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Cardinality of Hajós Graphs and Hajós Fuzzy Graphs on Fan Graph, Lollipop Graph, Friendship Graph, Tadpole Graph and Crown Graph

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Abstract. Hajos Fuzzy graph is a new fuzzy graph obtained by applying a binary operation, named Hajos construction, on two fuzzy graphs. The Hajos construction on two (fuzzy) graphs produces many different (fuzzy) graphs depending on the choice of vertices and edges. The cardinality of Hajos (fuzzy) graphs is the total number of Hajos (fuzzy) graphs from two given Hajos (fuzzy) graphs. Here, the cardinality of Hajós fuzzy graphs are determined for any two fuzzy graphs based on the permutations and combinations method.is By this concept, the cardinality of Hajós (fuzzy) graphs is derived for the combinations of the two (fuzzy) graphs, such as the fan graph, lollipop graph, friendship graph, tadpole graph and crown graphs.

Keywords: Hajos Fuzzy Graph, Cardinality, Fan Graph, Lollipop Graph, Friendship Graph, Tadpole graph and Crown Graph.

AMS Mathematics Subject Classification (2010): 94D05,

I. Introduction

Graph theory is an emerging research field with numerous applications in real-life problems. Due to uncertainty in real-life problems which weighted crisp graphs cannot address, there is a need for fuzzy graph theory. Fuzzy graph theory is an extension of graph theory, which plays a crucial role in real-world circumstances like medical diagnosis, social network analysis, and natural language processing. Rosenfeld initially introduced fuzzy graph theory in 1975 [10]. Yamuna and Karthika have studied the Hajos stable graphs in a note on Hajos stable graphs [12]. Juan Carlos García-Altamirano, Mika Olsen, Jorge Cervantes-Ojeda studied the methods to obtain symmetric cycles and symmetric cycles of length 5 in their works How to construct the symmetric cycle of length 5 using Hajós construction with an adapted Rank Genetic Algorithm [5] and Computational complexity of Hajós constructions of symmetric odd cycles [6].

The preliminaries that we have studied to develop our new concepts are as follows:

Let G_1 and G_2 be any two graphs, (u_1, v_1) be an edge of G_1 and (u_2, v_2) be an edge of G_2. Then the Hajós construction produces a different graph H that combines the two graphs by

i. Merging vertices u_1 and u_2 into a single vertex u_{12}
ii. Eliminating the two edges (u_1, v_1) and (u_2, v_2)
iii. Adding a new edge (v_1, v_2).[12]

A fuzzy graph G is defined as an ordered pair G: (σ, μ) on graph G* : (V, E) where V is the set of vertices, a vertex is also called a node, and E is the set of edges which is a subset of V ×V, the membership functions are σ : V → [0, 1] and μ: V ×V → [0, 1].**[10]**
A path graph is a graph whose vertices can be arranged as $v_1, v_2, \cdots, v_n$ so that the edges of the path are $v_i v_{i+1}$, $i = 1, 2, \ldots, n - 1$. [1]

Friendship graph, is a set of n triangles having a common central vertex.[11]
A fan graph Fm,n is defined as the graph join K_m+ P_n where is the empty graph on m nodes and P_n is the path graph on n nodes. The case m=1 corresponds to the usual fan graphs, while m=2 corresponds to the double fan graphs, etc.,[4]

A lollipop graph by combining the complete graph K_m and the path graph P_n with a bridge (edge) so that the lollipop graph notation is $L_{m,\ n}$ for any natural number m and n.[7]

A *crown graph* (also known as a *cocktail party graph*) $H_{n,\ n}$ is a graph obtained from the complete bipartite graph $K_{n,\ n}$ by removing a perfect matching. Formally, $V(H_{n,\ n}) = \{ 1', \cdots, n', 1, \cdots, n\}$ and $E(H_{n,\ n}) = \{ ij' / i \neq j'\}$.[8]

Tadpole graph $(T_{m,n})$ is defined as a graph obtained by combining a vertex of cycle graph C_m with one of the leaf of path graph P_n. Suppose that the vertices and edges in the tadpole graph are notated as follows. $V(T_{m,\ n}) = \{u_i;\ i = 1,\ 2, \cdots, n\} \cup \{v_j;\ j = 1, 2, \cdots, n\}$ $E(T_{m,\ n}) = \{u_1 u_m,\ u_i u_{i+1};\ i = 1, 2, \cdots, m-1\} \cup \{u_1 v_1\} \cup \{ v_i v_{i+1};\ i = 1, 2, \cdots, n-1\}$.[2]

The total number of Hajos graphs that can be constructed from two graphs G_1* and G_2* is defined as the cardinality of Hajos graph from G_1* and G_2*. It is denoted by C(H[G_1*; G_2*]) or by $|$H[G_1 *; G_2 *]$|$, where H[G_1*; G_2*] denotes the set of all Hajos graphs from G_1* and G_2*.[9]

The total number of Hajos fuzzy graphs that can be obtained from two fuzzy graphs G_1 and G_2 is defined as the cardinality of Hajos fuzzy graph from G_1 and G_2. It is denoted by C(H[G_1; G_2]) or by $|$H[G_1; G_2]$|$, where H[G_1; G_2] denotes the set of all Hajos fuzzy graphs from G_1 and G_2.

(i) If G_i* and G_i'* are isomorphic graphs, i = 1, 2, then C(H[G_1*; G_2*]) = C(H[G_1'*; G_2'*]).
(ii) (ii) If G_i and G_i' are isomorphic fuzzy graphs, i = 1, 2, then C(H[G_1; G_2]) = C(H[G_1' ; G_2']).[9]

2. Methodology

Here, the Hajós construction is applied on fuzzy graphs. Hajós construction is one of the effective ways that are used to combine two networks. The Hajós fuzzy graph is a fuzzy graph obtained by applying the binary operation, namely Hajós construction, on two fuzzy graphs. The cardinality of Hajós (fuzzy) graph are obtained.

Definition 2.1. Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\ \mu_2)$ be two fuzzy graphs on $G_1^*:(V_1, E_1)$ and $G_2^*:(V_2, E_2)$ respectively. Let u_1v_1 be any edge in G_1 and u_2v_2 be any edge in G_2. The *Hajós fuzzy graph* of G_1 and G_2 with respect to the edge-vertices $u_1v_1 - u_1$ and $u_2v_2 - u_2$, denoted by $H[G_1(u_1v_1;u_1), G_2(u_2v_2;u_2)]:(\sigma,\mu)$ on $H^*:(V,E)$ where $V = (V_1 - \{u_1\}) \cup (V_2 - \{u_2\}) \cup \{u_1 \cdot u_2\}$ *and* $E = (E_1 - \{u_1v_1\}) \cup (E_2 - \{u_2v_2\}) \cup \{v_1v_2\}$ is constructed as follows:

(i) Merge the vertices u_1 in G_1 and u_2 in G_2 into a new single vertex $u_1 \cdot u_2$ with membership value $\sigma(u_1 \cdot u_2) = \sigma_1(u_1) \vee \sigma_2(u_2)$.

(ii) Remove the edges u_1v_1 in G_1 and u_2v_2 in G_2.

(iii) Insert a new edge v_1v_2 in H with membership value $\mu(v_1v_2) = \mu_1(u_1v_1) \wedge \mu_2(u_2v_2)$.

The edges $vw \in E_i - \{u_iv_i\}$ are of two types:

$vw \in E_i,\ v \neq u_i,\ w \neq u_i;\ \ v \in V_i,\ w = u_1 \cdot u_2$, $i = 1, 2$. Assign the membership values for these edges as

$$\mu(vw) = \begin{cases} \mu_1(vw), & if\ vw \in E_1,\ v \neq u_1, w \neq u_1 \\ \mu_2(vw), & if\ vw \in E_2, v \neq u_2,\ w \neq u_2 \\ \mu_1(vw), & if\ v \in V_1,\ w = u_1 \cdot u_2 \\ \mu_2(vw), & if\ v \in V_2,\ w = u_1 \cdot u_2 \end{cases}$$

(iv) For all the vertices $v \neq u_i$ i=1, 2, assign their membership value as

$$\sigma(v) = \begin{cases} \sigma_1(v), & if\ v \in V_1 \\ \sigma_2(v), & if\ v \in V_2 \end{cases}$$

Let us verify that σ and μ satisfy the conditions of fuzzy graph.

Let vw be any edge of H. Then $vw \in E_i - \{u_iv_i\}$ or $vw = v_1v_2$.

If $vw \in E_1$, $v \neq u_1,\ w \neq u_1, \mu(vw) = \mu_1(vw) \leq \sigma_1(v) \wedge \sigma_1(w) = \sigma(v) \wedge \sigma(w)$.

If $vw \in E_2$, $v \neq u_2,\ w \neq u_2, \mu(vw) = \mu_2(vw) \leq \sigma_2(v) \wedge \sigma_2(w) = \sigma(v) \wedge \sigma(w)$.

For any edge $v\,(u_1 \cdot u_2)$ with $v \in V_1$, $\mu(v(u_1 \cdot u_2)) = \mu_1(vu_1) \leq \sigma_1(v) \wedge \sigma_1(u_1) \leq \sigma_1(v) \wedge (\sigma_1(u_1) \vee \sigma_1(u_2)) = \sigma(v) \wedge \sigma((u_1 \cdot u_2))$.

Similarly for any edge $v(u_1 \cdot u_2)$ with $v \in V_2$, $\mu(v(u_1 \cdot u_2)) \leq \sigma(v) \wedge \sigma((u_1 \cdot u_2))$.

For the new edge v_1v_2 in H,

$$\mu(v_1v_2) = \mu(u_1v_1) \wedge \mu(u_2v_2) \leq \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2) \leq \sigma_1(v_1) \wedge \sigma_2(v_2)$$

$$= \sigma(v_1) \wedge \sigma(v_2).$$

Therefore for all the edges vw in H, $\mu(vw) \leq \sigma(v) \wedge \sigma(w)$

Hence $H[G_1(u_1v_1;u_1), G_2(u_2v_2;u_2)]:(\sigma,\mu)$ is a fuzzy graph, called the *Hajós fuzzy graph* on G*(V, E).

Example 2.1. F_2 and F_4 are two fuzzy graphs from which u_3 and V_6 are identified respectively and the edges u_3u_4 and $V_5 V_6$ are deleted to form a new edge namely u_4V_5 whose membersip value is $\mu(u_4V_5) = 0.1$ and new vertex $u_3 \cdot V_6$ formed with a membership value $\sigma(u_3 \cdot V_6) = 0.8$

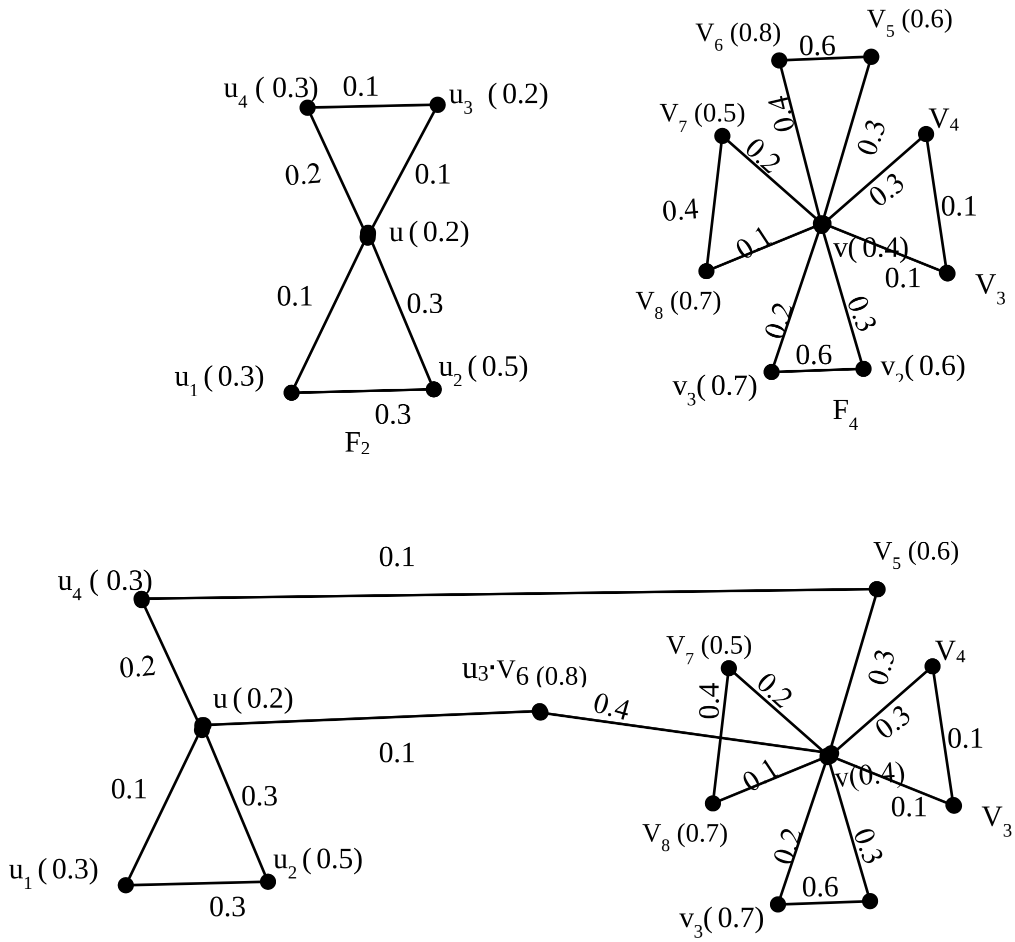

Figure 1: Hajos fuzzy graph formed from F_2

3. Results and discussions

3.1. Cardinality of Hajós graphs and Hajós fuzzy graphs on fan graph, lollipop graph, friendship graph, tadpole graph and crown graph

In this chapter, the cardinality of Hajós Fuzzy Graphs is derived from the combinations of the graphs, such as the Fan Graph, Lollipop Graph, Friendship Graph, Tadpole graph and Crown Graphs.

Theorem 3.1. *The number of Hajós graphs that can be constructed from the underlying crisp graphs G_1* and G_2* is the same as the number of Hajós fuzzy graphs that can be constructed from the fuzzy graphs G_1 and G_2.*

Theorem 3.2. Let $G_i^*(V_i, E_i)$, i = 1, 2 be two Fan Graphs $F_{n_i,\ m_i}$. Then the number of Hajós graphs that can be obtained from G_i^*, i = 1, 2 is $4(m_1 n_1 + n_1 - 1)\,(m_2 n_2 + n_2 - 1)$.

Proof: Let V₁ = $\{ v_{11}, v_{12}, \cdots, v_{1m_1}, v_{21}, v_{22}, \cdots, v_{2n_1}\}$ and V₂ = $\{u_{11}, u_{12}, \cdots, u_{1m_2}, u_{21}, u_{22}, \cdots, u_{2n_2}\}$ be the vertex sets of G_1* and G_2* respectively, where $v_{11}v_{12}\cdots v_{1m_1}$ and $u_{11}u_{12}\cdots u_{1m_2}$ are the paths of the G_1* and G_2* respectively. Let the remaining vertices $v_{21}, v_{22}, \cdots, v_{2n_1}$ of G_1* and $u_{21}, u_{22}, \cdots, u_{2n_2}$$G_2$* be the vertices of $\overline{K_{n_1}}$ and $\overline{K_{n_2}}$ respectively. Let $i \in \{1, 2, \cdots, m_1\}$, $j \in \{1, 2, \cdots, n_1\}$, $k \in \{1, 2, \cdots, m_2\}$ and $l \in \{1, 2, \cdots, n_2\}$.

For each identification of vertices in the construction of Hajós graphs, there exist different cases as follows:

(i) v_{1i} is identified with u_{1k},where both are end vertices (ii) v_{1i} is an end vertex is identified with u_{1k} is an internal vertex (iii) v_{1i} is an internal vertex identified with u_{1k} is an end vertex, (iv) v_{1i} is identified with u_{1k}, where both are internal vertices (v) v_{2j} is identified with u_{2l}, (vi) v_{2j} is identified with u_{1k} is an internal vertex, (vii) v_{2j} is identified with u_{1k} is an end vertex, (viii) v_{1i} is an end vertex is identified with u_{2l}, (ix) v_{1i} is an internal vertex is identified with u_{2l}.

Case (i): v_{1i} is identified with u_{1k}, where both are end vertices.
For each edge incident at v_{1i}, there are $(n_2 + 1)$ choices of edges incident at u_{1k} for constructing Hajós graphs. Therefore the number of Hajós graphs corresponding to an edge incident at v_{1i} is $(n_2 + 1)$. Since the number of edges incident at v_{1i} is $(n_1 + 1)$, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is $(n_1 + 1)$. As there are 2 end vertices in the paths of $G_1 *$ and $G_2 *$, the total number of Hajós graphs obtained by identifying v_{1i} with u_{1k} is $4(n_1 + 1)(n_2 + 1)$.

Case (ii): v_{1i} is an end vertex is identified with u_{1k} is an internal vertex .
For each edge incident at v_{1i}, there are $(n_2 + 2)$ choices of edges incident at u_{1k}. Therefore, the number of Hajós graph corresponding to an edge incident at v_{1i}, is $(n_2 + 2)$. Since the number of edges incident at v_{1i} is $(n_1 + 1)$, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is $(n_1 + 1)(n_2 + 2)$. As there are 2 end vertices in the path of $G_1 *$ and $(m_2 - 2)$ internal vertices in $G_2 *$, the total number of Hajós graphs obtained by identifying v_{1i}, with u_{1k} is $2(n_1 + 1)(n_2 + 2)(m_2 - 2)$.

Case (iii): v_{1i} is an internal vertex identified with u_{1k} is an end vertex.
For each edge incident at v_{1i}, there are $(n_2 + 1)$ choices of edges incident at u_{1k}. Therefore, the number of Hajós graph corresponding to an edge incident at v_{1i} is $(n_2 + 1)$. Since $(n_1 + 2)$ edges are incident at v_{1i}, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is $(n_2 + 1)(n_1 + 2)$. As there are $(m_1 - 2)$ internal vertices in the path of $G_1 *$ and 2 end vertices in $G_2 *$, the total number of Hajós graphs obtained by identifying v_{1i} with u_{1k} is $2(m_1 - 2)(n_2 + 1)(n_1 + 2)$.

Case (iv): v_{1i} is identified with u_{1k}, where both are internal vertices
For each edge incident at v_{1i}, there are $(n_2 + 2)$ choices of edges incident at u. Therefore the number of Hajós graph corresponding to an edge incident at v_{1i} is $(n_2 + 2)$. Since the number of edges incident at v_{1i} is $(n_1 + 2)$, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is $(n_1 + 2)(n_2 + 2)$. As there are $(m_1 - 2)$ and $(m_2 - 2)$

vertices in the internal vertices of G_{1^*} and $G_2 *$. Hence the total number of Hajós graphs obtained by identifying v_{1i} with u_{1k} is $(m_1 - 2)(n_1 + 2)(m_2 - 2)(n_2 + 2)$.

Case (v): v_{1i} is an end vertex is identified with u_{2l}
For each edge incident at v_{1i}, there are m_2 choice of the edges incident at u_{2l}. Therefore the number of Hajós graph corresponding to an edge incident at v_{1i} is m_2. Since the number of edges incident at v_{1i} is $(n_1 + 1)$, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is $(n_1 + 1)\, m_2$. As there are 2 end vertices *in* G_1 and n_2 vertices of $\overline{K_{n_2}}$ in G_2, the total number of Hajós graphs obtained by identifying v_{1i} with u_{2k} is $2(n_1 + 1)\, m_2 n_2$.

Case (vi): v_{1i} is an internal vertex is identified with u_{2l}.
For each edge incident at v_{1i}, there are m_2 choices of edges incident at u_{2l}. Therefore the number of Hajós graphs corresponding to an edge incident at v_{1i} is m_2. The number of edges incident at`2 v_{1i} is $(n_1 + 2)$. Therefore the number of Hajós graphs corresponding to all the edges incident at v_{1i} is $(n_1 + 2)m_2$. As there are $(m_1 - 2)$ internal vertices *in* $G_1 *$ and n pendant vertices in G_2*, the total number of Hajós graphs obtained by identifying v_{2i} with u_{1k} is $(m_1 - 2)(n_1 + 2)m_2 n_2$.

Case (vii): v_{2j} is identified with u_{2l}
For each edge incident at v_{2j}, there are m_2 choices of the edges incident at u_{2l}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is m_2. Since m_1 edges are incident at v_{2j}, the number of Hajós graphs corresponding to all the edges incident at v_{2j} is $m_1 m_2$. As there are n_1 and n_2 pendant vertices in G_1 and G_2, the total number of Hajós graphs obtained by identifying v_{2j} with u_{2l} is $m_1 m_2 n_1 n_2$.

Case (viii): v_{2j} is identified with u_{1k} is an internal vertex
For each edge incident at v_{2j}, there are $(n_2 + 2)$ choice of edges incident at u_{1k}. Therefore the number of Hajós graph corresponding to an edge incident at v is $(n_2 + 2)$. Since the number of edges incident at v_{2j} is m_1, the number of Hajós graphs corresponding to all the edges incident at v_{2j} is $m_1(n_2 + 2)$. As there are n_1 pendant vertices in G_2 and $(m_2 - 2)$ vertices in the internal vertices of G_{1^*}, the total number of Hajós graphs obtained by identifying v_{2j} with u_{1k} is $m_1 n_1 (m_2 - 2)(n_2 + 2)$.

Case (ix): v_{2j} is identified with u_{1k} is an end vertex
For each edge incident at v_{2j}, there are $(n_2 + 1)$ choices of edges incident at u_{1k}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is $(n_2 + 1)$. The number of edges incident at v_{2j} is m_1. Therefore the number of Hajós graphs corresponding to all the edges incident at v_{2j} is $m_1(n_2 + 1)$. As there are n_1 pendant vertices *in* $G_1 *$ and 2 end vertex in $G_2 *$, the total number of Hajós graphs obtained by identifying v_{2j} with u_{1k} is $2m_1 n_1 (n_2 + 1)$.

From the above nine cases, the total number of Hajós graph from G_1 and G_2 is $4(n_1 + 1)(n_2 + 1)$ $+2(n_1 + 1)(n_2 + 2)(m_2 - 2) + 2(m_1 - 2)(n_2 + 1)(n_1 + 2)$+$(m_1 -$

$2)(n_1+2)(m_2-2)(n_2+2)+m_1m_2n_1n_2+m_1n_1(m_2-2)(n_2+2)+2m_1n_1(n_2+1)+2(n_1+1)m_2n_2+(m_1-2)(n_1+2)m_2n_2 = 4(n_1m_1+m_1-1)(m_2n_2+m_2-1)$.

Theorem 3.3. Let $G_i(V_i, E_i)$, i = 1, 2 be two fuzzy graphs on fan graphs $F_{n_i,\ m_i}$ respectively. Then the number of Hajós graphs that can be obtained from G_i^*, i = 1, 2 is $4(m_1n_1+m_1-1)(m_2n_2+m_2-1)$.

Theorem 3.4. Let $G_i^*(V_i, E_i)$, i = 1, 2 be two Lollipop graphs $L_{n_1,\ m_1}$ and $L_{n_2,\ m_2}$. Then the number of Hajós graphs that can be obtained from G_i^*, i = 1, 2 is $(n_1(n_1-1)+2m_1)(n_2(n_2-1)+2m_2)$.

Proof: Let V_1 = $\{v_{11}, v_{12}, \cdots, v_{1m_1}, v_{21}, v_{22}, \cdots, v_{2n_1}\}$ and V_2 = $\{u_{11}, u_{12}, \cdots, u_{1m_2}, u_{21}, u_{22}, \cdots, u_{2n_2}\}$ be the vertex sets of G_1^* and G_2^* respectively, where $v_{11}v_{12}\cdots v_{1m_1}$ and $u_{11}u_{12}\cdots u_{1m_2}$ are the paths of the G_1^* and G_2^* respectively. Let the remaining vertices $v_{21}, v_{22}, \cdots, v_{2n_1}$ of G_1^* and $u_{21}, u_{22}, \cdots, u_{2n_2}$ G_2^* be the vertices of K_{n_1} and K_{n_2} respectively. Let $i \in \{1, 2, \cdots, m_1\}$, $j \in \{1, 2, \cdots, n_1\}$, $k \in \{1, 2, \cdots, m_2\}$ and $l \in \{1, 2, \cdots, n_2\}$.

For each identification of vertices in the construction of Hajós graphs, there exist different cases as follows:

(i) v_{1i} is identified with u_{1k},where both are end vertices (ii) v_{1i} is an end vertex is identified with u_{1k} is an internal vertex (iii) v_{1i} is an internal vertex identified with u_{1k} is an end vertex, (iv) v_{1i} is identified with u_{1k}, where both are internal vertices (v) v_{2j} is identified with u_{2l}, (vi) v_{2j} is identified with u_{1k} is an internal vertex, (vii) v_{2j} is identified with u_{1k} is an end vertex, (viii) v_{1i} is an end vertex is identified with u_{2l}, (ix) v_{1i} is an internal vertex is identified with u_{2l}.

Case (i): v_{1i} is identified with u_{1k}, where both are end vertices

Subcase (i.a): v_{1i} is identified with u_{1k}, where both are end vertices and both are with a bridge connecting K_{n_1} & K_{n_2}.

For each edge incident at v_{1i}, there are 2 choices of edges incident at u_{1k} for constructing Hajós graphs. Therefore the number of Hajós graphs corresponding to an edge incident at v_{1i} 2. Since the number of edges incident at v_{1i} is 2, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is 4. As there exist 1 end vertex of the path that is incident with a bridge of G_1* and 1 end vertex of the path that is incident with a bridge in G_2*, the total number of Hajós graphs obtained by identifying v_{1i} with u_{1k} is 4.

Subcase (i.b): v_{1i} is identified with u_{1k}, where both are end vertices, and both are without a bridge connecting K_{n_2}.

For each edge incident at v_{1i}, there exist only one edge incident at u_{1k} for constructing Hajós graphs. Therefore the number of Hajós graphs corresponding to an edge incident at v_{1i} 1. Since the number of edges incident at v_{1i} is 1, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is 1. As there exist 1 end vertex of the path that is incident with the bridge in G_1* and 1 end vertex of the path that doesn't incident with the bridge in G_2*, the total number of Hajós graphs obtained by identifying v_{1i} with u_{1k} is 1.

Subcase (i.c): v_{1i} is identified with u_{1k}, where both are end vertices and v_{1i} is a vertex incident
with a bridge connecting K_{n_1} and u_{1k} is a vertex that doesn't incident with the bridge connecting K_{n_2}.
For each edge incident at v_{1i}, there exist only one edge incident at u_{1k} for constructing Hajós graphs. Therefore the number of Hajós graphs corresponding to an edge incident at v_{1i} is 1. Since the number of edges incident at v_{1i} is 2, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is 2. As there exist 1 end vertex of the path that is incident with the bridge in G_1* and 1 end vertex of the path that does not incident with the bridge in G_2*, the total number of Hajós graphs obtained by identifying v_{1i} with u_{1k} is 2.

Subcase (i.d): v_{1i} is identified with u_{1k}, where both are end vertices and v_{1i} is a vertex that does not incident with a bridge connecting K_{n_1} and u_{1k} is a vertex that incident with the bridge connecting K_{n_2}.

For each edge incident at v_{1i}, there are 2 choices of edges incident at u_{1k} for constructing Hajós graphs. Therefore the number of Hajós graphs corresponding to an edge incident at v_{1i} is 2. Since the number of edges incident at v_{1i} is 1, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is 2. As there exist 1 end vertex of the path that does not incident with the bridge in G_1* and 1 end vertex of the path that is incident with the bridge in G_2*, the total number of Hajós graphs obtained by identifying v_{1i} with u_{1k} is 2.

Case (ii): v_{1i} is an end vertex is identified with u_{1k} is an internal vertex

Subcase(ii.a): v_{1i} is an end vertex incident with the bridge connecting K_{n_1} is identified with u_{1k} is an internal vertex
For each edge incident at v_{1i}, there are 2 choices of edges incident at u_{1k}. Therefore, the number of Hajós graph corresponding to an edge incident at v_{1i} is 2. Since the number of edges incident at v_{1i} is 2, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is 4. As there are 1 end vertex in the path of G_1* that incident with the bridge and (m_2-2) internal vertices in the path of G_2*, the total number of Hajós graphs obtained by identifying v_{1i}, with u_{1k} is $4(m_2-2)$.

Subcase(ii.b): v_{1i} is an end vertex that does not incident with the bridge connecting K_{n_1} is identified with u_{1k} is an internal vertex
For each edge incident at v_{1i}, there are 2 choices of edges incident at u_{1k}. Therefore, the number of Hajós graph corresponding to an edge incident at v_{1i} is 2. Since the number of edges incident at v_{1i} is 1, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is 2. As there are 1 end vertex in the path of G_1* that incident with the bridge and (m_2-2) internal vertices in the path of G_2*, the total number of Hajós graphs obtained by identifying v_{1i}, with u_{1k} is $2(m_2-2)$.

Case (iii): v_{1i} is an internal vertex identified with u_{1k} is an end vertex.
Subcase (iii.a): v_{1i} is an internal vertex identified with u_{1k} is an end vertex, u_{1k} with a bridge connecting K_{n_2}.
For each edge incident at v_{1i}, there are 2 choices of edges incident at u_{1k}. Therefore, the number of Hajós graph corresponding to an edge incident at v_{1i} is 2. Since 2 edges are incident at v_{1i}, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is 4. As there are $(m_1 - 2)$ internal vertices in the path of $G_1 *$ and 1 end vertex with a bridge connecting K_{n_2} in $G_2 *$, the total number of Hajós graphs obtained by identifying v_{1i} with u_{1k} is $4(m_1 - 2)$.

Subcase (iii.b): v_{1i} is an internal vertex identified with u_{1k} is an end vertex, u_{1k} without a bridge connecting K_{n_2}.
For each edge incident at v_{1i}, there exist only one edge incident at u_{1k}. Therefore, the number of Hajós graph corresponding to an edge incident at v_{1i} is 1. Since 2 edges are incident at v_{1i}, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is 2. As there are $(m_1 - 2)$ internal vertices in the path of $G_1 *$ and and 1 end vertex without a bridge connecting K_{n_2} in $G_2 *$, the total number of Hajós graphs obtained by identifying v_{1i} with u_{1k} is $2(m_1 - 2)$.

Case (iv): v_{1i} is identified with u_{1k}, where both are internal vertices
For each edge incident at v_{1i}, there are 2 choices of edges incident at u_{1k}. Therefore the number of Hajós graph corresponding to an edge incident at v_{1i} is 2 . Since the number of edges incident at v_{1i} is 2, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is 4. As there are $(m_1 - 2)$ and $(m_2 - 2)$ vertices in the internal vertices of the paths of G_1* and $G_2 *$, the total number of Hajós graphs obtained by identifying v_{1i} with u_{1k} is $4(m_1 - 2)(m_2 - 2)$.

Case (v): v_{1i} is an end vertex is identified with u_{2l}
Subcase (v.a): When v_{1i} is an end vertex with a bridge connecting K_{n_1} is identified with u_{2l}, *where* u_{2l} with a bridge connecting P_{m_2}.
For each edge incident at v_{1i}, there are n_2 choice of the edges incident at u_{2l}. Therefore the number of Hajós graph corresponding to an edge incident at v_{1i} is n_2. Since the number of edges incident at v_{1i} is 2, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is $2n_2$. As there exist 1 end vertex that incident with the bridge *in* $G_1 *$ and 1 vertex of K_{n_2} in $G_2 *$, with a bridge connecting P_{m_2}, the total number of Hajós graphs obtained by identifying v_{1i} with u_{2l} is $2n_2$.

Subcase (v.b): v_{1i} is an end vertex with a bridge connecting K_{n_1} is identified with u_{2l}, *where* u_{2l} without a bridge connecting P_{m_2}.
For each edge incident at v_{1i}, there are $(n_2 - 1)$ choice of the edges incident at u_{2l}. Therefore the number of Hajós graph corresponding to an edge incident at v_{1i} is $(n_2 - 1)$. Since the number of edges incident at v_{1i} is 2, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is $2(n_2 - 1)$. As there exist 1 end vertex that is incident

with the bridge $in\ G_1*$ and (n_2-1) vertices in K_{n_2} of G_2* without a bridge connecting P_{m_2}, the total number of Hajós graphs obtained by identifying v_{1i} with u_{2l} is $2(n_2-1)^2$.

Subcase (v.c): When v_{1i} is an end vertex that does not incident with bridge connecting K_{n_1} is identified with u_{2l}, $where\ u_{2l}$ with a bridge connecting P_{m_2}.
For each edge incident at v_{1i}, there are n_2 choice of the edges incident at u_{2l}. Therefore the number of Hajós graph corresponding to an edge incident at v_{1i} is n_2. Since the number of edges incident at v_{1i} is 1, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is n_2. As there exist 1 end vertex that does not incident with the bridge in G_1* and 1 vertex of K_{n_2} in G_2*, with a bridge connecting P_{m_2}, the total number of Hajós graphs obtained by identifying v_{1i} with u_{2l} is n_2.

Subcase (v.d): v_{1i} is an end vertex without a bridge connecting K_{n_1} is identified with u_{2l}, $where\ u_{2l}$ without a bridge connecting P_{m_2}.
For each edge incident at v_{1i}, there are (n_2-1) choice of the edges incident at u_{2l}. Therefore the number of Hajós graph corresponding to an edge incident at v_{1i} is (n_2-1). Since the number of edges incident at v_{1i} is 1, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is (n_2-1). As there exist 1 end vertex that does not incident with the bridge $in\ G_1*$ and (n_2-1) vertices in K_{n_2} of G_2* without a bridge connecting P_{m_2}, the total number of Hajós graphs obtained by identifying v_{1i} with u_{2l} is $(n_2-1)^2$.

Case (vi): v_{1i} is an internal vertex is identified with u_{2l}
Subcase (vi.a): v_{1i} is an internal vertex is identified with u_{2l}, $where\ u_{2l}$ with a bridge connecting P_{m_2}.
For each edge incident at v_{1i}, there are n_2 choices of edges incident at u_{2l}. Therefore the number of Hajós graphs corresponding to an edge incident at v_{1i} is n_2. The number of edges incident at v_{1i} is 2. Therefore the number of Hajós graphs corresponding to all the edges incident at v_{1i} is $2n_2$. As there are (m_1-2) internal vertices $in\ G_1*$ and 1 vertex with a bridge connecting P_{m_2} in K_{n_2} of G_2*, the total number of Hajós graphs obtained by identifying v_{2i} with u_{1k} is $2(m_1-2)n_2$.

Subcase (vi.b): v_{1i} is an internal vertex is identified with u_{2l}, $where\ u_{2l}$ without a bridge connecting P_{m_2}.
For each edge incident at v_{1i}, there are (n_2-1) choices of edges incident at u_{2l}. Therefore the number of Hajós graphs corresponding to an edge incident at v_{1i} is (n_2-1). The number of edges incident at v_{1i} is 2. Therefore the number of Hajós graphs corresponding to all the edges incident at v_{1i} is $2(n_2-1)$. As there are (m_1-2) internal vertices $in\ G_1*$ and (n_2-1) vertices in K_{n_2} of G_2* without a bridge connecting P_{m_2}, the total number of Hajós graphs obtained by identifying v_{2i} with u_{1k} is $2(m_1-2)(n_2-1)^2$.

Case (vii): v_{2j} is identified with u_{2l}
Subcase (vii.a): v_{2j} a vertex that is incident with a bridge connecting P_{m_1} is identified with u_{2l}, $where\ u_{2l}$ with a bridge connecting P_{m_2}.

For each edge incident at v_{2j}, there are n_2 choices of the edges incident at u_{2l}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is n_2. Since n_1 edges are incident at v_{2j}, the number of Hajós graphs corresponding to all the edges incident at v_{2j} is $n_1 n_2$. As there exists 1 vertex in K_{n_1} of $G_1 *$ that incident with the bridge connecting P_{m_1} and 1 vertex of K_{n_2} with a bridge connecting P_{n_1} of $G_2 *$, the total number of Hajós graphs obtained by identifying v_{2j} with u_{2l} is $n_1 n_2$.

Subcase (vii.b): v_{2j} a vertex without a bridge connecting P_{m_1} is identified with u_{2l}, $where\ u_{2l}$ without a bridge connecting P_{m_2}.
For each edge incident at v_{2j}, there are $(n_2 - 1)$ choices of the edges incident at u_{2l}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is $(n_2 - 1)$. Since $(n_1 - 1)$ edges are incident at v_{2j}, the number of Hajós graphs corresponding to all the edges incident at v_{2j} is $(n_1 - 1)(n_2 - 1)$. As there exist $(n_1 - 1)$ vertices with the bridge and $(n_2 - 1)$ vertices in $K_{n_1}$and K_{n_2} of $G_1 *$ and $G_2 *$ respectively, the total number of Hajós graphs obtained by identifying v_{2j} with u_{2l} is $(n_1 - 1)^2 (n_2 - 1)^2$.

Subcase (vii.c): v_{2j} a vertex that is incident with a bridge connecting P_{m_1} is identified with u_{2l}, $where\ u_{2l}$ without a bridge connecting P_{m_2}.
For each edge incident at v_{2j}, there are $(n_2 - 1)$ choices of the edges incident at u_{2l}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is $(n_2 - 1)$. Since n_1 edges are incident at v_{2j}, the number of Hajós graphs corresponding to all the edges incident at v_{2j} is $n_1(n_2 - 1)$. As there exists 1 vertex in K_{n_1} of $G_1 *$ that incident with the bridge connecting P_{m_1} and $(n_2 - 1)$ vertices in K_{n_2} without a bridge connecting P_{n_1} of $G_2 *$, the total number of Hajós graphs obtained by identifying v_{2j} with u_{2l} is $n_1(n_2 - 1)^2$.

Subcase (vii.d): v_{2j} a vertex that is incident without a bridge connecting P_{m_1} is identified with u_{2l}, $where\ u_{2l}$ with a bridge connecting P_{m_2}.
For each edge incident at v_{2j}, there are n_2 choices of the edges incident at u_{2l}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is n_2. Since $(n_1 - 1)$ edges are incident at v_{2j}, the number of Hajós graphs corresponding to all the edges incident at v_{2j} is $n_2(n_1 - 1)$. As there exists $(n_1 - 1)$ vertices in K_{n_1} of $G_1 *$ that does not incident with the bridge connecting P_{m_1} and 1 vertex in K_{n_2} with a bridge connecting P_{n_2} of $G_2 *$, the total number of Hajós graphs obtained by identifying v_{2j} with u_{2l} is $n_2(n_1 - 1)^2$.

Case (viii): v_{2j} is identified with u_{1k}, u_{1k} is an internal vertex
Subcase (viii.a): v_{2j} a vertex that is incident with a bridge connecting P_{m_1} is identified with u_{1k}, u_{1k} is an internal vertex
For each edge incident at v_{2j}, there are 2 choice of edges incident at u_{1k}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is 2. Since the number of

edges incident at v_{2j} is n_1, the number of Hajós graphs corresponding to all the edges incident at v_{2j} is $2n_1$. As there exist 1 vertex in K_{n_1} with a bridge connecting P_{n_1} of G_1* and (m_2-2) vertices in the internal vertices of G_{2*}, the total number of Hajós graphs obtained by identifying v_{2j} with u_{1k} is $2n_1(m_2-2)$.

Subcase (viii.b): v_{2j} a vertex that does not incident with a bridge connecting P_{m_1} is identified with u_{1k}, u_{1k} is an internal vertex
For each edge incident at v_{2j}, there are 2 choice of edges incident at u_{1k}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is 2. Since the number of edges incident at v_{2j} is (n_1-1), the number of Hajós graphs corresponding to all the edges incident at v_{2j} is $2(n_1-1)$. As there are (n_1-1) vertices in K_{n_1} with a bridge connecting P_{n_1} of G_1* and (m_2-2) vertices in the internal vertices of G_{2*}, the total number of Hajós graphs obtained by identifying v_{2j} with u_{1k} is $2(m_2-2)(n_1-1)^2$.

Case (ix): v_{2j} is identified with u_{1k} is an end vertex
Subcase (ix.a): v_{2j} is a vertex that incident with a bridge connecting $P_{m_1}$is identified with u_{1k} is an end vertex, where u_{1k} with a bridge connecting K_{n_2}.
For each edge incident at v_{2j}, there are 2 choices of edges incident at u_{1k}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is 2. The number of edges incident at v_{2j} is n_1. Therefore the number of Hajós graphs corresponding to all the edges incident at v_{2j} is $2n_1$. As there exist 1 vertex *in* $K_{n_1}$of G_1* and 1 end vertex in $P_{m_2}$of G_2* without a bridge connecting K_{n_2}, the total number of Hajós graphs obtained by identifying v_{2j} with u_{1k} is $2n_1$.

Subcase (ix.b): v_{2j} is a vertex incident with a bridge connecting P_{m_1} is identified with u_{1k} is an end vertex, where u_{1k} without a bridge connecting K_{n_2}.
For each edge incident at v_{2j}, there exist only one edge incident at u_{1k}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is 1. The number of edges incident at v_{2j} is n_1. Therefore the number of Hajós graphs corresponding to all the edges incident at v_{2j} is n_1. As there exist only one vertex in that incident to the bridge with $K_{n_1}$of G_1* and 1 end vertex in $P_{m_2}$of G_2* without a bridge connecting K_{n_2}, the total number of Hajós graphs obtained by identifying v_{2j} with u_{1k} is n_1.

Subcase (ix.c): v_{2j} is a vertex that incident without a bridge connecting $P_{m_1}$is identified with u_{1k} is an end vertex, where u_{1k} with a bridge connecting K_{n_2}.
For each edge incident at v_{2j}, there are 2 choices of edges incident at u_{1k}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is 2. The number of edges incident at v_{2j} is (n_1-1). Therefore the number of Hajós graphs corresponding to all the edges incident at v_{2j} is $2(n_1-1)$. As there exist (n_1-1) vertices *in* $K_{n_1}$of G_1* and 1 end vertex in $P_{m_2}$of G_2* with a bridge connecting K_{n_2}, the total number of Hajós graphs obtained by identifying v_{2j} with u_{1k} is $2(n_1-1)^2$.

Subcase (ix.d): v_{2j} is a vertex that incident without a bridge connecting P_{m_1} is identified with u_{1k} is an end vertex, where u_{1k} without a bridge connecting K_{n_2}.
For each edge incident at v_{2j}, there exist only one edge incident at u_{1k}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is 1. The number of edges incident at v_{2j} is $(n_1 - 1)$. Therefore the number of Hajós graphs corresponding to all the edges incident at v_{2j} is $(n_1 - 1)$. As there exist $(n_1 - 1)$ vertices *in* K_{n_1} of G_1* and 1 end vertex in P_{m_2} of G_2* without a bridge connecting K_{n_2}, the total number of Hajós graphs obtained by identifying v_{2j} with u_{1k} is $(n_1 - 1)^2$.
From the above nine cases, the total number of Hajós graph from G1* and G2* is $(n_1(n_1 - 1) + 2m_1)(n_2(n_2 - 1) + 2m_2)$.

Theorem 3.5. Let $G_i(V_i, E_i)$, i = 1, 2 be two fuzzy Graphs on Lollipop Graphs $L_{n_1,\ m_1}$ and $L_{n_2,\ m_2}$ respectively. Then the number of Hajós graphs that can be obtained from G_i, i = 1, 2 is $(n_1(n_1 - 1) + 2m_1)(n_2(n_2 - 1) + 2m_2)$.

Theorem 3.6. Let $G_1*(V_1, E_1)$ be two Tadpole graphs $T_{n_1,\ m_1}$ and $T_{n_2,\ m_2}$ respectively. Then the number of Hajós graphs that can be obtained from G_i*, i = 1, 2 is $4(n_1 + m_1)(n_2 + m_2)$.
Proof: Let $V_1 = \{v_{11}, v_{12}, \cdots, v_{1m_1}, v_{21}, v_{22}, \cdots, v_{2n_1}\}$ and $V_2 = \{u_{11}, u_{12}, \cdots, u_{1m_2}, u_{21}, u_{22}, \cdots, u_{2n_2}\}$ be the vertex sets of G1* and G2* respectively, where $v_{11}v_{12} \cdots v_{1m_1}$ and $u_{11}u_{12} \cdots u_{1m_2}$ are the paths P_{m_1} and P_{m_2} of the G1* and G2* respectively. Let the remaining vertices $v_{21}, v_{22}, \cdots, v_{2n_1}$ of G1* and $u_{21}, u_{22}, \cdots, u_{2n_2}$ G2* be the vertices of C_{n_1} and C_{n_2} respectively. Let $i \in \{1, 2, \cdots, m_1\}$, $j \in \{1, 2, \cdots, n_1\}$, $k \in \{1, 2, \cdots, m_2\}$ and $l \in \{1, 2, \cdots, n_2\}$.
For each identification of vertices in the construction of Hajós graphs, there exist different cases as follows:

(i) v_{1i} is identified with u_{1k},where both are end vertices (ii) v_{1i} is an end vertex is identified with u_{1k} is an internal vertex (iii) v_{1i} is an internal vertex identified with u_{1k} is an end vertex, (iv) v_{1i} is identified with u_{1k}, where both are internal vertices (v) v_{2j} is identified with u_{2l}, (vi) v_{2j} is identified with u_{1k} is an internal vertex, (vii) v_{2j} is identified with u_{1k} is an end vertex, (viii) v_{1i} is an end vertex is identified with u_{2l}, (ix) v_{1i} is an internal vertex is identified with u_{2l}.

For the four cases, **(i)** v_{1i} is identified with u_{1k},where both are end vertices **(ii)** v_{1i} is an end vertex is identified with u_{1k} is an internal vertex **(iii)** v_{1i} is an internal vertex identified with u_{1k} is an end vertex, **(iv)** v_{1i} is identified with u_{1k}, where both are internal vertices, the proof follows from the theorem 3.5.
Case (v): v_{1i} is an end vertex is identified with u_{2l}

Subcase (v.a): When v_{1i} is an end vertex with a bridge connecting K_{n_1} is identified with u_{2l}, *where* u_{2l} with a bridge connecting P_{m_2}.
For each edge incident at v_{1i}, there are 3 choice of the edges incident at u_{2l}. Therefore the number of Hajós graph corresponding to an edge incident at v_{1i} is 3. Since the number of edges incident at v_{1i} is 2, the number of Hajós graphs corresponding to all the edges

incident at v_{1i} is 6. As there exist 1 end vertex that incident with the bridge *in* $G_1 *$ and 1 vertex of C_{n_2} in $G_2 *$, with a bridge connecting P_{m_2}, the total number of Hajós graphs obtained by identifying v_{1i} with u_{2l} is 6.

Subcase (v.b): v_{1i} is an end vertex with a bridge connecting K_{n_1} is identified with u_{2l}, $where\ u_{2l}$ without a bridge connecting P_{m_2}.

For each edge incident at v_{1i}, there are 2 choice of the edges incident at u_{2l}. Therefore the number of Hajós graph corresponding to an edge incident at v_{1i} is 2 Since the number of edges incident at v_{1i} is 2, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is 4. As there exist 1 end vertex that is incident with the bridge *in* $G_1 *$ and $(n_2 - 1)$ vertices in C_{n_2} of G_2* without a bridge connecting P_{m_2}, the total number of Hajós graphs obtained by identifying v_{1i} with u_{2k} is $4(n_2 - 1)$.

Subcase (v.c): When v_{1i} is an end vertex that does not incident with bridge connecting K_{n_1} is identified with u_{2l}, $where\ u_{2l}$ with a bridge connecting P_{m_2}.

For each edge incident at v_{1i}, there are 3 choice of the edges incident at u_{2l}. Therefore the number of Hajós graph corresponding to an edge incident at v_{1i} is 3. Since the number of edges incident at v_{1i} is 1, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is 3. As there exist 1 end vertex that does not incident with the bridge *in* $G_1 *$ and 1 vertex of C_{n_2} in $G_2 *$, with a bridge connecting P_{m_2}, the total number of Hajós graphs obtained by identifying v_{1i} with u_{2l} is 3.

Subcase (v.d): v_{1i} is an end vertex without a bridge connecting K_{n_1} is identified with u_{2l}, $where\ u_{2l}$ without a bridge connecting P_{m_2}.

For each edge incident at v_{1i}, there are 2 choice of the edges incident at u_{2l}. Therefore the number of Hajós graph corresponding to an edge incident at v_{1i} is 2 Since the number of edges incident at v_{1i} is 1, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is 2. As there exist 1 end vertex that does not incident with the bridge *in* $G_1 *$ and $(n_2 - 1)$ vertices in C_{n_2} of G_2* without a bridge connecting P_{m_2}, the total number of Hajós graphs obtained by identifying v_{1i} with u_{2k} is $2(n_2 - 1)$.

Case (vi): v_{1i} is an internal vertex is identified with u_{2l}

Subcase (vi.a): v_{1i} is an internal vertex is identified with u_{2l}, $where\ u_{2l}$ with a bridge connecting P_{m_2}.

For each edge incident at v_{1i}, there are 3 choices of edges incident at u_{2l}. Therefore the number of Hajós graphs corresponding to an edge incident at v_{1i} is 3. The number of edges incident at v_{1i} is 2. Therefore the number of Hajós graphs corresponding to all the edges incident at v_{1i} is 6. As there are $(m_1 - 2)$ internal vertices *in* $G_1 *$ and 1 vertex with a bridge connecting P_{m_2} in C_{n_2} of G_2*, the total number of Hajós graphs obtained by identifying v_{2i} with u_{1k} is $6(m_1 - 2)$.

Subcase (vi.b): v_{1i} is an internal vertex is identified with u_{2l}, $where\ u_{2l}$ without a bridge connecting P_{m_2}.

For each edge incident at v_{1i}, there are 2 choices of edges incident at u_{2l}. Therefore the number of Hajós graphs corresponding to an edge incident at v_{1i} is 2. The number of edges incident at v_{1i} is 2. Therefore the number of Hajós graphs corresponding to all the edges incident at v_{1i} is 4. As there are $(m_1 - 2)$ internal vertices *in* $G_1 *$ and (n pendant vertices in G_2* without a bridge connecting P_{m_2}, the total number of Hajós graphs obtained by identifying v_{2i} with u_{1k} is $4(m_1 - 2)(n_2 - 1)$.

Case (vii): v_{2j} is identified with u_{2l}

Subcase (vii.a): v_{2j} a vertex that is incident with a bridge connecting P_{m_1} is identified with u_{2l}, *where* u_{2l} with a bridge connecting P_{m_2}.

For each edge incident at v_{2j}, there are 3 choices of the edges incident at u_{2l}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is 3. Since 3 edges are incident at v_{2j}, the number of Hajós graphs corresponding to all the edges incident at v_{2j} is 9. As there exists 1 vertex in K_{n_1} of $G_1 *$ that incident with the bridge connecting P_{m_1} and 1 vertex of C_{n_2} with a bridge connecting P_{n_1} of $G_2 *$, the total number of Hajós graphs obtained by identifying v_{2j} with u_{2l} is 9.

Subcase (vii.b): v_{2j} a vertex without a bridge connecting P_{m_1} is identified with u_{2l}, *where* u_{2l} without a bridge connecting P_{m_2}.

For each edge incident at v_{2j}, there are 2 choices of the edges incident at u_{2l}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is 2. Since 2 edges are incident at v_{2j}, the number of Hajós graphs corresponding to all the edges incident at v_{2j} is 4. As there exist $(n_1 - 1)$ vertices with the bridge in K_{n_1} of $G_1 *$ and $(n_2 - 1)$ vertices in C_{n_2} of $G_2 *$ respectively, the total number of Hajós graphs obtained by identifying v_{2j} with u_{2l} is $4(n_1 - 1)(n_2 - 1)$.

Subcase (vii.c): v_{2j} a vertex that is incident with a bridge connecting P_{m_1} is identified with u_{2l}, *where* u_{2l} without a bridge connecting P_{m_2}.

For each edge incident at v_{2j}, there are 2 choices of the edges incident at u_{2l}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is 2. Since 3 edges are incident at v_{2j}, the number of Hajós graphs corresponding to all the edges incident at v_{2j} is 6. As there exists 1 vertex in C_{n_1} of $G_1 *$ that incident with the bridge connecting P_{m_1} and $(n_2 - 1)$ vertices in C_{n_2} without a bridge connecting P_{n_1} of $G_2 *$, the total number of Hajós graphs obtained by identifying v_{2j} with u_{2l} is $6(n_2 - 1)$.

Subcase (vii.d): v_{2j} a vertex that is incident without a bridge connecting P_{m_1} is identified with u_{2l}, *where* u_{2l} with a bridge connecting P_{m_2}.

For each edge incident at v_{2j}, there are 3 choices of the edges incident at u_{2l}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is 3. Since 2 edges are incident at v_{2j}, the number of Hajós graphs corresponding to all the edges incident at v_{2j} is 6. As there exists $(n_1 - 1)$ vertices in C_{n_1} of $G_1 *$ that does not incident with the bridge connecting P_{m_1} and 1 vertex in C_{n_2} with a bridge connecting P_{n_2} of $G_2 *$, the total number of Hajós graphs obtained by identifying v_{2j} with u_{2l} is $6(n_1 - 1)$.

Case (viii): v_{2j} is identified with u_{1k}, u_{1k} is an internal vertex

Subcase (viii.a): v_{2j} a vertex that is incident with a bridge connecting P_{m_1} is identified with u_{1k}, u_{1k} is an internal vertex

For each edge incident at v_{2j}, there are 2 choice of edges incident at u_{1k}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is 2. Since the number of edges incident at v_{2j} is 3, the number of Hajós graphs corresponding to all the edges incident at v_{2j} is 6. As there exist 1 vertex in C_{n_1} with a bridge connecting P_{n_1} of $G_1 *$ and $(m_2 - 2)$ internal vertices in G_{2^*}, the total number of Hajós graphs obtained by identifying v_{2j} with u_{1k} is $6(m_2 - 2)$.

Subcase (viii.b): v_{2j} a vertex that does not incident with a bridge connecting P_{m_1} is identified with u_{1k}, u_{1k} is an internal vertex

For each edge incident at v_{2j}, there are 2 choice of edges incident at u_{1k}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is 2. Since the number of edges incident at v_{2j} is 2, the number of Hajós graphs corresponding to all the edges incident at v_{2j} is 4. As there are $(n_1 - 1)$ vertices in C_{n_1} with a bridge connecting P_{m_1} of $G_1 *$ and $(m_2 - 2)$ internal vertices in G_{2^*}, the total number of Hajós graphs obtained by identifying v_{2j} with u_{1k} is $4(m_2 - 2)(n_1 - 1)$.

Case (ix): v_{2j} is identified with u_{1k} is an end vertex

Subcase (ix.a): v_{2j} is a vertex that incident with a bridge connecting $P_{m_1}$is identified with u_{1k} is an end vertex, where u_{1k} with a bridge connecting C_{n_2}.

For each edge incident at v_{2j}, there are 2 choices of edges incident at u_{1k}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is 2. The number of edges incident at v_{2j} is 3. Therefore the number of Hajós graphs corresponding to all the edges incident at v_{2j} is 6. As there exist 1 vertex *in* $C_{n_1}$of $G_1 *$ and 1 end vertex in $P_{m_2}$of $G_2 *$ does not incident at the bridge connecting C_{n_2}, the total number of Hajós graphs obtained by identifying v_{2j} with u_{1k} is 6.

Subcase (ix.b): v_{2j} is a vertex incident with a bridge connecting P_{m_1} is identified with u_{1k} is an end vertex, where u_{1k} without a bridge connecting C_{n_2}.

For each edge incident at v_{2j}, there exist only one edge incident at u_{1k}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is 1. The number of edges incident at v_{2j} is 3. Therefore the number of Hajós graphs corresponding to all the edges incident at v_{2j} is 3. As there exist only one vertex in that incident to the bridge with $C_{n_1}$of $G_1 *$ and 1 end vertex in $P_{m_2}$of $G_2 *$ without a bridge connecting C_{n_2}, the total number of Hajós graphs obtained by identifying v_{2j} with u_{1k} is 3.

Subcase (ix.c): v_{2j} is a vertex that incident without a bridge connecting $P_{m_1}$is identified with u_{1k} is an end vertex, where u_{1k} with a bridge connecting C_{n_2}.

For each edge incident at v_{2j}, there are 2 choices of edges incident at u_{1k}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is 2. The number of edges incident at v_{2j} is 2. Therefore the number of Hajós graphs corresponding to all the edges incident at v_{2j} is 4. As there exist (n_1-1) vertices *in* $C_{n_1}$of G_1* and 1 end vertex in $P_{m_2}$of G_2* with a bridge connecting K_{n_2}, the total number of Hajós graphs obtained by identifying v_{2j} with u_{1k} is $4(n_1-1)$.

Subcase (ix.d): v_{2j} is a vertex that incident without a bridge connecting $P_{m_1}$is identified with u_{1k} is an end vertex, where u_{1k} without a bridge connecting C_{n_2}.
For each edge incident at v_{2j}, there exist only one edge incident at u_{1k}. Therefore the number of Hajós graph corresponding to an edge incident at v_{2j} is 1. The number of edges incident at v_{2j} is 2. Therefore the number of Hajós graphs corresponding to all the edges incident at v_{2j} is 2. As there exist (n_1-1) vertices *in* $C_{n_1}$of G_1* and 1 end vertex in $P_{m_2}$of G_2* without a bridge connecting C_{n_2}, the total number of Hajós graphs obtained by identifying v_{2j} with u_{1k} is $2(n_1-1)$.

From the above nine cases, the total number of Hajós graph from G$_1$* and G$_2$* is $4(n_1+m_1)(n_2+m_2)$.

Theorem 3.7. Let G$_i$(V$_i$, E$_i$), i = 1, 2 be two fuzzy graphs on Tadpole graphs $T_{n_1,\ m_1}$ *and* $T_{n_2,\ m_2}$ respectively. Then the number of Hajós graphs that can be obtained from G$_i$, i = 1, 2 is $4(n_1+m_1)(n_2+m_2)$.

Theorem 3.8. Let G$_i$*(V$_i$, E$_i$), i = 1, 2 be two Friendship graphs F_{n_i} respectively. Then the number of Hajós graphs that can be obtained from G$_i$*, i = 1, 2 is $12n_2\ (n_1\ +m_1)$.
Proof: Let V$_1$ = $\{v, v_1, v_2, \cdots, v_{2n_1}\}$ and V$_2$ = $\{u, u_1, u_2, \cdots, u_{2n_2}\}$ be the vertex sets of G$_1$* and G$_2$* respectively, where v and u are the central vertices of G$_1$* and G$_2$* respectively. Let the remaining vertices $v_1, v_2, \cdots, v_{2n_1}$ and $u_1, u_2, \cdots, u_{2n_2}$ be the non-central vertices of F_{n_1} & F_{n_2} respectively. Let $i \in \{1, 2, \cdots, 2n_1\}$, and $l \in \{1, 2, \cdots, 2n_2\}$.
For each identification of vertices in the construction of Hajós graphs, there exist different cases as follows:

(i) v is identified with u (ii) v is identified with u_l (iii) v_i identified with u (iv) v_i is identified with u_l

Case (i): v is identified with u.
For each edge incident at v, there are $2n_2$ choices of edges incident at u for constructing Hajós graphs. Therefore the number of Hajós graphs corresponding to an edge incident at v is $2n_2$. Since the number of edges incident at v is $2n_1$, the number of Hajós graphs corresponding to all the edges incident at v is $4n_1n_2$. As there exist 1 central vertex in G_1* and 1 central vertex in G_2*, the total number of Hajós graphs obtained by identifying v with u is $4n_1n_2$.

Case (ii): v is identified with u_l
For each edge incident at v, there are 2 choices of edges incident at u_l for constructing Hajós graphs. Therefore the number of Hajós graphs corresponding to an edge incident at

v_{1i} is 2. Since the number of edges incident at v is $2n_1$, the number of Hajós graphs corresponding to all the edges incident at v is $4n_1$. As there exist 1 central vertex in G_1* and $2n_2$ non-central vertices in G_2*, the total number of Hajós graphs obtained by identifying v_{1i} with u is $8n_1n_2$.

Case (iii): v_i is identified with u

For each edge incident at v_i, there are $2n_2$ choices of edges incident at u for constructing Hajós graphs. Therefore the number of Hajós graphs corresponding to an edge incident at v_i is $2n_2$. Since the number of edges incident at v_i is 2, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is $4n_2$. As there exist $2n_1$ non-central vertices in G_1* and 1 central vertex in G_2*, the total number of Hajós graphs obtained by identifying v_{1i} with u_{1k} is $8n_1n_2$.

Case (iv): v_i is identified with u_l

For each edge incident at v_i, there are 2 choices of edges incident at u_l for constructing Hajós graphs. Therefore the number of Hajós graphs corresponding to an edge incident at v_{1i} is 2. Since the number of edges incident at v_i is 2, the number of Hajós graphs corresponding to all the edges incident at v_i is 4. As there exist $2n_1$ non-central vertices in G_1* and $2n_2$ non-central vertices in G_2*, the total number of Hajós graphs obtained by identifying v_{1i} with u_l is $16n_1n_2$.

From the above six cases, the total number of Hajós graph from G_1* and G_2* is $36n_1n_2$.

Theorem 3.9. Let $G_i(V_i, E_i)$, i = 1, 2 be two fuzzy graphs on Friendship graphs F_{n_1} and F_{n_2}. Then the number of Hajós fuzzy graphs that can be obtained from G_i, i = 1, 2 is $36n_1n_2$.

Theorem 3.10. Let $G_i*(V_i, E_i)$ be two Crown Graphs H_{n_1,n_1} and H_{n_2,n_2} respectively. Then the number of Hajós graphs that can be obtained from G_i*, i = 1, 2 is $4n_1n_2(n_1 - 1)(n_2 - 1)$.

Proof: Let $V_1 = \{v_{11}, v_{12}, \cdots, v_{1n_2}, v_{21}, v_{22}, \cdots, v_{2n_2}\}$ and $V_2 = \{u_{11}, u_{12}, \cdots, u_{1n_2}, u_{21}, u_{22}, \cdots, u_{2n_2}\}$ be the vertex sets of G_1* and G_2* respectively, where v is the central vertex of the G_1* and the remaining vertices $v_1, v_2, \cdots, v_{2n_1}$ of G_1* be the vertices of K_{n_1}. Let $V_{21} = \{u_{11}, u_{12}, \cdots, u_{1n_2}\}$ and $V_{22} = \{u_{21}, u_{22}, \cdots, u_{2n_2}\}$ be the two partitions of V_2 respectively. Let $i \in \{1, 2, \cdots, n_1\}$, and $l \in \{1, 2, \cdots, n_2\}$.

For each identification of vertices in the construction of Hajós graphs, there exist different cases as follows:

(i) v_{1i} is identified with u_{1l} (ii) v_{1i} is identified with u_{2l} (iii) v_{2i} is identified with u_{1l} (iv) v_{2i} is identified with u_{2l}

Case (i): v_{1i} is identified with u_{1l}

For each edge incident at u_{1l}, there are $(n_2 - 1)$ choices of edges incident at u_{1l} for constructing Hajós graphs. Therefore the number of Hajós graphs corresponding to an edge incident at v is $(n_2 - 1)$. Since the number of edges incident at v_{1i} is $(n_1 - 1)$, the number of Hajós graphs corresponding to all the edges incident at v_{1i} is $(n_1 - 1)(n_2 - 1)$. As there exist n_1 vertices in V_{11} of G_1* and n_2 vertices in V_{21} of G_2*, the total number of Hajós graphs obtained by identifying v_{1i} with u_{1l} is $n_1n_2(n_1-1)\,(n_2-1)$.

Case (ii): v_{1i} is identified with u_{2l}.
Proof is similar to that of case (i), the total number of Hajos graphs obtained by identifying v with u_{2l} is $n_1n_2(n_1-1)\,(n_2-1)$.

Case (iii): v_{2i} is identified with u_{1l}
For each edge incident at v_{2i} , there are (n_2-1) choices of edges incident at u_{1l} for constructing Hajós graphs. Therefore the number of Hajós graphs corresponding to an edge incident at v_{2i} is (n_2-1). Since the number of edges incident at v_{2i} is (n_1-1), the number of Hajós graphs corresponding to all the edges incident at v_{1i} is $(n_1-1)(n_2-1)$. As there are n_1 vertices in V_{12} of G_1* and n_2 vertices in V_{21} of G_2*, the total number of Hajós graphs obtained by identifying v_{2i} with u_{2l} is $n_1n_2(n_1-1)\,(n_2-1)$.

Case (iv) v_{2i} is identified with u_{2l}
Proof is similar to that of case (iii), the total number of Hajos graphs obtained by identifying v_{2i} with u_{2l} is $n_1n_2(n_1-1)\,(n_2-1)$.
From the above four cases, the total number of Hajós graphs that can be obtained from G_1* and G_2* is $4n_1n_2(n_1-1)\,(n_2-1)$.

Theorem 3.11. Let $G_i(V_i, E_i)$, i = 1, 2 be two fuzzy graphs on Crown graphs H_{n_i,n_i} respectively. Then the number of Hajós fuzzy graphs that can be obtained from G_i, i = 1, 2 is $12n_1n_2\,(n_2-1)$.

4. Conclusion

In this paper, the cardinality of Hajós graphs, on Fan, Lollipop Graph, Tadpole Graph, Friendship Graph and Crown Graph are derived. The cardinality of Hajós fuzzy graphs is same as the cardinality of Hajos graphs. By applying the fact the cardinality of Hajos fuzzy graphs on Fan, Lollipop Graph, Tadpole Graph, Friendship Graph and Crown Graph are also obtained.

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