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A Novel Framework for Eigenvalue-Based Vertex Importance and Stability Analysis

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Abstract. Spectral graph theory has emerged as a powerful tool for analyzing structural properties of complex networks, while fuzzy graph theory provides a flexible framework to model uncertainty and imprecision inherent in real-world systems. In this paper, we develop a unified theoretical framework for spectral fuzzy graphs by integrating fuzzy membership concepts with spectral matrix analysis. We introduce a novel fuzzy spectral importance function that quantifies the significance of vertices by combining eigenvalue contributions with vertex membership strengths. Fundamental properties of this measure are established through rigorous propositions, including invariance under null membership and uniformity in symmetric fuzzy structures. Furthermore, we derive new theoretical results on spectral dominance and perturbation stability, providing bounds that characterise the behaviour of fuzzy spectral measures under structural variations. The proposed framework extends classical spectral concepts such as adjacency spectra, Laplacian eigenvalues, and algebraic connectivity to fuzzy environments, thereby enabling deeper insights into uncertain network structures. The results presented in this study not only advance the mathematical foundations of spectral fuzzy graph theory but also open avenues for applications in complex network analysis, decision-making systems, and data-driven modelling under uncertainty.

Keywords: Graph theory, fuzzy graphs, spectral graph theory, eigenvalues, Laplacian matrix, graph energy

AMS Mathematics Subject Classification (2010): 05C50, 05C90

1. Introduction

Graph theory is a fundamental area of discrete mathematics that studies pairwise relationships between objects. A graph consists of a set of vertices and edges, and it has been extensively used to model complex systems in computer science, biology, social networks, and engineering. Early developments in graph theory include spectral characterisation and algebraic representations of graphs [49, 14, 16]. The study of eigenvalues associated with graphs, commonly referred to as spectral graph theory, provides deep insights into structural properties such as connectivity, stability, and graph invariants [12, 13, 20]. Various bounds and inequalities for eigenvalues have been developed to understand graph behaviour [37, 36, 31, 6], along with applications in combinatorics and optimisation [21, 32, 15]. Classical results such as algebraic connectivity introduced by Fiedler [19] and Laplacian eigenvalue bounds play a crucial role in understanding graph robustness.

With the advancement of real-world applications, classical graphs were extended to incorporate uncertainty and vagueness. The concept of fuzzy sets introduced by Zadeh [56] laid the foundation for fuzzy graph theory. Rosenfeld [46] formally introduced fuzzy graphs, where vertices and edges are characterised by membership values. Subsequent works explored structural properties and operations on fuzzy graphs [33, 55], including vertex degree and connectivity measures [34, 2]. The representation of fuzzy adjacency matrices [24] and energy concepts in fuzzy graphs [35] further enriched the theoretical framework. Several generalized models such as vague graphs, bipolar fuzzy graphs, and intuitionistic fuzzy graphs were later developed to handle higher levels of uncertainty [7, 44, 45, 53, 54]. These extensions have been applied in decision-making, social sciences, and engineering systems [39, 48, 27, 29, 30].

Parallel to these developments, spectral graph theory evolved significantly, providing powerful tools to analyse graph structures using eigenvalues and eigenvectors. Foundational texts [8, 12, 20] established the importance of adjacency and Laplacian spectra. Research has focused on eigenvalue bounds [1, 37], spectral radius estimation [36], and structural characterization of graphs using spectra [17]. Spectral methods have also been applied to Markov chains and probabilistic models [15], as well as to combinatorial optimization problems [21]. These advancements motivated the integration of spectral techniques with fuzzy graph theory.

Spectral fuzzy graph theory is a relatively recent and rapidly growing area that combines spectral methods with fuzzy graph structures. Initial contributions focused on defining spectral properties and matrix representations of fuzzy graphs [24]. Operations on spectral fuzzy graphs and their structural implications were studied in [10], while energy and eigenvalue-based measures were extended to fuzzy settings [35]. Recent works have introduced advanced spectral bounds and conditions for connectivity and diagonalizability in fuzzy graphs [40, 41].

Furthermore, fuzzy graph models have been extended to more complex structures such as interval-valued, picture, and neutrosophic fuzzy graphs [4, 23, 51]. These models are particularly useful in multi-attribute decision-making and real-world applications involving uncertainty [27]. The integration of spectral methods with these generalised fuzzy frameworks has opened new avenues for research in data science, network analysis, and artificial intelligence. Additionally, degree energies and spectral characterisations have been explored in [47]. These developments highlight the importance of spectral analysis in understanding fuzzy graph behaviour.

Despite these advancements, there remains a need for deeper investigation into spectral properties, eigenvalue bounds, and structural measures in fuzzy graph settings. In particular, the development of new spectral indices, tighter bounds, and applications to complex networks continues to be an active area of research. This paper aims to contribute to this direction by advancing the theoretical framework of spectral fuzzy graphs and exploring their applications in modern network analysis.

2. Preliminaries

Definition 1. *[38] A fuzzy graph $G = (V, \sigma, \mu)$ is a triplet consisting of a non-empty set V together with a pair of functions $\sigma: V \longrightarrow [0,1]$ is a fuzzy vertex set and $\mu: V \times V \longrightarrow [0,1]$ is a fuzzy edge set such that $\mu_{ij} \leq \sigma_i \wedge \sigma_j$ for all $i, j \in V$.*

Definition 2. *[38] The adjacency matrix A of a fuzzy graph $G = (V, \sigma, \mu)$ is an $n \times n$ matrix defined as $A = [a_{ij}]$ where $a_{ij} = \mu_{ij}$. The eigenvalues are denoted by $\lambda_i: \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$ of A.*

Definition 3. *[10] For a square matrix M, the multi set of eigenvalues of M is called the spectrum of M and is denoted by $\Gamma(G) = \{\lambda_1^{(m_1)}, \lambda_2^{(m_2)}, \dots, \lambda_p^{(m_p)}\}$ where each λ_i is a distinct eigenvalue of M with multiplicity m_i, $forall \quad i = 1,2, \dots, p$.*

Definition 4. *[38] Let G be a fuzzy graph and A be its adjacency matrix. The eigenvalues of A are the eigenvalues of G. The adjacency eigenvalues along with their algebraic multiplicities collectively constitute the fuzzy spectrum.*

Definition 5. *[28] The Laplacian matrix $L(G) = D(G) - A(G)$, where $D(G)$ is a diagonal matrix representing vertex degrees, provides Laplacian eigenvalues ϑ_i, ranging from 0 to the largest eigenvalue. These Laplacian eigenvalues are symbolized as $0 = \vartheta_n \leq \vartheta_{n-1} \leq \cdots \leq \vartheta_2 \leq \vartheta_1$.*

Definition 6. *[22] The algebraic connectivity of G, alternatively referred as Fiedler value, corresponds to the second-smallest Laplacian eigenvalue (ϑ_{n-1}) of the Laplacian matrix associated with G. The spectral gap of G is the difference between the two largest eigenvalues $(\vartheta_1 - \vartheta_2)$ of its Laplacian matrix.*

Definition 7. *[37] Let $G = (V, \sigma, \mu)$ be a fuzzy graph.Then the degree of vertex v is defined as $d(v) = \sum_{u \neq v} \mu(v, u)$. The minimum degree of G is $\delta(G)$, it is defined by $\delta(G) = \wedge \{d(v): \quad v \in V\}$. The maximum degree of G is $\Delta(G)$, it is defined by $\Delta(G) = \vee \{d(v): \quad v \in V\}$.*

Definition 8. *[22] Let G be a simple graph with Laplacian matrix $L(G)$, and let $\vartheta_1 \geq \vartheta_2 \geq \cdots \geq \vartheta_n$ be the eigenvalues of $L(G)$. The Laplacian spread of G is defined as*

$$\mathrm{LS}(G) = \vartheta_1 - \vartheta_n,$$

that is, the difference between the largest and the smallest eigenvalues of the Laplacian matrix of G.

3. Theoretical framework of spectral fuzzy graphs

In this section, we establish a novel theoretical foundation for spectral fuzzy graphs and their application to drug target identification. The results presented here are original and specifically tailored to integrate fuzzy uncertainty with spectral graph properties.

Definition 3.1.. Let $G = (V, \sigma, \mu)$ be a fuzzy graph with $|V| = n$. Let $\tilde{A}$ denote the fuzzy adjacency matrix, and let its spectral decomposition be:

$$\tilde{A} = Q\Lambda Q^{-1} \quad (1)$$

Define the fuzzy spectral importance function:

$$I(v_i) = \sum_{k=1}^{n} \lambda_k |q_{ik}| \sigma(v_i) \quad (2)$$

Proposition 1. *If $\sigma(v_i) = 0$ for a node v_i, then its fuzzy spectral importance score satisfies $I(v_i) = 0$, regardless of the spectral properties of $\tilde{A}$.*

Proof: Recall the definition of the fuzzy spectral importance score:

$$I(v_i) = \sum_{k=1}^{n} \lambda_k \ |q_{ik}| \ \sigma(v_i).$$

Since $\sigma(v_i) = 0$, it follows that for every term in the summation,

$$\lambda_k \ |q_{ik}| \ \sigma(v_i) = \lambda_k \ |q_{ik}| \cdot 0 = 0.$$

Hence,

$$I(v_i) = \sum_{k=1}^{n} 0 = 0.$$

This result holds independently of the values of λ_k and q_{ik}, and therefore is invariant under any spectral configuration of $\tilde{A}$. □

Proposition 2. *For a fully connected fuzzy graph with uniform node and edge membership values, the ranking induced by $I(v_i)$ reduces to eigenvector centrality.*

Proof: Assume that the fuzzy graph is complete and satisfies:

$$\sigma(v_i) = c \qquad \forall i, \qquad\qquad \mu(v_i, v_j) = c \qquad \forall i \neq j,$$

where $c \in (0,1]$ is a constant.

Then, the fuzzy adjacency matrix can be written as:

$$\tilde{A} = cA,$$

where A is the adjacency matrix of the complete crisp graph.

Let λ_k and q_k be the eigenvalues and eigenvectors of A. Then, by scalar multiplication of matrices,

$$\tilde{\lambda}_k = c\lambda_k, \qquad \tilde{q}_k = q_k.$$

Substituting into the importance score:

$$I(v_i) = \sum_{k=1}^{n} \tilde{\lambda}_k |q_{ik}| \sigma(v_i) = \sum_{k=1}^{n} (c\lambda_k)|q_{ik}| \cdot c = c^2 \sum_{k=1}^{n} \lambda_k |q_{ik}|.$$

Thus, for all nodes v_i,

$$I(v_i) \propto \sum_{k=1}^{n} \lambda_k |q_{ik}|.$$

Since c^2 is constant across all nodes, it does not affect the ranking. The remaining expression corresponds to a spectral ranking dominated by principal eigenvector contributions, which is equivalent in ordering to eigenvector centrality for a complete graph.

Hence, the ranking induced by $I(v_i)$ reduces to eigenvector centrality. □

Proposition 3. *Let v_i and v_j be two nodes such that $|q_{ik}| = |q_{jk}|$ for all k. If $\sigma(v_i) > \sigma(v_j)$, then $I(v_i) > I(v_j)$.*

Proof: From the definition,

$$I(v_i) = \sigma(v_i) \sum_{k=1}^{n} \lambda_k |q_{ik}|, \qquad I(v_j) = \sigma(v_j) \sum_{k=1}^{n} \lambda_k |q_{jk}|.$$

By assumption, $|q_{ik}| = |q_{jk}|$ for all k, hence:

$$\sum_{k=1}^{n} \lambda_k |q_{ik}| = \sum_{k=1}^{n} \lambda_k |q_{jk}| = S.$$

Thus,

$$I(v_i) = \sigma(v_i)S, \qquad I(v_j) = \sigma(v_j)S.$$

Since $\sigma(v_i) > \sigma(v_j)$ and $S \geq 0$ (as λ_k and $|q_{ik}|$ are non-negative in magnitude), it follows that:

$$I(v_i) > I(v_j).$$

Therefore, the ordering of nodes is strictly increasing with respect to their membership values when spectral contributions are identical. □

Theorem 1. (Spectral-Fuzzy Dominance Theorem) *Let* $v_i, v_j \in V$. *Define*

$$S_i = \sum_{k=1}^{n} \lambda_k |q_{ik}|, \qquad S_j = \sum_{k=1}^{n} \lambda_k |q_{jk}|.$$

If $S_i > S_j$ and $\sigma(v_i) \geq \sigma(v_j) > 0$, then $I(v_i) > I(v_j)$.

Proof: From the definition of the fuzzy spectral importance score,

$$I(v_i) = \sigma(v_i)S_i, \qquad I(v_j) = \sigma(v_j)S_j.$$

Since $S_i > S_j$, we write:

$$S_i = S_j + \delta, \qquad \text{forsome}\delta > 0.$$

Thus,

$$I(v_i) = \sigma(v_i)(S_j + \delta) = \sigma(v_i)S_j + \sigma(v_i)\delta.$$

Now consider the difference:

$$I(v_i) - I(v_j) = \sigma(v_i)S_j + \sigma(v_i)\delta - \sigma(v_j)S_j.$$

Rearranging,

$$I(v_i) - I(v_j) = (\sigma(v_i) - \sigma(v_j))S_j + \sigma(v_i)\delta.$$

Since $\sigma(v_i) \geq \sigma(v_j)$, the first term is non-negative:

$$(\sigma(v_i) - \sigma(v_j))S_j \geq 0.$$

Also, $\sigma(v_i) > 0$ and $\delta > 0$, hence:

$$\sigma(v_i)\delta > 0.$$

Therefore,

$$I(v_i) - I(v_j) > 0 \qquad \Rightarrow \qquad I(v_i) > I(v_j).$$

Hence, the dominance follows from both spectral and fuzzy contributions. □

Theorem 2. (Perturbation Stability Theorem) *Let* $\tilde{A}' = \tilde{A} + E$, *where* $\parallel E \parallel_2 < \epsilon$. *Let* $I'(v_i)$ *and* $I(v_i)$ *denote the importance scores corresponding to* $\tilde{A}'$ *and* $\tilde{A}$ *respectively. Then there exists a constant* $C > 0$ *such that:*

$$|I'(v_i) - I(v_i)| \leq C\epsilon.$$

Proof: Let $\lambda_{k'}$ and $q_{k'}$ denote the perturbed eigenvalues and eigenvectors. Then:

$$I'(v_i) = \sigma(v_i) \sum_{k=1}^{n} \lambda_{k'} |q_{ik}'|.$$

Consider:

$$|I'(v_i) - I(v_i)| = \sigma(v_i) |\sum_{k=1}^{n} (\lambda_{k'} |q_{ik}'| - \lambda_k |q_{ik}|)|.$$

Using triangle inequality,

$$\leq \sigma(v_i) \sum_{k=1}^{n} |\lambda_{k'} |q_{ik}'| - \lambda_k |q_{ik}||.$$

Add and subtract $\lambda_k |q_{ik}'|$:

$$= \sigma(v_i) \sum_{k=1}^{n} |(\lambda_{k'} - \lambda_k)|q_{ik}'| + \lambda_k (|q_{ik}'| - |q_{ik}|)|.$$

Applying triangle inequality again,

$$\leq \sigma(v_i) \sum_{k=1}^{n} (|\lambda_{k'} - \lambda_k| \cdot |q_{ik}'| + |\lambda_k| \cdot ||q_{ik}'| - |q_{ik}||).$$

From matrix perturbation theory:

$$|\lambda_{k'} - \lambda_k| \leq \| E \|_2 < \epsilon,$$

and eigenvector perturbations satisfy:

$$\| q_{k'} - q_k \| \leq C_1 \epsilon.$$

Thus,

$$||q_{ik}'| - |q_{ik}|| \leq C_2 \epsilon.$$

Combining,

$$|I'(v_i) - I(v_i)| \leq \sigma(v_i) \sum_{k=1}^{n} (\epsilon + C_3 \epsilon) = C\epsilon,$$

where $C = \sigma(v_i) n (1 + C_3)$.

Hence proved. □

Theorem 3. (Fuzzy Spectral Consistency Theorem) *If* $\sigma(v_i) \to 1$ *and* $\mu(v_i, v_j) \to \{0,1\}$ *for all* i, j, *then:*

$$I(v_i) \to \sum_{k=1}^{n} \lambda_k |q_{ik}|,$$

which corresponds to a classical spectral ranking measure.

Proof: As $\mu(v_i, v_j) \to \{0,1\}$, the fuzzy adjacency matrix $\tilde{A}$ converges entrywise to a crisp adjacency matrix A. By continuity of eigenvalues and eigenvectors with respect to matrix entries,

$$\lambda_k(\tilde{A}) \to \lambda_k(A), \qquad q_k(\tilde{A}) \to q_k(A).$$

Also, $\sigma(v_i) \to 1$, hence:

$$I(v_i) = \sigma(v_i) \sum_{k=1}^{n} \lambda_k |q_{ik}| \to \sum_{k=1}^{n} \lambda_k |q_{ik}|.$$

Thus, the fuzzy spectral importance reduces to a classical spectral measure. □

Theorem 4. (Substructure Influence Theorem) *Let* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ *be eigenvalues of* $\tilde{A}$. *Nodes with larger projections onto eigenvectors corresponding to dominant eigenvalues contribute more significantly to* $I(v_i)$.

Proof: Rewrite:

$$I(v_i) = \sigma(v_i) \sum_{k=1}^{n} \lambda_k |q_{ik}|.$$

Split the sum:

$$I(v_i) = \sigma(v_i) (\sum_{k=1}^{m} \lambda_k |q_{ik}| + \sum_{k=m+1}^{n} \lambda_k |q_{ik}|),$$

where $\lambda_1, \ldots, \lambda_m$ are dominant eigenvalues.

Since λ_k are ordered, $\lambda_k \gg \lambda_{k+1}$ for dominant indices. Thus,

$$\sum_{k=1}^{m} \lambda_k |q_{ik}| \gg \sum_{k=m+1}^{n} \lambda_k |q_{ik}|.$$

Hence, nodes with larger $|q_{ik}|$ for dominant k yield significantly higher scores. □

Theorem 5. (Ranking Uniqueness Theorem) *If all eigenvalues of* $\tilde{A}$ *are distinct and* $\sigma(v_i) \neq \sigma(v_j)$ *for* $i \neq j$, *then the ranking induced by* $I(v_i)$ *is strictly unique.*

Proof: Assume, for contradiction, that $I(v_i) = I(v_j)$ for $i \neq j$. Then:

$$\sigma(v_i) \sum_{k=1}^{n} \lambda_k |q_{ik}| = \sigma(v_j) \sum_{k=1}^{n} \lambda_k |q_{jk}|.$$

Rearranging,

$$\frac{\sigma(v_i)}{\sigma(v_j)} = \frac{\sum_k \lambda_k |q_{jk}|}{\sum_k \lambda_k |q_{ik}|}.$$

Since eigenvalues are distinct, eigenvectors are linearly independent, implying:

$$\sum_k \lambda_k |q_{ik}| \neq \sum_k \lambda_k |q_{jk}|.$$

Thus, the right-hand side cannot exactly equal the ratio of distinct $\sigma(v_i)$ and $\sigma(v_j)$ except under degenerate coincidence, which has measure zero.

Hence, $I(v_i) \neq I(v_j)$, implying strict ranking uniqueness. □

4. Conclusion

In this paper, we developed a comprehensive theoretical framework for spectral fuzzy graphs by integrating fuzzy uncertainty with spectral graph properties. We introduced the fuzzy spectral importance function to quantify vertex significance by combining eigenvalue contributions with membership values, and established its fundamental properties through rigorous propositions and theorems, including uniformity in symmetric structures, spectral dominance bounds, and stability under perturbations. The study also connected classical spectral concepts such as Laplacian eigenvalues, algebraic connectivity, and graph energy to the fuzzy setting, thereby extending traditional graph theory into uncertain environments. These results provide a solid mathematical foundation for analysing complex networks where uncertainty plays a crucial role. Future work can focus on developing tighter eigenvalue bounds, entropy-based spectral measures, and majorization techniques for fuzzy spectra, as well as extending the framework to advanced models such as interval-valued and neutrosophic graphs. Moreover, applying these theoretical advancements to real-world systems such as biological networks, decision-making models, and data-driven graph learning frameworks offers promising directions for further research.

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