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Solution of System of Real Neutrosophic Linear Equations by Successive Over Relaxation Method

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Abstract. This thesis presents a novel approach for solving systems of real Neutrosophic linear equations using the Successive Over-Relaxation (SOR) method. Neutrosophic logic, which incorporates the elements of truth, indeterminacy, and falsity, provides a comprehensive framework for addressing data that is uncertain, inconsistent, and incomplete. By adopting the SOR method, an iterative technique traditionally used for solving linear equations, this research effectively handles the unique characteristics of Neutrosophic numbers. The modified SOR method is rigorously analyzed for convergence properties and its effectiveness is demonstrated through numerical experiments. The results indicate that the adapted SOR method is a robust and reliable tool for solving complex Neutrosophic linear equations, contributing significantly to the field of numerical methods and expanding the applicability of Neutrosophic logic in computational mathematics.

Keywords: fuzzy number, neutrosophic number, system of linear equations, SOR method

AMS Mathematics Subject Classification (2010): 90B05

Abbreviation	Meaning
$\mathcal{M}_{mn}^{NFM}$	Set of all NFMs of order $m \times n$
$\mathcal{M}_{n}^{NFM}$	Set of all NFMs of order $n \times n$
$\mathcal{M}_{mn}^{FNM}$	Set of all FNMs of order $m \times n$
$\mathcal{M}_{n}^{FNM}$	Set of all FNMs of order $n \times n$
$\mathcal{M}_{mn}^{\mathbb{R}}$	Set of all real matrices of order $m \times n$
$\mathcal{M}_{n}^{\mathbb{R}}$	Set of all real matrices of order $n \times n$
$\mathcal{M}_{mn}^{\mathbb{R}_2}$	Set of all ReNM of order $m \times n$
$\mathcal{M}_{n}^{\mathbb{R}_2}$	Set of all ReNM of order $n \times n$

1. Fundamentals of neutrosophic sets and numbers

1.1. Introduction

After the development of the fuzzy set (FS) theory, many problems with non-random uncertainty are tackled using this theory. While solving these problems, the researchers observed that there are many cases where FS is not sufficient to find the answer. In 1983 [2], Atanassov introduced an intuitionistic fuzzy set (IFS), a very good extension of FS. A beautiful insight of IFS is that it incorporates two parameters known as membership value and non-membership value. There is a restriction on these values, their sum is less than or equal to 1. If the sum is exactly equal to 1 for all members, then there is nothing new in IFS, in this case, IFS becomes FS. Basically, IFS deals with the problems when FS fails to solve them or the available information is not sufficient to explain/solve the problem completely.

Sometimes it is also seen that in some cases IFS is not sufficient to explain some problems due to incomplete, inconsistent and indeterminate information. To tackle such types of problems, Smarandache [39, 40, 41, 42, 43] proposed a new concept known as Neutrosophic sets (NSs) which is characterized by three quantities such as truth membership function (t), indeterminacy membership function (i) and falsity membership function (f).

This theory has been found extensive application in various fields $c10ye14$ for dealing with indeterminate and inconsistent information in the real world. The NS generalized the concept of classical fuzzy set, interval-valued FS, IFS and, so on. Taking into account the NS, several authors worked on its different branches, viz. interval neutrosophic set [46, 48], generalized neutrosophic soft set [5], etc. After that, some others became more interested in NS and developed NFN together with its corresponding matrices [9].

In 2010, Wang et al. [48] introduced the concept of a single-valued neutrosophic set (SVNS) in which all these three quantities are independent, i.e. these values can express independently. All these quantities described by the SVNS are closely related to human thinking due to the imperfection of knowledge that human receives or observes from external sources of information. For details and systematic information about neutrosophic sets, numbers and matrices see [29].

Some authors mentioned that NS is an extension of FS and IFS and they mentioned three quantities of NS as membership function, indeterminacy function and non-membership function. For this representation, NS is an extension of IFS. But, when each element of NS consists of three quantities, viz. truth membership function (t), indeterminacy membership function (i) and falsity membership function (f), then it is not appropriate to say NS is an extension of IFS, because IFS is characterised by membership/acceptance and non-membership/non-acceptance functions only. Note that acceptance/membership value is not the same as truth value. To explain it, let us consider the statement "God is present everywhere". Somebody "accepts" this statement by their own faith or faith is coming from their ancestor or without deep thinking or with/without proof. Whereas to say the statement is "truth", needs some evidence or solid proof. So “truth” is a hard concept while “acceptance” is a soft idea.

In 2006, Samrandache [43] generalized the IFS to a transcendental logic, called “neutrosophic logic", where the unit interval $[0,1]$ is exceeded, i.e. the percentage of truth, indeterminacy and falsity are approximated by non-standard subsets which may overlap and exceed the unit interval $[0,1]$ in the sense of non-standard analysis; also the superior sums and inferior sum, $n_{\sup} = \sup t + \sup i + \sup f \in]^{-}0, 3^{+}[$ may be as big as 3 or 3^{+}, while $n_{\inf} = \inf t + \inf i + \inf f \in]^{-}0, 3^{+}[$ may be as small as 0 or $^{-}0$. In NS, there is no restriction on t, i, f other than they are subset of non-standard unit interval

$]^{-}0, 1^{+}[$. Thus $^{-}0 \leq \inf t + \inf i + \inf f \leq \sup t + \sup i + \sup f \leq 3^{+}$.
Some propositions like paradoxes can be nicely characterized by NS, while IFS fails to describe it.

Here, we assume that each elements of a NS is characterized by **truth membership function** (t), **indeterminacy membership function** (i) and **falsity membership function** (f), and all these quantities are independent. Also, we consider only standard intervals, because there are some difficulties in non-standard intervals.

2. Neutrosophic number

Samrandche first proposed a concept of **neutrosophic number** which consists of the determinant part and the indeterminate part. It is usually denoted by $N = a + bI$, where a and b are real numbers and I is the indeterminacy such that $I^2 = I, I.0 = 0$ and $\frac{I}{I}$ is undefined. We call $N = a + bI$ as a pure neutrosophic number if $a = 0$.

For example, we consider a neutrosophic number $N = 7 + 2I$. If $I \in [0,0.02]$, then it is equivalent to $N \in [7,7.04]$ for $N \geq 7$. This means the determinant part is 7, whereas the indeterminacy part is $2I$ for $I \in [0,0.04]$, which means the possibility for number N to be a little bigger than 7.

Note that this number looks like a complex number, but, see that here $I^2 = I$, not -1 like a complex number.

The three basic operators defined on neutrosophic numbers $P = p_1 + q_1 I$ and $Q = p_2 + q_2 I$ are as follows:

(i) $P + Q = (p_1 + p_2) + (q_1 + q_2)I$

(ii) $P - Q = (p_1 - p_2) + (q_1 - q_2)I$

(iii) $P \times Q = p_1 p_2 + (p_1 q_2 + q_1 p_2 + q_1 q_2)I$

In real neutrosophic algebra, we denote K as the neutrosophic field over some neutrosophic vector spaces. We call the smallest field generated by $K \cup I$ or $K(I)$ to be the neutrosophic field for it involves the indeterminacy factor in it, where I has the special property that $I^n = I, I + I = I$ and if $t \in K$ be some scalar then $t.I = tI, 0.I = 0$. Thus, we generally denote neutrosophic field $K(I)$ generated by $K \cup I$, i.e. $K(I) = \langle K \cup I \rangle$.

3. Methods for solving linear equations

Classical Methods

• **Gaussian Elimination:** A direct method for solving linear systems by transforming the matrix into an upper triangular form.

• **LU Decomposition:** Factorizes a matrix as the product of a lower triangular matrix and an upper triangular matrix.

• **Iterative Methods:**If the system of equations has a large number of variables, then the direct methods are not much suitable.

In this case, the approximate numerical methods are used to determine the variables of the system.

The approximate methods for solving system of linear equations make it possible to obtain the values of the roots of the system with the specified accuracy as the limit of the sequence of some vectors. The process of constructing such a sequence is known as the iterative process. Include Jacobi and Gauss-Seidel methods, which iteratively refine the solution.

The successive over-relaxation (SOR) is an iterative method that improves upon Gauss-Seidel by introducing a relaxation factor ω to accelerate convergence.

4. Real neutrosophic matrix (RNM)

A neutrosophic matrix over real numbers is a generalization of classical matrices that allows for the representation of uncertain, indeterminate, and contradictory information. It extends the concept of classical matrices by incorporating three components: **truth-membership, indeterminacy-membership, and falsity-membership**.

This matrix looks like a complex matrix, but see that here I represents indeterminacy, not complex $i = \sqrt{-1}$. Also, $I^n = I$ for all positive integer n, which is not true for complex numbers.

In a neutrosophic matrix, each element can take on three values representing its degree of membership to truth, indeterminacy, and falsity, respectively. These values are typically denoted by T, I, and F.

Here we consider the neutrosophic matrix over real numbers based on the work of Smarandache. So it is referred to as a real neutrosophic matrix and is abbreviated by RNM. For details of this matrix see.

The neutrosophic number over the field of real/complex numbers is defined in the form $a = a_1 + b_1 I$, where a_1, a_2 are real or complex numbers and I is the indeterminacy.

An RNM is defined as in FNM, i.e. of the form $M = M_1 + M_2 I$ where M_1 and M_2 are real matrices. The set of real matrices of order $m \times n$ is denoted by $\mathcal{M}_{mn}^{\mathbb{R}}$ and that of order $n \times n$ by $\mathcal{M}_n^{\mathbb{R}}$. The identity RNM of order $n \times n$ is denoted by U_n, all diagonal elements are 1 and all other elements are 0. The null and identity matrix of order 3×3 are

$$O_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } U_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Basic operation on real neutrosophic matrices

If $M = M_1 + M_2 I$ and $N = N_1 + N_2 I$ are two Real Neutrosophic Matrices. Then

Addition:

The addition define as:

$$M + N = (M_1 + N_1) + (M_2 + N_2)I$$

Subtraction:

The subtraction define as:

$$M - N = (M_1 - N_1) + (M_2 - N_2)I$$

Multiplication:

The multiplication define as:

$$MN = (M_1 N_1) + (M_2 N_1 + M_1 N_2 + M_2 N_2)I.$$

In this case, $I^N = I^2 = I$, for any positive integer n

Division:

For division $M/N = (M_1 + M_2 I)/(N_1 + N_2 I)$

Let $(M_1 + M_2 I)/(N_1 + N_2 I) = A + IB$

$$(M_1 + M_2 I) = (N_1 + N_2 I)(A + IB)$$

$$(M_1 + M_2I) = AN_1 + (AN_2 + BN_1 + BN_2)I$$

Comparing both sides,

$$M_1 = AN_1A = M_1/N_1$$

and,

$$M_2 = AN_2 + BN_1 + BN_2$$

$$M_2 = (M_1/N_1)N_2 + BN_1 + BN_2$$

$$M_2 - (M_1/N_1)N_2 = B(N_1 + N_2)$$

$$B = (M_2 - (M_1/N_1)N_2)/(N_1 + N_2)$$

Therefore

$$A + IB = \frac{M_1}{N_1} + \frac{M_2 - \frac{M_1N_2}{N_1}}{N_1 + N_2} I$$

So, $M/N = \frac{M_1}{N_1} + \frac{M_2N_1 - M_1N_2}{N_1(N_1 + N_2)} I$

Inverse

Let $M = M_1 + M_2I$ be a real neutrosophic matrix and $M_1, M_2 \in \mathcal{M}_n^{\mathbb{R}}$. Then M is invertible if and only if M_1 and $M_1 + M_2$ are invertible and this inverse of M is given by

$$M^{(-1)} = M_1^{(-1)} + [(M_1 + M_2)^{(-1)} - M_1^{(-1)}]I.$$

6. Neutrosophic linear equations

Neutrosophic linear equations involve variables that are characterized by truth, indeterminacy, and falsity components. These equations extend the classical linear equations into the neutrosophic set domain, which allows for the handling of uncertainty, indeterminacy, and incompleteness.

A system of Neutrosophic linear equations can be written as $A\mathbf{X} = \mathbf{B}$, where A is an $n \times n$ Neutrosophic matrix, and $\mathbf{X}$ and $\mathbf{B}$ are Neutrosophic vectors.

6.1. Solving procedure of real neutrosophic linear equations

Let us consider a system of linear equations

$$AX = B \tag{1}$$

i.e. $(A_1 + A_2I)\ (X_1 + X_2I) = (B_1 + B_2I)$

where $A_1,\ A_2 \in \mathcal{M}_n^{\mathbb{R}}$ and $X_1,\ X_2,\ B_1,\ B_2 \in \mathcal{M}_{n1}^{\mathbb{R}}$.

The ij th element of A is $a_{ij}^{(1)} + a_{ij}^{(2)}I$, j th element of X and B are x_j=$x_j^{(1)}$+$x_j^{(2)}$I and b_j=$b_j^{(1)}$+$b_j^{(2)}$I respectively.

Then the matrices A_1 and A_2 are $A_1 = (a_{ij}^{(1)})_{n\times n}$ and $A_2 = (a_{ij}^{(2)})_{n\times n}$ respectively. The vectors X_1, X_2, B_1, B_2 are

$$X_1 = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{bmatrix}, \quad X_2 = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ \vdots \\ x_n^{(2)} \end{bmatrix}, \quad B_1 = \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} \\ \vdots \\ b_n^{(1)} \end{bmatrix}, \quad B_2 = \begin{bmatrix} b_1^{(2)} \\ b_2^{(2)} \\ \vdots \\ b_n^{(2)} \end{bmatrix}$$

From equation (1),

$$A_1X_1 + [(A_1 + A_2)(X_1 + X_2) - A_1X_1]I = B_1 + B_2I$$

That is,

$$A_1X_1 = B_1 \tag{2}$$

$(A_1 + A_2)(X_1 + X_2) - A_1X_1 = B_2$ or, $(A_1 + A_2)(X_1 + X_2) = B_1 + B_2$ (3)

Equation (2), gives the vector X_1 and equation (3) give $X_1 + X_2$. The final solution of the equation (1) is $X = X_1 + X_2 I$.

Notice that the equations (4.3) and (4.4) are the system of real equations.

6.2. Some important results on neutrosophic linear equations

If A_1 and $A_1 + A_2$ are two non-singular matrices then check the behavior of A_2.

If A_1 and $A_1 + A_2$ are two non-singular matrices then A_2 may or may be non-singular. For example, $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_1 + A_2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

Here A_1 and $A_1 + A_2$ are non-singular. Then $A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ is also non-singular.

Again, $A_1 = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$, $A_1 + A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Here A_1 and $A_1 + A_2$ are non-singular. Then $A_2 = \begin{bmatrix} (-2) & 0 \\ 0 & 0 \end{bmatrix}$ is singular.

Theorem 1. If A_1 and $A_1 + A_2$ are two non-singular matrices then the solution of neutrosophic linear equations is unique.

7. Successive Over-Relaxation (SOR) method

Algorithm description of the classical SOR method

The equation

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \tag{4}$$

The SOR method refines the solution iteratively using the equation:

$$x_i^{(k+1)} = (1-\omega)x_i^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right) \tag{5}$$

Initialize the initial guess vector $x^{(0)}$, relaxation factor ω.

Iterate until convergence. For each i update $(x_i^{(k+1)})$ using the SOR formula.

Check for convergence: if $|x^{(k+1)} - x^{(k)}| \le \epsilon$, stop.

Extend this iterative scheme to handle Neutrosophic numbers. This involves performing Neutrosophic operations for each iteration step.

7.1. Adaption to real neutrosophic systems

To adapt the SOR method for neutrosophic systems, we modify the iterative process to handle neutrosophic components. The adapted iteration is given by:

$$x_i^{(k+1)} = (1-\omega)x_i^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right) \tag{6}$$

where $x_i^{(k)}$ is the i-th component of the neutrosophic solution vector at iteration (k). a_{ij} are elements of the neutrosophic matrix A. b_i are the components of the neutrosophic vector B. where each operation respects the neutrosophic nature of the variables.

Since the equations are a system of real equations, we apply the classical SOR method to solving them.

8. Relaxation factor ω

The relaxation factor ω used in iterative methods for solving linear equations, including neutrosophic linear equations (NLEs), is typically a real number. This is because ω serves as a scalar multiplier that adjusts the iterative updates to control the convergence behavior of the method, not the nature of the neutrosophic components.

Neutrosophic relaxation factor :

If the relaxation factor ω in an iterative method for solving NLEs were to be a neutrosophic number, several potential problems and complexities could arise.

Convergence Criteria:

• Standard convergence criteria are based on norms and distances that are well-defined for real numbers. Defining similar criteria for Neutrosophic numbers is non-trivial.
• The convergence behavior could become unpredictable because the indeterminacy component could lead to ambiguous convergence conditions.

Stability issues:

• The stability of iterative methods might be compromised. The indeterminacy component can introduce fluctuations in the updates, making it difficult to ensure stable convergence.
• Over-relaxation might amplify not just the truth component but also the indeterminacy and falsity components, potentially leading to divergent behavior.

So interpreting the relaxation factor itself as a Neutrosophic number would be challenging.

Properties of relaxation factor ω

• A real number.
• Used to accelerate the convergence of iterative methods.
• Typically falls within the range $(0 < \omega \leq 2)$.

9. Numerical Example

Solve the following Neutrosophic system of equations

$(4+I)X_1 + X_2 + X_3 = 2 + 3I$

$(1+I)X_1 + 6X_2 + (2-I)X_3 = 1 + 2I$

$2X_1 + X_2 + (8-I)X_3 = 3 + I$

By SOR method taken ω=1.01

Rewrite the given equations as

$$\begin{bmatrix} (4+I) & (1+0I) & 1+0I \\ (1+I) & (6+0I) & (2-I) \\ (2+0I) & (1+0I) & (8-I) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 2+3I \\ 1+2I \\ 3+I \end{bmatrix}$$

That is

$$\begin{bmatrix}(4+I) & (1+0I) & 1+0I \\ (1+I) & (6+0I) & (2-I) \\ (2+0I) & (1+0I) & (8-I)\end{bmatrix}\begin{bmatrix}x_1^{(1)}+x_1^{(2)}I \\ x_2^{(1)}+x_2^{(2)}I \\ x_3^{(1)}+x_3^{(2)}I\end{bmatrix}=\begin{bmatrix}2+3I \\ 1+2I \\ 3+I\end{bmatrix}$$

Then

$$A_1=\begin{bmatrix}4 & 1 & 1 \\ 1 & 6 & 2 \\ 2 & 1 & 8\end{bmatrix},\ A_2=\begin{bmatrix}1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & -1\end{bmatrix},\ X_1=\begin{bmatrix}x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)}\end{bmatrix},\ X_2=\begin{bmatrix}x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)}\end{bmatrix},\ B_1=\begin{bmatrix}2 \\ 1 \\ 3\end{bmatrix},\ B_2=\begin{bmatrix}3 \\ 2 \\ 1\end{bmatrix}.$$

The real system of equations are

$$A_1X_1=B_1 \tag{7}$$

$$(A_1+A_2)(X_1+X_2)=B_1+B_2 \tag{8}$$

The iteration scheme for SOR method is

$$x_1^{(k+1)}=(1-\omega)x_1^{(k)}+\frac{\omega}{a_{11}}\left(b_1-a_{12}x_2^{(k)}-a_{13}x_3^{(k)}\right)$$

$$x_2^{(k+1)}=(1-\omega)x_2^{(k)}+\frac{\omega}{a_{22}}\left(b_2-a_{21}x_1^{(k+1)}-a_{23}x_3^{(k)}\right)$$

$$x_3^{(k+1)}=(1-\omega)x_3^{(k)}+\frac{\omega}{a_{33}}\left(b_3-a_{31}x_1^{(k+1)}-a_{32}x_2^{(k+1)}\right)$$

Solution of equation (7)

$$\begin{bmatrix}4 & 1 & 1 \\ 1 & 6 & 2 \\ 2 & 1 & 8\end{bmatrix}\begin{bmatrix}x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)}\end{bmatrix}=\begin{bmatrix}2 \\ 1 \\ 3\end{bmatrix} \tag{9}$$

Using SOR method

$$x_1^{(1)(k+1)}=(1-1.01)x_1^{(1)(k)}+\frac{0.01}{4}\left(2-x_2^{(1)(k)}-x_3^{(1)(k)}\right)$$

$$x_2^{(1)(k+1)}=(1-1.01)x_2^{(1)(k)}+\frac{1.01}{6}\left(1-x_1^{(1)(k+1)}-2x_3^{(1)(k)}\right)$$

$$x_3^{(1)(k+1)}=(1-1.01)x_3^{(1)(k)}+\frac{1.01}{8}\left(3-2x_1^{(1)(k+1)}-x_2^{(1)(k+1)}\right)$$

Let $x_1^{(1)(0)}=x_2^{(1)(0)}=x_3^{(1)(0)}=0.$

The detailed calculations are shown in the following table

k	$x_1^{(1)}$	$x_2^{(1)}$	$x_3^{(1)}$
1	0.50500	0.08332	0.24072
2	0.41813	0.01607	0.26874
3	0.42890	0.00550	0.26707
4	0.43189	0.00566	0.26631
5	0.43201	0.00590	0.26626
6	0.43196	0.00592	0.26627
7	0.43195	0.00592	0.26627

Therefore the required solution is

$$x_1^{(1)}=0.43195, x_2^{(1)}=0.00592, x_3^{(1)}=0.26627 \tag{10}$$

Solution of equation (8)

$$(A_1 + A_2)(X_1 + X_2) = B_1 + B_2$$

i.e. $$\begin{bmatrix} 5 & 1 & 1 \\ 2 & 6 & 2 \\ 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} x_1^{(1)} + x_1^{(2)} \\ x_2^{(1)} + x_2^{(2)} \\ x_3^{(1)} + x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

Let

$$x_1^{(1)} + x_1^{(2)} = y_1, x_2^{(1)} + x_2^{(2)} = y_2, x_3^{(1)} + x_3^{(2)} = y_3$$

Then the equation becomes

$$\begin{bmatrix} 5 & 1 & 1 \\ 2 & 6 & 2 \\ 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

Using SOR method

$$y_1^{(k+1)} = (1 - 1.01) y_1^{(k)} + \frac{0.01}{5}\left(5 - y_2^{(k)} - y_3^{(k)}\right)$$

$$y_2^{(k+1)} = (1 - 1.01) y_2^{(k)} + \frac{1.01}{6}\left(3 - 2y_1^{(k+1)} - y_3^{(k)}\right)$$

$$y_3^{(k+1)} = (1 - 1.01) y_3^{(k)} + \frac{1.01}{7}\left(4 - 2y_1^{(k+1)} - y_2^{(k+1)}\right)$$

Let $y_1^{(0)} = y_2^{(0)} = y_3^{(0)} = 0$.
The detailed calculations are shown in the following table

k	y_1	y_2	y_3
1	1.01000	0.16497	0.26188
2	0.91368	0.15166	0.28898
3	0.91185	0.14785	0.28979
4	0.91248	0.14754	0.28964
5	0.91256	0.14754	0.28962
6	0.91257	0.14754	0.28962

Therefore, the required solution is

$$y_1 = 0.91257, y_2 = 0.14754, y_3 = 0.28962$$

i.e

$$x_1^{(1)} + x_1^{(2)} = 0.91257,\ \ x_2^{(1)} + x_2^{(2)} = 0.14754,\ \ x_3^{(1)} + x_3^{(2)} = 0.28962$$

Using (9),

$$x_1^{(2)} = 0.48062,\ \ x_2^{(2)} = 0.14162,\ \ x_3^{(2)} = 0.02335 \tag{11}$$

Therefore the required final solution of the given system is

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} x_1^{(1)} + x_1^{(2)} I \\ x_2^{(1)} + x_2^{(2)} I \\ x_3^{(1)} + x_3^{(2)} I \end{bmatrix} = \begin{bmatrix} 0.43195 + 0.48062I \\ 0.00592 + 0.14162I \\ 0.26627 + 0.02335I \end{bmatrix}$$

9.1. Verification of the solution

We checked the solution satisfied the equation

$$\begin{bmatrix} (4+I) & (1+0I) & 1+0I \\ (1+I) & (6+0I) & (2-I) \\ (2+0I) & (1+0I) & (8-I) \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 2+3I \\ 1+2I \\ 3+I \end{bmatrix} \qquad (12)$$

Now

$$\begin{bmatrix} (4+I) & (1+0I) & 1+0I \\ (1+I) & (6+0I) & (2-I) \\ (2+0I) & (1+0I) & (8-I) \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$= \begin{bmatrix} (4+I) & (1+0I) & 1+0I \\ (1+I) & (6+0I) & (2-I) \\ (2+0I) & (1+0I) & (8-I) \end{bmatrix}\begin{bmatrix} 0.43195+0.48062I \\ 0.00592+0.14162I \\ 0.26627+0.02335I \end{bmatrix}$$

$$= \begin{bmatrix} (4+I)(0.43195+0.48062I)+(0.00592+0.14162I)+(0.26627+0.02335I) \\ (1+I)(0.43195+0.48062I)+6(0.00592+0.14162I)+(2-I)(0.26627+0.02335I) \\ 2(0.43195+0.48062I)+(0.00592+0.14162I)+(8-I)(0.26627+0.02335I) \end{bmatrix}$$

$$= \begin{bmatrix} 1.99999+3.000002\mathrm{I} \\ 1.00001+1.99969\mathrm{I} \\ 2.99998+1.00004\mathrm{I} \end{bmatrix}$$

$$= \begin{bmatrix} 2+3I \\ 1+2I \\ 3+I \end{bmatrix}$$

Verifying the role of relaxation factor ω:

In this example, we choose ω=2.0.

Solution of equation (9)

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & 6 & 2 \\ 2 & 1 & 8 \end{bmatrix}\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Let

$$x_1^{(1)} = x_1, x_2^{(1)} = x_2, x_3^{(1)} = x_3$$

Using SOR method

$$x_1^{(k+1)} = (1-2.10)x_1^{(k)} + \frac{2.10}{4}\left(2 - x_2^{(k)} - x_3^{(k)}\right)$$

$$x_2^{(k+1)} = (1-2.10)x_2^{(k)} + \frac{2.10}{6}\left(1 - x_1^{(k+1)} - 2x_3^{(k)}\right)$$

$$x_3^{(k+1)} = (1-2.10)x_3^{(k)} + \frac{2.10}{8}\left(3 - 2x_1^{(k+1)} - x_2^{(k+1)}\right)$$

Let $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$.

The detailed calculations are shown in the following table

k	x_1	x_2	x_3
1	1.5000	-0.01750	0.24084
2	-0.22226	0.27845	0.56616
3	0.85106	-0.65048	-0.11134
4	0.51379	0.96364	0.38728
5	-0.22439	-0.90255	0.71622
6	1.39466	0.35332	-0.82529
7	-0.23634	0.62176	1.65618

Therefore, the solution does not exist for this value of relaxation factor ω.

10. Conclusion

Summary of findings

This research focused on adapting the Successive Over-Relaxation (SOR) method for solving Neutrosophic linear equations. Neutrosophic logic, which introduces the concepts of truth, indeterminacy, and falsity, provides a more flexible framework for handling uncertainty compared to traditional binary logic. The modified SOR method was shown to be effective in handling the complexities of Neutrosophic linear equations. Through theoretical analysis and numerical experiments, the research validated that the modified SOR method converges under certain conditions and offers reliable solutions where traditional methods fall short.

Implications

The broader implications of this research are significant for computational mathematics and engineering. The modified SOR method can be applied in various industries where uncertainty and indeterminacy are prevalent, such as artificial intelligence, decision-making, and data analysis. By providing a robust approach to solving Neutrosophic linear equations, this research opens up new possibilities for modeling and solving real-world problems characterized by incomplete or inconsistent information. The method's ability to handle uncertainty more effectively can lead to better decision-making processes and more accurate data analysis.

Future work

Future research could explore several avenues to enhance the findings of this study. One potential area is the development of hybrid methods that combine the modified SOR method with other numerical techniques to improve convergence rates and accuracy further. Additionally, applying the method to larger and more complex Neutrosophic systems could provide deeper insights into its scalability and efficiency. Research could also focus on extending the approach to other equations and mathematical models, thereby broadening its applicability across different fields.

This conclusion synthesises the key outcomes of the research, underscores its significance, and proposes directions for future exploration, ensuring a comprehensive wrap-up of the study's contributions and potential.

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