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Double Dominating Energy of Some Graphs

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Abstract. In this paper the double dominating energy of a graph is introduced. The double dominating energy of a crown graph, cocktail party graph and complete graph are computed.

Keywords: Energy, double dominating set, double dominating matrix, double dominating energy.

AMS Mathematics Subject Classification (2010): 05C10

1. Introduction

The concept of energy of a graph was introduced by I. Gutman [1], in the year 1978. Let G be a graph with n vertices and m edges and let $A = (a_{ij})$ be the adjacency matrix of the graph. The eigen values $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ of A, assumed in non-decreasing order $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n$ are the eigen values of the graph G. Since A(G) is real and symmetric, its eigen values are real numbers. The energy E(G) of G is defined to be the sum of the absolute values of its eigen values of G. That is $E(G) = \sum_{i=1}^{n} |\lambda_i|$.

In HMO Theory, the total energy of the π electrons is equal to the sum of the energies of all π -electrons in the considered molecule. It can be calculated from the eigen values of the underlying molecular graph [2, 8].

Similar to energies like Laplacian energy, distance energy, minimum covering energy, incidence energy [3,4, 5, 6, 7], the double dominating energy is defined in this paper and same is found out for some graphs.

2. Double dominating energy

Let G be a simple graph of order n with vertex set $V = \{v_1, v_2, v_3, \ldots ,v_n\}$ and edge set E. A subset $D' \subseteq V$ is a double dominating set if D' is a dominating set and every vertex of $V - D'$ is adjacent to atleast two vertices in D'. The Double Domination number $\gamma_{\times 2}(G)$ is the minimum cardinality taken over all the minimal double dominating sets of G.

Let D′ be the minimum double dominating set of a graph G. The minimum double dominating matrix of G is the n × n matrix defined by $A_{D'}(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 1 & \text{if } i = j \text{ and } v_i \in D' \\ 0 & \text{otherwise} \end{cases}$$

The characteristic polynomial of $A_{D'}(G)$ is denoted by $fn(G, \lambda) = \det(\lambda I - A_D(G))$.

The minimum double dominating eigen values of the graph G are the eigen values of $A_{D'}(G)$.

Since $A_{D'}(G)$ is real and symmetric, its eigen values are real numbers and are labelled in non-increasing order $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n$. The minimum double dominating energy of G is defined as

$$E_{D'}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Example 2.1. Let G be a cycle C_4 on 4 vertices u_1, u_2, u_3, u_4 with minimum double dominating set $D' = \{u_1, u_3\}$. Then

$$A_{D'}(C_4) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of $A_{D'}(C_4)$ is $\lambda^4 - 2\lambda^3 - 3\lambda^2 + 4\lambda$, the minimum double dominating eigen values are 0, 1, $\frac{1+\sqrt{17}}{2}$, $\frac{1-\sqrt{17}}{2}$, and the minimum double dominating energy is $E_{D'}(C_4) = 1+\sqrt{17}$.

3. Properties of double dominating energy

Theorem 3.1. Let G be a graph with n vertices and m edges.

If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the eigen values of $A_{D'}(G)$, then $\sum_{i=1}^{n} \lambda_i^2 = 2|E| + |D'|$.

Proof :

The sum of square of the eigen values of $A_{D'}(G)$ is the trace of $A_{D'}(G)^2$.

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} a_{ji} \\ &= 2\sum_{i<j} (a_{ij})^2 + \sum_{i=1}^{n} (a_{ij})^2 = 2|E| + |D'| = 2m + |D'|. \end{aligned}$$

Theorem 3.2. Let G be a simple graph with n vertices, m edges and let D′ be a double dominating set of G and $F = |\det A_{D'}(G)|$ then

$$\sqrt{2m + |D'| + n(n\text{-}1)f^{2/n}} \leq E_{D'}(G) \leq \sqrt{n(2m + |D'|)}$$

Proof :

Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n$ be the eigen values of $A_{D'}(G)$. By Cauchy-Schwarz inequality,

$$\sum_{i=1}^{n}\left(a_i b_i\right)^2 \le \sum_{i=1}^{n}\left(a_i^2\right)\left(\sum_{i=1}^{n}\left(b_i^2\right)\right)$$

let$a_i = 1$, $b_i = |\lambda_i|$,

$$E_{D'}(G)^2 = \left(\sum_{i=1}^{n}|\lambda_i|\right)^2 \le n\left(\sum_{i=1}^{n}|\lambda_i|^2\right) = n\sum_{i=1}^{n}\lambda_i^2 = n(2m + |D'|$$

$$E_{D'}(G) \le \sqrt{n(2m + |D'|}$$

$$[E_{D'}(G)]^2 = \left(\sum_{i=1}^{n}|\lambda_i|\right)^2 = \sum_{i=1}^{n}|\lambda_i|^2 + \sum_{i \ne j}|\lambda_i|\ |\lambda_j|$$

From the inequality between the arithmetic and geometric mean, we obtain

$$\frac{1}{n(n-1)}\sum_{1 \ne j}|\lambda_i|\ |\lambda_j| \ge \left(\prod_{i \ne j}|\lambda_i|\ |\lambda_j|\right)^{\frac{1}{n(n-1)}}$$

$$\begin{aligned}
\sum_{i \ne j}|\lambda_i||\lambda_j| &\ge n(n-1)\left[\prod_{i=1}^{n}|\lambda_i|^{2(n-1)}\right]^{\frac{1}{n(n-1)}} \\
&\ge n(n-1)\left[\prod_{i=1}^{n}|\lambda_i|\right]^{\frac{2}{n}} \\
&\ge n(n-1)\left[\prod_{i=1}^{n}|\lambda_i|\right]^{\frac{2}{n}} \\
&\ge n(n-1)\left|\prod_{i=1}^{n}|\lambda_i|\right|^{\frac{2}{n}} \\
&\ge n(n-1)\left|\det A_{D'}(G)\right|^{\frac{2}{n}}
\end{aligned}$$

$$\sum_{1 \ne j}|\lambda_i|\ |\lambda_j| \ge n(n-1)F^{\frac{2}{n}}$$

$$\begin{aligned}
[E_{D'}(G)]^2 &\ge \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)F^{2/n} \\
&\ge (2m + |D'| + n(n-1)F^{2/n} \\
[E_{D'}(G)] &\ge \sqrt{(2m + |D'| + n(n-1)F^{2/n}}
\end{aligned}$$

Bapat and Pati showed that if the graph energy is a rational number, then it is an even integer [9]. The analogous result for minimum double dominating energy is given in the following theorem.

Theorem 3.3. Let G be a graph with a minimum double dominating set D′. If the minimum double dominating energy $E_{D'}(G)$ is a rational number, then$E_{D'}(G) = |D'|(\text{mod}2)$.

Proof: Let $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ be the minimum double dominating eigen values of a graph G of which $\lambda_1, \lambda_2, \ldots, \lambda_r$ are positive and the remaining are non-positive, then

$$
\begin{aligned}
E_{D'}(G) &= (\lambda_1 + \lambda_2 + \ldots\ldots + \lambda_r) - (\lambda_{r+1} + \ldots\ldots + \lambda_n) \\
&= 2(\lambda_1 + \lambda_2 + \ldots\ldots + \lambda_r) - (\lambda_1 + \lambda_2 + \ldots\ldots + \lambda_n) \\
&= 2(\lambda_1 + \lambda_2 + \ldots\ldots + \lambda_r) - \sum_{i=1} \lambda_i \\
&= 2(\lambda_1 + \lambda_2 + \ldots\ldots + \lambda_r) - |D'| \\
E_{D'}(G) &= |D'|(\mathrm{mod}\,2)
\end{aligned}
$$

4. Double dominating energies of some families of graphs

Definition 4.1. The crown graph S_n^0 for an integer $n \geq 2$ is the graph with vertex set $\{u_1, u_2, \ldots\ldots u_n, v_1, v_2, \ldots\ldots v_n\}$ and the edge set $\{u_i v_j : 1 \leq i, j \leq n, i \neq j\}$.

Theorem 4.2. For $n \geq 4$, the double dominating energy of the crown graph S_n^o is equal to

$$2 + 2(n-3) + \sqrt{n^2 - 2n + 9} + \sqrt{n^2 + 2n - 7}$$

Proof:

The crown graph S_n^0 with vertex set $v = \{u_1, u_2, \ldots\ldots u_n, v_1, v_2, \ldots\ldots v_n\}$, the minimum double dominating set $D' = \{u_1, u_2, v_1, v_2,\}$.

$$
\text{Then } A_{D'}\left(S_n^0\right) = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\
0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 1 \\
. & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 0 \\
0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 1 & \cdots & 1 & 0 & 1 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
. & . & . & . & . & . & . & . & . & . \\
1 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}
$$

characteristic polynomial is

$$
\begin{bmatrix}
\lambda-1 & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\
0 & \lambda-1 & 0 & \cdots & 0 & -1 & 0 & -1 & \cdots & -1 \\
0 & 0 & \lambda & \cdots & 0 & -1 & -1 & 0 & \cdots & -1 \\
. & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & \cdots & \lambda & -1 & -1 & -1 & \cdots & 0 \\
0 & -1 & -1 & \cdots & -1 & \lambda-1 & 0 & 0 & \cdots & 0 \\
-1 & 0 & -1 & \cdots & -1 & 0 & \lambda-1 & 0 & \cdots & 0 \\
-1 & -1 & 0 & \cdots & -1 & 0 & 0 & \lambda & \cdots & 0 \\
. & . & . & . & . & . & . & . & . & . \\
-1 & -1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & \lambda
\end{bmatrix}
$$

characteristic equation is

$$\lambda(\lambda-2)(\lambda+1)^{n-3}(\lambda-1)^{n-3}(\lambda^2+(n-3))\lambda-(2n-4))(\lambda^2-(n-1)\lambda-2)=0$$

Minimum double dominating eigen values are

$\lambda = 0$, $\lambda = 2$, $\lambda = -1$, ($n - 3$ times), $\lambda = 1$ ($n - 3$ times)

$$\lambda = \frac{(n-1) \pm \sqrt{n^2-2n+9}}{2} \text{ (one time each)}$$

$$\lambda = \frac{(3-n) \pm \sqrt{n^2+2n-7}}{2} \text{ (one time each)}$$

Minimum double dominating energy

$$E_{D'}(S_n^0) = 2 + 2(n-3) + \sqrt{n^2-2n+9} + \sqrt{n^2+2n-7}$$

Theorem 4.3. The double dominating energy of the complete graph K_n is equal to $(n-3) + \sqrt{n^2-2n+9}$.

Proof:

The complete graph K_n with vertex set $v = \{v_1, v_2, \ldots .v_n\}$, the minimum double dominating set $D' = \{v_1, v_2\}$. Then

$$A_{D'}(K_n) = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 \\ . & . & . & . & . & . \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

characteristic polynomial is

$$\begin{bmatrix} \lambda-1 & -1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda-1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & -1 & \lambda & \cdots & -1 \\ . & . & . & . & . & . \\ -1 & -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}$$

characteristic equation is

$$\lambda(\lambda+1)^{n-3}(\lambda^2-(n-1)\lambda-2)=0$$

Minimum double dominating eigen values are

$\lambda = 0$, $\lambda = -1$ ($n - 3$ times)

$$\lambda = \frac{(n-1) \pm \sqrt{n^2-2n+9}}{2} \text{ (one time each)}$$

Minimum double dominating energy

$$E_{D'}(K_n) = (n-3) + \sqrt{n^2-2n+9}$$

Definition 4.4. The cocktail party graph is denoted by $K_{n\times 2}$, is a graph having the vertex set $V = \bigcup_{i=1}^{n} \{u_i, v_i\}$ and the edge set

$$E = \{u_iu_j,\ v_iv_j : i \neq j\} \cup \{u_iv_j,\ v_iu_j : 1 \leq i < j \leq n\}$$

Theorem 4.5. The minimum double dominating energy of cocktail party graph K_{nx2} is $(2n - 3) + \sqrt{4n^2 - 4n + 9}$.

Proof:

Let K_{nx2} be the cocktail party graph with vertex set $V = \bigcup_{i=1}^{n} \{u_i, v_i\}$. The minimum double dominating set is $D' = \{u_1, v_1\}$. Then

$$A_{D'}\left(K_{n\times 2}\right) = \begin{bmatrix} 1 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & \cdots & \cdots & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & \cdots & \cdots & 1 & 1 & 1 & 1 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 1 & 1 & 1 & 1 & \cdots & \cdots & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & \cdots & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & 0 & 0 \end{bmatrix}$$

characteristic polynomial is

$$\begin{bmatrix} \lambda-1 & 0 & -1 & -1 & \cdots & \cdots & -1 & -1 & -1 & -1 \\ 0 & \lambda-1 & -1 & -1 & \cdots & \cdots & -1 & -1 & -1 & -1 \\ -1 & -1 & \lambda & 0 & \cdots & \cdots & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & \lambda & \cdots & \cdots & -1 & -1 & -1 & -1 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -1 & -1 & -1 & -1 & \cdots & \cdots & \lambda & 0 & -1 & -1 \\ -1 & -1 & -1 & -1 & \cdots & \cdots & 0 & \lambda & -1 & -1 \\ -1 & -1 & -1 & -1 & \cdots & \cdots & -1 & -1 & \lambda & 0 \\ 1 & -1 & -1 & -1 & \cdots & \cdots & -1 & -1 & 0 & \lambda \end{bmatrix}$$

characteristic equation is

$$\lambda^{n-1}(\lambda-1)(\lambda+2)^{n-2}(\lambda^2-(2n-3)\lambda-2n)=0$$

Minimum double dominating eigen values are

$\lambda = 0(n - 1 \text{ times})$,$\lambda = 1$,$\lambda = -2(n\text{-}2 \text{ times})$

$$\lambda = \frac{(2n-3) \pm \sqrt{4n^2-4n+9}}{2} \text{ (one time each)}$$

Minimum double dominating energy

$$E_{D'}(K_{n\times 2}) = 1+2(n-2)+\sqrt{4n^2-4n+9} = (2n - 3) + \sqrt{4n^2-4n+9}.$$

5. Conclusion

Thus in this paper, the new energy namely the double dominating energy is defined and has been found for some graphs.

REFERENCES

1. I. Gutman, The energy of a graph, *Ber. Math-Statist.Sekt. for Schungsz. Graz,* 103 (1978) 1-22.
2. A. Graovac, I. Gutman and N. Trinajstic, Topological approach to the chemistry of conjugated molecules, Springer-Verlag, Berlin, 1977.
3. I. Gutman, B. Zhou, Laplacean energy of a graph, *Lin. Algebra Appl*., 414 (2006) 29-37.
4. G. Indulal, I. Gutman and A. Vijayakumar, On distance energy of graph, *MATCH Commun. Math. Comput. Chem.,* 60 (2008) 461 –472.
5. C. Adiga, A. Bayad, J.Gutman, S.A. Srinivas, The minimum covering energy of a graph, *Kragujevac J. Sci.*, 34 (2012) 39 – 56.
6. M.R. Joo, D. Yandeh and M. Kiani, Mirzakhah, Incidence energy of a graph, *MATCH commun. Math. Comput. Chem.*, 62 (2009) 561 – 572.
7. J. Liu, B. Liu, A Laplacean – energy like invariant of a graph, *MATCH Commun. Math. Comput. Chem.,* 59 (2008) 355 – 372.
8. I. Gutman, Topological studies on hetero conjugated molecules. alternant systems with one heteroatom, *Theor. Chim. Acta,* 50 (1979) 287 – 297.
9. R.B.Bapat and S.Pati, Energy of a graph is neveran odd integer, *Bull. Kerala Math. Asscc.,* 1 (2011) 129-132.