' / b_L \leq b' \leq b_U\}$ (2.4)
is determine further, when the solution set is not empty, it is shown that $xA' = b'$ has one maximum solution and a finite number of minimum solutions.

Lemma 2.6. (Theorem 1 of [2]) x is a solution of $xA' = b'$ where $A' \in A^I = \{ A' / A' \in [A_L,A_U]\}$ and $b' \in b^I = \{ b' / b' \in [b_L,b_U]\}$ if and only if $x \in x^I$ for some $x^I \in \Omega(A^I, b^I)$ where $x^I = \{ x' / x_L \leq x' \leq x_U\}$.

Lemma 2.7. For the equation $xA = b$ where $x = [x_j / j \in N_m]$, $b = [b_k / k \in N_n]$ and $A \in F_{mn}$, $\Omega(A, b) \neq \phi$, if and only if $\hat{x} = [\hat{x}_j / j \in N_m]$, defined as $\hat{x}_j = \min \sigma(a_{ik}, b_k)$
where $\sigma(a_{ik}, b_k) = \begin{cases} b_k, & \text{if } a_{ik} > b_k \\ 1, & \text{otherwise} \end{cases}$

3. New Algorithm for solving *I –V* Fuzzy relational Equations

In this section, new algorithm is proposed to solve the Interval valued *(I – V)* fuzzy relational equation $P \cdot Q = R$ with *max-min* composition and max product composition. The algorithm operates systematically and graphically on a matrix pattern to get all the interval solution P *in* the ***(I –V)*** fuzzy relational equation $P.Q = R$. An example is given to illustrate its effectiveness.

Let interval valued Fuzzy matrix $Q = [q_{jk}]_{mxn}$ be called the state – matrix and Interval valued fuzzy vector $r = [r_k]$ be called the output vector with $q_{jk} \in [0, 1]$, $r_k = [0,1]$ for all $j \in J$ and $k \in K$, where $j = [\ 1,2,....m]$ and $K= \{1,2,.....,n\}$, $m,n \in N$. The problem is to determine all vectors $p \in P = \{ p = [p_j]_{1xm} \ / p_j \in [\ 0,\ 1\ ]\}$ satisfying Interval valued fuzzy relational equation.

$$p \cdot Q = r \tag{3.1}$$

Then by using Lemma 2.2, $p \cdot Q = r$ can be expressed as the following fuzzy relational equations

$$p_L \cdot Q_L = r_L \tag{3.2}$$

and

$$p_U \cdot Q_U = r_U \tag{3.3}$$

where '·' denotes the *max-min* composition with $\max_{j \in J} \min (p_j , q_{jk}) = r_k$ or *max-product* composition with $\max_{j \in J} (p_j \cdot q_{jk}) = r_k$ for all $k \in K$.

Lemma 3.1. (Lemma 3.1 of [8]) The (I-V) fuzzy relational equation of the form (3.1) is consistent if and only if the corresponding fuzzy relational Eqs. (3.2) and (3.3) are consistent.

The following algorithm and Theorems are the generalization of results found in [5].

Algorithm 3.2.

Following are the algorithms for solving Interval valued fuzzy relational Eq. (3.1) with *max-min* (or *max-product*) composition:

Step 1: Verify the consistency of the Eq. (3.1) by using Lemma 3.1 and Lemma 2.5.

Step 2: Rank the elements of r with decreasing order and find the maximum solution $\overline{p}$ by using Lemma 2.7, established in [9].

Step 3: Construct $M=[m_{jk}]$, $j =1,2,....,m$, $k=1,2,....,n$. where $m_{jk} \triangleq (\overline{p}_j, q_{jk})$.

Let $M =[\ M_L, M_U\ ]$. Therefore $m_{jk} = [m_{jkL}, m_{jkU}]$.

Hence $M_L = [m_{jkL}]$ and $M_U =[m_{jkU}\ ]$.

This interval matrix M is called the matrix pattern.

Step 4: Interval matrix m_{jk}, which satisfies $\min(\overline{p}_j, q_{jk}) = r_k$ (or $p_j.\ q_{jk} = r_k$), and then let the marked m_{jk} be denoted by $\overline{m}_{jk}$.

Step 5: If k_1 is the smallest k in all marked m_{jk}, then set $\underline{p}_{j1}$ to be the smaller one of the two elements in $\overline{m}_{j1k1}$ (or set $\underline{p}_{j1}$ to be $\overline{p}_{jl}$)

Step 6: Delete the j_1^{th} row and k_1^{th} column of M and then delete all the columns that contain marked m_{j1k}, where $k \neq k_1$.

Step 7: In all remained and marked $\overline{m}_{jk}$ find the smallest k and set it to be k_2, then let $\underline{p}_{j2}$ be the smaller one of the two elements in $\overline{m}_{j2k2}$ (or let $\underline{p}_{j2}$ be $\overline{p}_{j2}$).

Step 8: Delete the j_2^{th} row and the k_2^{th} column of M and then delete all the columns that contain marked $\overline{m}_{j2k}$, where $k \neq k_2$.

Step 9: Repeat Step *7 & 8* until no marked m_{jk} remained.

Step 10: The other $\underline{p}_j$ which are not set in steps 5 to *8*, are set to be zero.

Lemma 3.3. If the interval valued fuzzy relational equation is of the form as (3.1), for given $m \times n$ interval matrix Q and $1 \times n$ interval vector r, the interval minimum solution $\underline{p}$ can be obtained by the above algorithm.

Proof. Owing the fact that the whole interval minimum solution of $\underline{p}$ can be derived step by step from the above algorithm, we can prove it in a straight way.

The steps 1 and 2 are the standard procedures that has been illustrated in [8] for (*I-V)* fuzzy relational equation and in [9] for the corresponding fuzzy relational equations. Since $0 \leq \underline{p}_j \leq \overline{p}_j$, step 3 is in fact, to put all the possible interval solution elements together. That is, with step 3, we would not miss any possible interval solution in the solving procedure. Thus, for deriving the whole minimum solutions $\underline{p}$, the procedure in step *3* is necessary. In step 4*,* according to step 3 and Lemma 2.7, we have the following deductions. That is after checking $min\ (p_{jL},\ q_{jkL}) = r_{kL}$ and $min\ (p_{jU},\ q_{jkU}) = r_{kU}$, it as noted that the position of m_{jkL} and m_{jkU} make $min\ (\overline{p}_{jL},\ q_{jkL}) = r_{kL}$ $min\ (\overline{p}_{jU},\ q_{jkU}) = r_{KU}$ happen and the results in $\overline{m}_{jkL}$ and $\overline{m}_{jkU}$ must be an element of the minimum solutions. Therefore by Lemma 2.3(ii), $min\ (\overline{p}_j,\ q_{jk}) = r_k$ it is noted that the position of $\overline{m}_{jk}$ make $min\ (\overline{p}_j,\ q_{jk}) = r_k$ happen and the results in $\overline{m}_{jk}$ must be an element of the minimum solutions. Accordingly, we should mark all these elements. In step 5*,* we need to check interval minimum solution from high rank. If it is true, by step 4, pick the minimum solution for the corresponding p_j. In step 6*:* obviously, since the interval minimum solution for j_1^{th} row of *M* (i.e. j_1^{th} element of $\underline{p}$) has been gotten, we delete all the other columns that contain marked $\overline{m}_{jk}$, where $k \neq k_1$. That is, the minimum solution for the j_1^{th} element of $\underline{p}$ would not be repeated. The analogous procedure in steps 7 to 9 is to guarantee running of the algorithm from the left to right and from the upper to the bottom in Interval matrix *M* and to get the whole interval minimum solutions. In step 10*,* if we cannot find the interval minimum solution for p_j the zero must be a solution naturally, the analogous proof can be done also for the *max–product* composition.

Remark 3.4. For the interval valued fuzzy relational equation *P.Q*=*R*, where *P* is $s \times m$ matrix and *R* is $s \times n$ matrix, the solution of the problem is obtained by solving the problem (3.1) repeatedly for each of the s-rows of P and the corresponding rows of R . Since the algorithms are similar no matter on solving *max-min* or *max–product* composition we only present an example with *max-min* composition to illustrate the procedure of the algorithm. This is illustrated in the following:

Example 3.5. Consider an interval valued fuzzy relational equation $P \cdot Q = R$ with *max-min* composition, where

$$Q = \begin{pmatrix} [0.5, 0.7] & [0.7, 0.9] & [0, 0.1] & [0, 0] \\ [0.4, 0.6] & [0.6, 0.8] & [0.4, 0.6] & [0, 0] \\ [0.2, 0.4] & [0.4, 0.6] & [0.5, 0.7] & [0.6, 0.8] \\ [0.1, 0.3] & [0.2, 0.4] & [0, 0] & [0.8, 1] \end{pmatrix}$$

$R = ([0.5, 0.7]\ \ [0.5, 0.7]\ \ [0, 0]\ \ [0, 0])$

By our representation (2.1) we have,

$$Q_L = \begin{pmatrix} 0.5 & 0.7 & 0.0 & 0.0 \\ 0.4 & 0.6 & 0.4 & 0.0 \\ 0.2 & 0.4 & 0.5 & 0.6 \\ 0.1 & 0.2 & 0.0 & 0.8 \end{pmatrix} \qquad Q_U = \begin{pmatrix} 0.7 & 0.9 & 0.0 & 0.0 \\ 0.6 & 0.8 & 0.6 & 0.0 \\ 0.4 & 0.6 & 0.7 & 0.8 \\ 0.3 & 0.4 & 0.0 & 1.0 \end{pmatrix}$$

R_L = [0.5 0.5 0.4 0.0] and R_U = [0.7 0.7 0.0 0.0]

First our task is to find the minimal solution $\underline{p} \in r^{1x4}$ in Eq. (3.2).

Step 1: It is known that the solution P exists by Lemma 2.5.

Step 2: By Corollary 3.8 of [8], the maximum solution p = [$0.5\ \ 0.5\ \ 0\ \ 0$]

Step 3: Build M_L as

$$M_L = \begin{pmatrix} (0.5, 0.5) & (0.5, 0.7) & (0.5, 0.0) & (0.5, 0.0) \\ (0.5, 0.4) & (0.5, 0.6) & (0.5, 0.4) & (0.5, 0.0) \\ (0.0, 0.2) & (0.0, 0.4) & (0.0, 0.5) & (0.0, 0.6) \\ (0.0, 0.1) & (0.0, 0.2) & (0.0, 0.0) & (0.0, 0.8) \end{pmatrix}$$

Step 4: Underline those elements which satisfies $min\ (\overline{p}_j, q_{jk}) = r_k$

$$M_L = \begin{pmatrix} \underline{(0.5, 0.5)} & \underline{(0.5, 0.7)} & (0.5,0) & \underline{(0.5, 0)} \\ (0.5, 0.4) & \underline{(0.5, 0.6)} & \underline{(0.5, 0.4)} & \underline{(0.5,0)} \\ (0, 0.2) & (0, 0.4) & (0, 0.5) & \underline{(0, 0.6)} \\ (0, 0.1) & (0, 0.2) & (0,0.0) & \underline{(0, 0.8)} \end{pmatrix}$$

Step 5: Set $\underline{p}_1 = min\ (0.5, 0.5) = 0.5;$ note here $j = l.$

Step 6: Delete the first row and the first column of M_L, and then delete all the columns that contain marked m_{lkL} , where $k \neq l.$

$$M_L = \begin{pmatrix} \underline{(0.5, 0.5)} & \underline{(0.5, 0.7)} & (0.5,0) & \underline{(0.5, 0)} \\ (0.5, 0.4) & \underline{(0.5, 0.6)} & \underline{(0.5, 0.4)} & \underline{(0.5,0)} \\ (0, 0.2) & (0, 0.4) & (0, 0.5) & \underline{(0, 0.6)} \\ (0, 0.1) & (0, 0.2) & (0,0.) & \underline{(0, 0.8)} \end{pmatrix}$$

Step 7: Set $\underline{p}_2 = min(0.5, 0.4) = 0.4;\ here\ j=2.$

Step 8: Delete the second row and the third column of M_L, and then delete all the columns that contain marked m_{2kL} , where $k \neq 3$.

$$M_L = \begin{pmatrix} \underline{(0.5, 0.5)} & \underline{(0.5, 0.7)} & (0.5,0) & \underline{(0.5, 0)} \\ (0.5, 0.4) & \underline{(0.5, 0.6)} & \underline{(0.5, 0.4)} & \underline{(0.5,0)} \\ (0, 0.2) & (0, 0.4) & (0, 0.5) & \underline{(0, 0.6)} \\ (0, 0.1) & (0, 0.2) & (0,0.) & \underline{(0, 0.8)} \end{pmatrix}$$

Step 9: Until now, we have set $\underline{p}_1 = 0.5$ and $\underline{p}_2 = 0.4$, then the other $\underline{p}_j$ are set to be zero, that is $\underline{p}_3 = 0$ and $\underline{p}_4 = 0$. Therefore, we have only one minimal solution of Eq. (3.2) as $\underline{p} = [0.5 \quad 0.4 \quad 0 \quad 0]$.

Similarly, we have found only one minimal solution of Eq. (3.3) as $\underline{p} = [0.7 \quad 0.6 \quad 0 \quad 0]$.

Therefore, by Lemma 2.2(ii) we have minimal solution of the interval valued fuzzy relational equation $P{\cdot}Q = R$ as $\underline{p}$ = ([0.5, 0.7]   [0.4, 0.6]   [0, 0]   [0, 0])

Acknowledgement.

I render my heartful thanks to *Prof. Dr. (Mrs.) AR. Meenakshi, Former AICTE - Emeritus Professor of Mathematics, Annamalai University*, for her expert guidance.

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