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Study on Strength of Strongest Dominating Sets in Fuzzy Graphs

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Abstract. A set S of vertices in a graph $G = (V.E)$ is a dominating set of G if every vertex of $V - S$ is adjacent to some vertex of S. A set S of vertices is an exact k-step dominating set if any vertex of G is at distance k from exact one vertex of S. In this paper, at the first we study on the strength of strongest 2-step dominating set in 2-corona fuzzy graphs and then we study on dominating set in fuzzy graphs especially fuzzy paths and fuzzy cycles.Then we introduce the concepts of strength of strongest dominating set in complement of fuzzy graphs.

Keywords: Dominating set, exact 1-step dominating set, strongest dominating set, complement of fuzzy graphs

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1. Introduction

Zadeh's paper developed a theory which proposed making the grade of membership of an element in a subset of a universal set a value in the closed interval of real numbers. Zadeh's ideas have found applications in many areas of science and technology. Theoretical mathematics have also been touched by fuzzy set theory. The ideas of fuzzy set theory have been introduced into topology, abstract algebra, geometry, graph theory, and analysis [17]. Fuzzy graphs were introduced by Rosenfeld, who has described the fuzzy analogue of several graph theoretic concepts like paths, cycles, trees and connectedness [23]. Nagoor Gani and Chandrasekaran defined dominating set and domination number in fuzzy graphs in [20]. Recently, several concepts in intuitionistic fuzzy graphs, picture fuzzy graphs, and cubic fuzzy graphs investigated in [18-22]. In this paper, we study domination in fuzzy graphs. We introduce the concept of strength of strongest dominating set by using membership values of vertices and edges in fuzzy graphs. We present some bounds for the strength of strongest dominating set in fuzzy graphs and then determine the strength of strongest dominating set in some families of fuzzy graphs including complete fuzzy graphs and complete bipartite fuzzy graphs. We also introduce the concept of strength of strongest k-step dominating set in fuzzy graphs, and present various bounds for the strength of strength of strongest 1-step dominating set in fuzzy graphs.

2. Definitions and notations

In this section we introduce some graph theory as well as fuzzy graph theory definitions and notations.

Let $G = (V,E)$ be a graph of order $n(G) = |V|$ and size $m(G) = |E(G)| = |E|$. The degree of a vertex v in G is number of edges that are adjacent to v and denote it by $\deg_G(v) = d_G(v)$. The maximum (minimum) degree among the vertices of G is denoted by $\Delta(G)$ ($\delta(G)$, respectively). A path P_n of G is a sequence $v_0v_1...v_n$ of vertices of G in which v_i is adjacent to v_{i+1} for $i= 1.2.....n - 1$. A subset S of V is called a *dominating set* in G if every vertex in $V\setminus S$ is adjacent to some vertex in S. The *domination number* of G is the minimum cardinality taken over all dominating sets in G and is denoted by $\gamma(G)$, or simply γ [3]. A subset S of V is called a total dominating set in G if every vertex in V is adjacent to some vertex in S. The *total domination number* of G is the minimum cardinality taken over all total dominating sets in G and is denoted by $\gamma_t(G)$, or simply γ_t. The literature on the subject of domination and total domination parameters in graphs have been surveyed and detailed in the books [8,9].

Two vertices u and v in a graph G for which $d(u,v) = k$, are said to k-step dominate each other. The set $N_k(v)$ denotes the set of vertices of G that are k-step dominated by v. Schultz [24] defined a set $S = \{v_1, v_2, ..., v_r\}$ of vertices in a graph G, for some integer $r \leq n$, as a step dominating set for G if there exist nonnegative integers $k_1,...,k_r$ such that the sets $\{N_{k_i}(v_i)\}$ form a partition of V (G). The minimum cardinality of a step dominating set is called the step domination number of G. A set S of vertices of G is called a k-step dominating set if $\cup_{v\in S} N_k(v) = V(G)$. A k-step dominating setSsuch that the sets $N_k(v)$, $v \in S$. are pairwise disjoint. Is called an exact k-step dominating set. If a graph G has an exact k-step dominating set, then G is called an exact k-step domination graph. The concepts of step domination and exact k-step domination were further studied in [4,5,10].

We next introduce fuzzy graph theory definitions and notations.

2.1. Fuzzy graph theory

We use the notations $\vee$ for supremum and $\wedge$ for infimum. A fuzzy subset of S is a mapping $\mu: S \to [0.1]$, where $[0.1]$ denotes the set $\{t \in R : 0 \leq t \leq 1\}$. We purpose μ as assigning to each element $x \in S$. a degree of membership $.0 \leq \mu(x) \leq 1$ [17]. A fuzzy graph $G = (V,\sigma,\mu)$ is a nonempty set V together with a pair of functions $\sigma : V \to [0.1]$ and $\mu : V \times V \to [0.1]$ such that for all $x.y \in V$, we have $\mu(x.y) \leq \sigma(x) \wedge \sigma(y)$. For the sake of notational convenience, we omit V in the sequel and use the notation $G = (\sigma,\mu)$. The underlying graph of fuzzy graph $G = (\sigma,\mu)$, is the graph with vertices and edges of $G = (\sigma,\mu)$ such that $\sigma(x) = 1$, for every vertex x of the fuzzy vertices of $G = (\sigma,\mu)$, and $\mu(x.y) = 1$, for every edge $(x.y)$ of the fuzzy edges of $G = (\sigma,\mu)$ and is denoted by $G^* = (\sigma,\mu)$ or G^*. The order p and size q of a fuzzy graph $G= (\sigma,\mu)$ are defined as $p = \sum_{x\in v} \sigma(x)$and $q = \sum_{xy\in E} u(x.y)$. Let $\sigma: V \to [0.1]$be a fuzzy subset of V . Then the complete fuzzy graph on σ is defined as $G(\sigma.u)$. where $u(x.y) > 0$ for all $x.y \in E$ and is denoted by K_σ. A fuzzy graph $G = (\sigma,\mu)$ is said to be bipartite if the set of vertices V can be partitioned into two nonempty sets V_1 and V_2 such that $\mu(v_1,v_2) = 0$ if $v_1 \in V_1$ and $v_2 \in V_1$ or $v_1 \in V_2$ and $v_2 \in V_2$. Further if $\mu(u.v) > 0$ for all $u \in V_1$ and $v \in V_2$, then G is called a complete bipartite fuzzy graph and is denoted by $K_{\sigma1,\sigma2}$, where

σ_1 and σ_2 are the restrictions of σ to V_1 and V_2, respectively.

Let $G = (\sigma,\mu)$ be a fuzzy graph on V and $S \subseteq f V$. Then the fuzzy cardinality of S is defined to be $\sum_{v \in S} \sigma(v)$ and is denoted by $|S|_f$. For a vertex x the set $N[x] = \{N(x) \cup \{x\}\}$ is the closed neighborhood of x. A vertex u of a fuzzy graph G is said to be an isolated vertex if $\mu(u,v) = 0$ for all $v \in V \setminus \{u\}$, that is, $N(u) = \emptyset$, [17]. Also $\mu(v.S)$ is equal to the minimum membership of edges between v and vertices of the set S. The degree of a vertex v is $\sum\sigma(u)$, where $u \in N(v)$. We denote by $\Delta_f(G)$ and $\delta_f(G)$ the maximum and minimum degree in fuzzy graph $G = (\sigma,\mu)$, respectively. For a vertex v, we define the depth of v as the minimum membership of edges adjacent to v and denote it by $d(v)$. A vertex with maximum membership in a fuzzy graph G is denoted by v_s and a vertex with minimum membership in fuzzy graph G is denoted by v_w. Also an edge with maximum membership is denoted by e_s and an edge by minimum membership is denoted by e_w. The edge e_{uv} is called strong edge if $\mu(e.v) = \wedge[\sigma(u),\sigma(v)]$. The complement of a fuzzy graph $G= (\sigma,\mu)$, denoted by $G^{-} = (\sigma,\mu^{-})$, is $G^{-} = (\sigma,\mu^{-})$, where $\mu^{-}(x.y) = \sigma(x) \wedge \sigma(y) - \mu(x.y)$, for all $x.y \in V$. If edge e_{uv} is strong edge, then $\mu(u.v) = 0$[14].

A path P in a fuzzy graph $G = (\sigma,\mu)$ is a sequence of distinct vertices $x_0.x_1.....x_n$ (except possibly x_0 and x_n) such that $\mu(x_{i-1}.x_i)>0.1 \le i \le n$. Here $n \ge 1$ is called the length of the path P. The consecutive pairs $(x_{i-1}.x_i)$ are called the edges of the path. The strength of the path of length k from x_0 to x_k, is defined as $\wedge_{i=1}^{k}\mu(x_{i-1},x_i)$ and is denoted by $\mu^k(x_0,x_k)$. In other words, the strength of a path is defined to be the weight of the weakest edge of the path.

A single vertex x may also be considered as a path. In this case, the path is of length 0. If a path has length 0, it is convenient to define its strength to be $\mu(x_0)$. It may be noted that any path of length $n > 0$ can be defined as a sequence of edges $(x_{i-1}.x_i)$, $1 \le i \le n$, satisfying the condition $\mu(x_{i-1}.x_i) > 0$ for $1 \le i \le n$, [17].

A vertex x in a fuzzy graph $G = (\sigma,\mu)$, dominates a vertex y if $\mu(x.y) > 0$. A subset S of vertices is called a *dominating set* in $G = (\sigma,\mu)$ if for every vertex $v \not\ni s$, there exists a vertex $u \in S$ such that u dominates v. The minimum fuzzy cardinality of a dominating set in $G = (\sigma,\mu)$ is called the *domination number* of fuzzy graph $G = (\sigma,\mu)$ and is denoted by $\gamma_f(G)$ or γ_f. A subset S of vertices is called a *total dominating set* in $G = (\sigma,\mu)$ if for every vertex v of $G = (\sigma,\mu)$, there exists $u \in S$ such that u dominates v. The minimum fuzzy cardinality of a total dominating set in $G = (\sigma,\mu)$ is called the *total domination number* of fuzzy graph $G = (\sigma,\mu)$ and is denoted by $\gamma_{tf}(G)$ or γ_{tf}. Domination in fuzzy graphs are studied in, for example [15,16,18,19,20].

Let $G = (\sigma,\mu)$ be a fuzzy graph. Analogue to k-step domination in graphs we can define k-step domination in fuzzy graphs as follows. A set $S \subseteq V(G)$ is a *k-step dominating set* of G if for every vertex v there exists at least one path of length k between v and the vertices of S, that is, there exists at least one vertex u of S such that $\mu^k(v,u) > 0$. A k-step dominating set S of G such that the sets $N_k(v) = \{u \in V(G)|\mu^k(v.u) > 0\}$, $v \in S$, are pair-wise disjoint, is called an *exact k-step dominating set*. If a fuzzy graph G has an exact k-step dominating set, then G is called an *exact k-step domination fuzzy graph*.

3. Strongest dominating set in fuzzy graphs

Following the definition of dominating sets in fuzzy graphs, we note that every vertex has a degree of membership in a fuzzy graph. So two different minimum dominating sets of a fuzzy graph may have non-equal fuzzy cardinality. Furthermore, given a dominating set *S* in a fuzzy graph $G = (\sigma,\mu)$, a vertex may be dominated by several vertices of *S* with different memberships. This motivate us to define the best dominating set for a fuzzy graph $G = (\sigma,\mu)$ by contemplate degree of membership of vertices and edges as follows. For a dominating set *S* and a vertex $v \in G \setminus S$, we define $max\{\mu(u.v)|u \in S.\ v \in G\setminus S\}$ as the *strength of dominance* on *v* and denote it by *sdom*(*v.S*). We also define $\min\{sdom(v.S)|v \in G \setminus S\}$ as the *dominate strength* of *S* and denote it by $sdom(G \setminus S,S)$. We denote by $S_s(G)$ the set of minimum dominating sets with maximum $sdom(G \setminus S,S)$. A set with maximum fuzzy cardinality between all minimum dominating sets of $S_s(G)$ is called the *strongest dominating set* and its fuzzy cardinality is called *strength of strongest dominating set* in $G = (\sigma,\mu)$ and is denoted it by *ssd*(*G*)[6].

In the following, we have theorems that are proved in [6] for the strength of strongest dominating set in fuzzy graphs.

Theorem 1 [6]. For every fuzzy graph $G = (\sigma,\mu)$, $\gamma f(G) \leq ssd(G)$.

Theorem 2 [**6**]. For any fuzzy graph $G = (\sigma,\mu)$ of order p, $ssd(G) \leq p$. and equality holds if and only if each vertex of G is an isolated vertex

Theorem 3[**6**]. For every fuzzy graph $G = (\sigma,\mu)$, $\gamma(G^*).\sigma(v_w) \leq ssd(G) \leq \gamma(G^*).\sigma(v_s)$.

Theorem 4[**6**] .Let $G = (\sigma,\mu)$ be a complete fuzzy graph K_σ. Then $ssd(K_\sigma) = \max\{\sigma(v)|v \in S\}$, where S is the set of vertices of K_σ with maximum dept.

Theorem 5[**6**]. Let $G = (\sigma,\mu)$ be a complete bipartite fuzzy graph $K_{\sigma_1.\sigma_2}$. Then $ssd(K_{\sigma_1.\sigma_2}) = \max\{\sigma_1(v) + \sigma_2(w)|\{v.w\} \in S\}$, where S is the set contains every pair of vertices $\{v.w\}$ such that $v \in \sigma_1.w \in \sigma_2$ and $d(v) \wedge d(w)$ maximum among all pair of vertices $\{v.w\}$ with $v \in \sigma_1$ and $w \in \sigma_2$. Also we have following theorems for strength of strongest exact 1-step dominating set in fuzzy graphs in [**6**].

Theorem 6[**6**]. Let $G = (\sigma,\mu)$ be an exact 1-step domination fuzzy graph. If v is an arbitrary vertex of G and S is an exact 1-step dominating set of G, then $sdom_1(v,S) = \mu(v,S)$.

Theorem 7[**6**]. For every exact 1-step domination fuzzy graph $G = (\sigma,\mu)$. we have $\gamma_{tf}(G) \leq ssd_1(G)$.

4. Strength of strongest 2-step dominating set

The 2-corona fuzzy graph $GOP_2 = (\sigma_c.\mu_c)$ of a fuzzy graph $G = (\sigma,\mu)$ is the fuzzy graph obtained from G by attaching a fuzzy path $P_2 = (v,\rho) = x_0.x_1.x_2$ of length 2 such that $\rho^2(x_0.x_2) > 0$, for every vertex $v \in GOP_2$, if $v \in G$, then $\sigma_c(v) = \sigma(v)$ and if $v = x_i \in P_2$, then $\sigma_c(v) =$

$\nu(x_i)$ .Also for every edge e $\in GOP_2$, if e $\in$G, then $\mu_c(v) = \mu(v)$, and if e $\in P_2$, then $\mu_c(v) = \rho(e)$.

In the following, we obtained the strength of strongest 2-step dominating set in 2-corona fuzzy graphs.

Theorem 8. Let G = (σ,μ) be a 2-step domination fuzzy graph with n vertices. If $GOP_2 = (\sigma_c.\mu_c)$ is a 2-corona fuzzy graph of G, then $ssd_2(GOP_2)$ = P.

Proof: Let S $\subseteq$V (G) be a 2-step dominating set of fuzzy graph G = (σ,μ) and $GOP_2 = (\sigma_c.\mu_c)$ be a 2-corona fuzzy graph of G that is obtained from G by attaching a path $P_2 = z_iu_iw_i$ of length 2 to each vertex of v_i of G, for i= 1.2,...,n, by merging z_i with v_i. For every vertex v_i of G there exists at least one path of length 2 between v_i and the vertices of S. For every vertex v_i of G, for i= 1.2,...,n, there exists at least one vertex t_k of S such that if $P_2 = v_i,v_j,t_k$ is the path of length 2 between v_i and vertex t_k of S, then $\mu^2(v_i.t_k) > 0$ and so $\mu(v_i.v_j) > 0$ and $\mu(v_j.t_k) > 0$. Let H = V (G). Since S $\subseteq$H is a 2-step dominating set of G, any vertex of G is 2-step dominated by H. On the other hand, every vertex w_i is at distance 2 of vertex $v_i \in$H.Also every vertex u_i is at distance 2 of vertex $v_j\in$H. So H = V (G) is a 2-step dominating set of fuzzy graph $GOP_2 = (\sigma_c.\mu_c)$. Because every vertex v_i of V (G), i= 1.2.....n, must be in every 2-step dominating set of fuzzy graph $GOP_2 = (\sigma_c.\mu_c)$ for 2-step dominating vertices w_i, so for every vertex v_i of V (G) we have the set V (G) \ $\{v_i\}$ is not a 2-step dominating set of $GOP_2 = (\sigma_c.\mu_c)$ and so H = V (G) is a minimal 2-step dominating set of $GOP_2 = (\sigma_c.\mu_c)$ .Also if $\acute{H}$ be another minimal 2-step dominating set of $GOP_2 = (\sigma_c.\mu_c)$, then every vertex v_i of V (G), i= 1.2.....n, must be in $\acute{H}$, for dominating vertices w_i. Therefore H $\subset \acute{H}$ and so $|H|_f < |\acute{H}|_f$.so H is the only minimum 2-step dominating set of $GOP_2 = (\sigma_c.\mu_c)$ and $ssd_2(GOP_2) = |H|_f = |V(G)|_f$= P.

5. New results for complement of fuzzy graphs

In these section first we study on strength of strongest dominating set in complement of fuzzy paths and fuzzy cycles. Then we introduce the concepts of strength of strongest dominating set in complement of other fuzzy graphs.

Theorem 9. Let P_n be fuzzy path. Then $ssd(\overline{P_n}) \leq 2\sigma(v_s)$.

Proof. Let path P_n be a fuzzy path that is a sequence of distinct vertices $x_0,x_1,...,x_n$ (except possiblyx_0 and x_n) such that $\mu(x_{i-1}.x_i) > 0$,$1 \leq i \leq n$. if P_n has no strong edge . Then we have $\bar{\mu}(x_{i-1}.x_i) > 0$,$1 \leq i \leq n$ in $\overline{P_n}$too .also every two vertex that are not injoyed. Are injoyed in $\overline{P_n}$. Therefore the vertex x_0 and also vertex x_n dominate all vertices in $\overline{P_n}$. Minimum dominating set a fuzzy path $\overline{P_n}$ has exactly on vertex. so $ssd(\overline{P_n}) \leq \sigma(v_s)$.
Now let the path P_n has at least one strong edge. If strong edge not be $(x_0.x_1)$ or $(x_{n-1}.x_n)$, then also the vertex x_0 or vertex x_n dominate all vertices in $\overline{P_n}$ and so minimum dominating set of fuzzy path $\overline{P_n}$ has exactly one vertex. So $ssd(\overline{P_n}) \leq \sigma(v_s)$ .But let the edge $(x_0.x_1)$ or $(x_{n-1}.x_n)$ be strong edge. Let $(x_0.x_1)$ be strong edge, so $\bar{u}(x_0.x_1) = \sigma(x_0) \wedge \sigma(x_1) - \mu(x_0.x_1)$ = 0 and there is no edge between x_0 and x_1. So we need two vertivesx_0 and x_1 in dominating set of $\overline{P_n}$ and we have $ssd(\overline{P_n}) \leq 2\sigma(v_s)$.

Theorem 10. Let C_n be a fuzzy cycle. Then ssd($\bar{C}_n$) ≤ 3σ(v_s).
Proof: Let the cycle C_n be a fuzzy cycle that is a sequence of distinct vertices $x_0,x_1,\dots,x_n=x_0$ such that $\mu(x_{i-1}.x_i) > 0$,1 ≤ i ≤ n, if C_n has no strong edge. Then the set of very vertex x_i, 1 ≤ i ≤ n-1 , is dominating set for $\bar{C}_n$.Therefor we have ssd($\bar{C}_n$) ≪ $\sigma(v_s)$. Now let the cycle C_n has at least one strong edge. If there exist the vertex such as x_k in cycle C_n that two edges that are injoyed by x_k not be strong, then $\{x_k\}$is dominating set for fuzzy cycle $\bar{C}_n$. So in these case ssd($\bar{P}_n$) ≪ $\sigma(v_s)$ .At the worst case if for every vertex in fuzzy cycle C_n two edges that are injoyed by these vetices be strong, then we need tree vertices for minimum dominating set of fuzzy cycle $\bar{C}_n$. So we have *ssd*($\bar{C}_n$) ≤ 3σ(v_s).

Now let the cycle C_n has at least one strong edge. If there exist the vertex such as x_k in cycle C_n that two edges that are injoyed by x_k not be strong, then $\{x_k\}$ is dominating set for fuzzy cycle $\bar{C}_n$ So in these case *ssd*($\bar{P}_n$) ≤ σ(v_s). At the worst case if for every vertex in fuzzy cycle C_n two edges that are injoyed by these vertices be strong, then we need tree vertices for minimum dominating set of fuzzy cycle $\bar{C}_n$. So we have *ssd*($\bar{C}_n$) ≤ 3σ(v_s).

Now we proved new theorems for domination in complement of fuzzy graphs and study on relationship between these parameters.

Theorem 11. Let G be a fuzzy graph without isolated vertex. Then ssd(G)+ssd($\bar{G}$) ≤ 2P.
Proof: Let *G* be a fuzzy graph without isolated vertex. Then *ssd*($\bar{G}$) = *max*$\{|S|_f \mid S \in S_s(\bar{G})\}$. It is distinct that *ssd*($\bar{G}$) ≤ *p*. We have *ssd*(G)+*ssd*($\bar{G}$) ≤ 2*P*.

Theorem 12. Let G be a fuzzy graph without strong edge and S be dominating set for G. Then S is dominating set for $\bar{G}$too.
Proof: Let *G* be a fuzzy graph without strong edge and *S* be dominating set for *G*. Then for every vertex such as *u* in *G* \ *S* there exist at least one vertex *v* in *S* that is adjacent with *u* by edge *uv* and membership *μ*(*u*,*v*) . Now let *u* be a vertex in $\bar{G}$ \ *S*. The vertex *v* in *S* is adjacent with *u* by edge *uv* and membership $\bar{\mu}(u.v) = \sigma(u)\wedge\sigma(v)-\mu(u.v) > 0$. So *S* is dominating set for $\bar{G}$ too.

Theorem 13. Let G be a fuzzy graph without strong edge. Then $\gamma_f(G) = \gamma_f(\bar{G})$

Theorem 14. Let G be a fuzzy graph that have at least one vertex that is joined by exact one vertex with strong edge. Then ssd($\bar{G}$)) ≤ 2σ(v_s)
Proof. Let G be a fuzzy graph and v be a vertex that is joined by exact one vertex u with strong edge. Then vertex v is joined by all vertices in $\bar{G}$ except u, because uv is strong edge. So {u,v} is dominating set for $\bar{G}$ with minimum cardinality. Let S = {u,v} be in the set S_s(G) that is the set of minimum dominating sets with maximum sdom(G\S,S).
Because v_s is the vertex by maximum membership, So we have ssd($\bar{G}$) ≤ 2σ(v_s).

Corollary15. Let G be a fuzzy graph that have at least one vertex that is joined by exact one vertex with strong edge. Then $\gamma_f(\bar{G}) = \min\{\sigma(v_i) + \sigma(u_i)\}$ that u_iv_i is strong edge.

Theorem 16. Let G be a fuzzy graph without strong edge by γ(G^*) = 1. Then ssd(G) ≤ σ(v_s).

Proof: Let G be a fuzzy graph without strong edge by $\gamma(G^*) = 1$. Because all edge uv of G has degree under $\sigma(u) \wedge \sigma(v)$, so $\mu^{-}(u.v) = \sigma(u) \wedge \sigma(v) - \mu(u.v) > 0$ and the vertex that is dominated G also dominate $\bar{G}$. Let set $S = \{v\}$ be dominating set of G. Then set $\{v\}$ is dominating set of $\bar{G}$ by minimum cardinality. So $sdom(G \backslash S.S) = \min\{sdom(u.S) | u \in G \setminus S\} = d(v)$. So S_s is the set of vertices that are dominating set for G by minimum depth. Therefor we have $ssd(G) = max\{\sigma(v_i) | \{v_i\} \in S_s\}$.

Theorem 17. Let G be a fuzzy graph and every vertex in G is injoyed by at least two vertices with edge by positive integer. Then

$$\sigma(v_w) + \delta_f(G) \leq ssd(G).$$

Proof: Let G be a fuzzy graph and every vertex in G is injoyed by at least two vertices with edge by positive integer and u be an arbitrary vertex of G. If all vertices adjacent with u be adjacent with another $n-1$ vertices of G by strong edges, then we have $N(u)$ vertices that are isolated in $\bar{G}$ and u must be adjacent by $V(\bar{G}) - N(u)$ vertices with edge by positive integer. So $\{u.N(u)\}$ is minimal dominating set for $\bar{G}$ and so $\gamma_f(\bar{G}) = \sigma(u) + \sum_{v \in N(u)} \sigma(v)$ for as much as v_w has minimum membership of vertices in G and $\bar{G}$, we have $\sigma(v_w) + \gamma_f(\bar{G}) = \sigma(v_w) + \gamma_f(G) \ll \gamma_f(\bar{G})$, we have $\sigma(v_w) + \gamma_f(G) \ll ssd(\bar{G})$.

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