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Solution of a System of Neutrosophic Linear Algebraic Equation by LU Decomposition Method

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Abstract. Neutrosophic numbers and matrices extend classical fuzzy set theory by incorporating three components to represent uncertainty: truth membership, indeterminacy membership, and falsity membership degrees. Real neutrosophic matrices consist of neutrosophic numbers as their entries, offering a structure to model uncertain and imprecise information across various applications. This paper investigates operations on real neutrosophic matrices, such as addition, multiplication, and scalar multiplication, and defines how these operations function within the neutrosophic framework.

The paper also examines the use of LU decomposition as an efficient method for solving systems of real neutrosophic linear algebraic equations. LU decomposition is a numerical technique that breaks down a matrix into lower and upper triangular matrices, enabling the solution of linear systems through forward and backward substitution.

Keywords: Fuzzy number, neutrosophic number, neutrosophic system of linear equations.

AMS Mathematics Subject Classification (2010): 90B05

1. Introduction

After the development of the fuzzy set (FS) theory, many non-random uncertainty problems were addressed using this framework. However, researchers soon observed that FS was inadequate in certain cases. In 1983, Atanassov introduced the intuitionistic fuzzy set (IFS) as an extension of FS [3]. IFS incorporates two parameters: membership value and non-membership value, with the restriction that their sum is less than or equal to 1. If the sum equals 1 for all elements, IFS reduces to FS. Essentially, IFS is applied when FS is insufficient or when the available information does not fully resolve the problem.

In some situations, IFS also proves inadequate due to incomplete, inconsistent, or indeterminate information. To handle such cases, Smarandache proposed neutrosophic sets (NSs), characterized by three components: truth membership function (t), indeterminacy membership function (i), and falsity membership function (f) [27, 28, 29, 30, 31].

NSs have found widespread application in dealing with indeterminate and inconsistent real-world information [37]. NSs generalize classical fuzzy sets, interval-valued FS, IFS, and related concepts. Various branches of NSs have since been developed, such as interval neutrosophic sets [33, 35] and generalized neutrosophic soft sets [4]. Subsequently, NS theory was expanded to include neutrosophic fuzzy numbers (NFNs) and corresponding matrices [8].

In 2010, Wang et al. introduced the single-valued neutrosophic set (SVNS), where the three components are independent and can vary freely [35]. These quantities, expressed by SVNS, align with human thinking, which often deals with imperfect or incomplete knowledge from external sources. For detailed information on neutrosophic sets, numbers, and matrices, refer to [22].

Some authors argue that NS is an extension of FS and IFS, characterizing the three components of NS as membership, indeterminacy, and non-membership functions. However, when each NS element consists of truth membership, indeterminacy membership, and falsity membership functions, NS differs from IFS, which is only characterized by membership (acceptance) and non-membership (non-acceptance) functions. Notably, membership (acceptance) is distinct from truth. In 2006, Smarandache extended IFS into a transcendental logic called "neutrosophic logic," allowing truth, indeterminacy, and falsity values to exceed the unit interval [0,1] [31].

In this paper, we assume that each element of NS is characterized by a truth membership function (t), an indeterminacy membership function (i), and a falsity membership function (f), and that these quantities are independent. We focus only on standard intervals, as non-standard intervals pose certain challenges.

Definition 1. *Let U be the universe of discourse and u an element of U. A SVNS W is represented by a truth membership function ($t_W(u)$), an indeterminacy membership function ($i_W(u)$), and a falsity membership function ($f_W(u)$). A* ***SVNS*** *W is defined as:*

$$W = \{x: (t_W(u), i_W(u), f_W(u)), u \in U\},$$

where $t_W(u)$, $i_W(u)$ and $f_W(u)$ belong to the interval [0,1].

Neutrosophic fuzzy logic has been applied to solve numerous decision-making problems [10, 12, 13, 14, 23], as well as issues related to social networks [18, 19, 20, 21].

2. Literature review

Neutrosophic set theory, introduced by Smarandache in 1995, extends the classical set theory by accommodating indeterminacy, vagueness, and inconsistency through the use of three membership functions: truth, indeterminacy, and falsehood. This framework has found applications in various fields where uncertainty is inherent, such as decision-making, pattern recognition, and artificial intelligence. One critical area of application is in solving systems of neutrosophic linear equations (SNLEs), which arise when dealing with uncertain data or imprecise information. The solution of SNLEs involves techniques that handle the complexities introduced by neutrosophic elements effectively.

To address SNLEs, researchers have explored several methodologies. One approach involves converting neutrosophic equations into crisp forms, enabling the use of existing linear algebraic techniques. For instance, Zhang et al. (2017) proposed a method to transform SNLEs into a system of crisp linear equations by defining new operations on neutrosophic numbers [39]. This transformation allows the application of traditional matrix-based methods like Gaussian elimination or LU decomposition.

Alternatively, researchers have developed direct solution methods that operate directly within the neutrosophic domain. For instance, Deli et al. (2019) introduced a computational algorithm based on neutrosophic Gaussian elimination, where operations are defined considering the neutrosophic characteristics of the coefficients and constants [7]. This approach preserves the inherent uncertainty throughout the solution process, providing a more accurate representation of the system's solution under uncertain conditions.

Moreover, the extension of existing numerical methods to accommodate neutrosophic elements has been explored. Li and Smarandache (2020) adapted the Gauss-Seidel method to solve SNLEs iteratively [17]. This method iteratively updates the neutrosophic solutions based on the updated neutrosophic coefficients and constants, converging to a solution that respects the uncertainties inherent in the original equations.

Furthermore, metaheuristic algorithms have been employed to tackle SNLEs due to their ability to handle complex and non-linear systems effectively. Alba et al. (2021) applied a genetic algorithm to find approximate solutions to SNLEs, optimizing the neutrosophic solutions iteratively through crossover and mutation operations [1]. This approach offers robustness against local optima and can handle large-scale systems with significant uncertainties.

In addition to computational methods, theoretical advancements have contributed to the understanding and solution of SNLEs. PÄƒcurar et al. (2018) developed a theoretical framework based on neutrosophic matrices and their properties, laying the foundation for analytical approaches to solving SNLEs [9]. This theoretical groundwork not only aids in the development of new solution techniques but also enhances the interpretation of results in neutrosophic algebra.

The application domains of SNLEs are vast, ranging from engineering to economics, where uncertainty and imprecision are prevalent. For instance, in control systems design, Zadeh (2016) highlighted the benefits of using neutrosophic set theory to model and control systems affected by uncertain and contradictory information [40]. By incorporating neutrosophic solutions to SNLEs, engineers can design robust controllers that are more resilient to uncertainties in the system dynamics.

In conclusion, the solution of systems of neutrosophic linear equations represents a significant advancement in addressing uncertainty in mathematical modeling. Researchers have developed a variety of methodologies, ranging from transformation techniques to specialized algorithms and theoretical frameworks. These approaches not only provide solutions to SNLEs but also contribute to the broader understanding of neutrosophic set theory and its applications. As research continues to evolve, further improvements in computational efficiency, solution accuracy, and theoretical underpinnings are expected, paving the way for enhanced decision-making in complex and uncertain environments.

3. Neutrosophic number

This section explores **real neutrosophic matrices** (RNMs) and their associated operations. A real neutrosophic matrix can be defined similarly to neutrosophic numbers, extending the concept to matrices, where each element is composed of a determinant and an indeterminate part.

A **real neutrosophic matrix** is of the form:

$$M = M_1 + M_2 I$$

where M_1 and M_2 are real matrices, and I represents indeterminacy, as described earlier in the context of neutrosophic numbers. The matrix elements of M_1 and M_2 are real numbers, and the indeterminacy I adds flexibility to represent uncertain or incomplete information in the matrix.

Notation:

- $\mathcal{M}_{mn}$ denotes the set of real matrices of order $m\times n$.
- $\mathcal{M}_{n}$ denotes the set of square real matrices of order $n\times n$.
- The **identity real neutrosophic matrix** of order $n \times n$ is denoted by U_n, where all diagonal elements are 1, and all off-diagonal elements are 0.

3.1. Operations on real neutrosophic matrices

Given two real neutrosophic matrices $A = A_1 + A_2 I$ and $B = B_1 + B_2 I$, the basic operations are defined as follows:

1. Addition:

$$A + B = (A_1 + B_1) + (A_2 + B_2)I$$

This operation adds the corresponding elements of the matrices, both in the determinant part and the indeterminate part.

2. Subtraction:

$$A - B = (A_1 - B_1) + (A_2 - B_2)I$$

Similar to addition, the elements in both parts (determinant and indeterminate) are subtracted.

3. Multiplication:

$$A \times B = A_1 B_1 + (A_2 B_1 + A_1 B_2 + A_2 B_2)I$$

Matrix multiplication follows the distributive property and includes contributions from both the determinant and indeterminate components.

4. Special types of real neutrosophic matrices

1. Identity Neutrosophic Matrix: The identity real neutrosophic matrix U_n is defined similarly to the identity matrix in classical linear algebra, with the indeterminate part set to zero:

$$U_n = I_n + 0 \cdot I$$

where I_n is the classical identity matrix.

2. Inverse Neutrosophic Matrix: For a real neutrosophic matrix M, its inverse M^{-1} (if it exists) is computed using methods similar to those for real matrices, but accounting for the indeterminate part. The exact process can be complex depending on the structure of the matrix.

Real neutrosophic matrices (RNMs) are useful in fields where uncertainty plays a crucial role in data representation, such as:

- Decision-making problems involving incomplete or inconsistent information.
- Modeling systems in areas like control theory, fuzzy systems, and social network analysis where indeterminacy cannot be ignored.

By incorporating the indeterminate part in matrices, RNMs provide an additional layer of flexibility compared to traditional matrices, making them a valuable tool in various computational and theoretical applications.

The null and identity matrices of order 3×3 are

$$O_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } U_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4.1. Basic operations on RNMs

The basic operations on RNMs $M = M_1 + M_2 I$ and $N = N_1 + N_2 I$ are

(i) $M + N = (M_1 + N_1) + (M_2 + N_2)I$

(ii) $M - N = (M_1 - N_1) + (M_2 - N_2)I$

(iii) $MN = (M_1 N_1) + (M_2 N_1 + M_1 N_2 + M_2 N_2)I$. In this case also, $I^n = I^2 = I$ for any positive integer n.

Assumed that the order of RNMs is compatible with the appropriate operations.

This matrix looks like a complex matrix, but see that here I represents indeterminacy, not complex $i = \sqrt{-1}$. Also, $I^n = I$ for all positive integer n, which is not true for complex numbers.

5. Solution of neutrosophic algebraic equations

In this section, the solutions of neutrosophic algebraic equations, viz. linear, non-linear and system of linear equations are investigated. At first, we consider a single equation on a real neutrosophic filed and then we discuss the same for a refined neutrosophic filed.

5.1. System of linear equations

Let us consider a system of linear equations

$$PY = Q \quad (1)$$

$$\text{or, } (P_1 + P_2 I)(Y_1 + Y_2 I) = (Q_1 + Q_2 I) \quad (2)$$

where $P_1, P_2 \in \mathcal{M}_n^{\mathbb{R}}$, $Y_1, Y_2, Q_1, Q_2 \in \mathcal{M}_{n1}^{\mathbb{R}}$.

The ijth element of P is $p_{ij} = p_{ij}^{(1)} + p_{ij}^{(2)} I$, jth element of Y and Q are $y_j = y_j^{(1)} + y_j^{(2)} I$ and $q_j = q_j^{(1)} + q_j^{(2)} I$ respectively.

Then the matrices P_1 and P_2 are $P_1 = (p_{ij}^{(1)})_{n \times n}$ and $P_2 = (p_{ij}^{(2)})_{n \times n}$ respectively . The vectors Y_1, Y_2, Q_1, Q_2 are

$$Y_1 = \begin{pmatrix} y_1^{(1)} \\ y_2^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ y_n^{(1)} \end{pmatrix}, \ Y_2 = \begin{pmatrix} y_1^{(2)} \\ y_2^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ y_n^{(2)} \end{pmatrix}, \ Q_1 = \begin{pmatrix} q_1^{(1)} \\ q_2^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ q_n^{(1)} \end{pmatrix}, \ Q_2 = \begin{pmatrix} q_1^{(2)} \\ q_2^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ q_n^{(2)} \end{pmatrix}$$

To solve the equation $PY = Q$ by **LU Decomposition Method** , first let $P = LU$, where $L = \begin{pmatrix} l_{11} & 0 & 0 & . & . & . & 0 \\ l_{21} & l_{22} & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ l_{n1} & l_{n2} & l_{n3} & . & . & . & l_{nn} \end{pmatrix}$ and

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & . & . & . & u_{1n} \\ 0 & u_{22} & u_{23} & . & . & . & u_{2n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & u_{nn} \end{pmatrix}$$

where $L, U \in \mathcal{M}_n^{\mathbb{R}}$.

The ijth element of L is $l_{ij} = l_{ij}^{(1)} + l_{ij}^{(2)} I$. The ijth element of U are $u_{ij} = u_{ij}^{(1)} + u_{ij}^{(2)} I$ for $i \neq j$ and $u_{ij} = 1 + 0I$ for $i = j$ ..

Then from Eq. (1)

$$LUY = Q \tag{3}$$

Again let $UY = Z$,where $Z = Z_1 + Z_2 I$ and $Z_1, Z_2 \in \mathcal{M}_{n1}^{\mathbb{R}}$.

And $Z_1 = \begin{pmatrix} z_1^{(1)} \\ z_2^{(1)} \\ . \\ . \\ . \\ z_n^{(1)} \end{pmatrix}, Z_2 = \begin{pmatrix} z_1^{(2)} \\ z_2^{(2)} \\ . \\ . \\ . \\ z_n^{(2)} \end{pmatrix}$.

Then from the Eq. (3)

$$LZ = Q \tag{4}$$

where Eq. (4) gives Z and again from $UY = Z$, we have Y. The final solution of the Eq. (1) is $Y = Y_1 + Y_2 I$.

General case: Let us consider the neutrosophic system of equations

$$\begin{pmatrix} p_{11}^{(1)}+p_{11}^{(2)}I & p_{12}^{(1)}+p_{12}^{(2)}I & . & . & . & p_{1n}^{(1)}+p_{1n}^{(2)}I \\ p_{21}^{(1)}+p_{21}^{(2)}I & p_{22}^{(1)}+p_{22}^{(2)}I & . & . & . & p_{2n}^{(1)}+p_{2n}^{(2)}I \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ p_{n1}^{(1)}+p_{n1}^{(2)}I & p_{n2}^{(1)}+p_{n2}^{(2)}I & . & . & . & p_{nn}^{(1)}+p_{nn}^{(2)}I \end{pmatrix}\begin{pmatrix} y_1^{(1)}+y_1^{(2)}I \\ y_2^{(1)}+y_2^{(2)}I \\ . \\ . \\ y_n^{(1)}+y_n^{(2)}I \end{pmatrix} = \begin{pmatrix} q_1^{(1)}+q_1^{(2)}I \\ q_2^{(1)}+q_2^{(2)}I \\ . \\ . \\ q_n^{(1)}+q_n^{(2)}I \end{pmatrix}$$

In this problem,

$$P_1 = \begin{pmatrix} p_{11}^{(1)} & p_{12}^{(1)} & . & . & . & p_{1n}^{(1)} \\ p_{21}^{(1)} & p_{22}^{(1)} & . & . & . & p_{2n}^{(1)} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ p_{n1}^{(1)} & p_{n2}^{(1)} & . & . & . & p_{nn}^{(1)} \end{pmatrix},$$

$$P_2 = \begin{pmatrix} p_{11}^{(2)} & p_{12}^{(2)} & . & . & . & p_{1n}^{(2)} \\ p_{21}^{(2)} & p_{22}^{(2)} & . & . & . & p_{2n}^{(2)} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ p_{n1}^{(2)} & p_{n2}^{(2)} & . & . & . & p_{nn}^{(2)} \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} y_1^{(1)} \\ y_2^{(1)} \\ . \\ . \\ . \\ y_n^{(1)} \end{pmatrix}, \; Y_2 = \begin{pmatrix} y_1^{(2)} \\ y_2^{(2)} \\ . \\ . \\ . \\ y_n^{(2)} \end{pmatrix}, \; Q_1 = \begin{pmatrix} q_1^{(1)} \\ q_2^{(1)} \\ . \\ . \\ . \\ q_n^{(1)} \end{pmatrix}, \; Q_2 = \begin{pmatrix} q_1^{(2)} \\ q_2^{(2)} \\ . \\ . \\ . \\ q_n^{(2)} \end{pmatrix}$$

Let $P = LU$, where $L = \begin{pmatrix} l_{11} & 0 & 0 & . & . & 0 \\ l_{21} & l_{22} & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ l_{n1} & l_{n2} & l_{n3} & . & . & l_{nn} \end{pmatrix}$ and

$$U=\begin{pmatrix} u_{11} & u_{12} & u_{13} & . & . & u_{1n} \\ 0 & u_{22} & u_{23} & . & . & u_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & u_{nn} \end{pmatrix}.$$

The ijth element of L is $l_{ij}=l_{ij}^{(1)}+l_{ij}^{(2)}I$. The ijth element of U is $u_{ij}=u_{ij}^{(1)}+u_{ij}^{(2)}I$ for $i\neq j$ and $u_{ij}=1+0I$ for $i=j$.

Now ,

$$P=\begin{pmatrix} p_{11}^{(1)}+p_{11}^{(2)}I & p_{12}^{(1)}+p_{12}^{(2)}I & . & . & . & p_{1n}^{(1)}+p_{1n}^{(2)}I \\ p_{21}^{(1)}+p_{21}^{(2)}I & p_{22}^{(1)}+p_{22}^{(2)}I & . & . & . & p_{2n}^{(1)}+p_{2n}^{(2)}I \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ p_{n1}^{(1)}+p_{n1}^{(2)}I & p_{n2}^{(1)}+p_{n2}^{(2)}I & . & . & . & p_{nn}^{(1)}+p_{nn}^{(2)}I \end{pmatrix}$$

$$=LU$$

$$=\begin{pmatrix} l_{11}^{(1)}+l_{11}^{(2)}I & 0+0I & 0+0I & . & . & 0+0I \\ l_{21}^{(1)}+l_{21}^{(2)}I & l_{22}^{(1)}+l_{22}^{(2)}I & 0+0I & . & . & 0+0I \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ l_{n1}^{(1)}+l_{n1}^{(2)}I & l_{n2}^{(1)}+l_{n2}^{(2)}I & l_{n3}^{(1)}+l_{n3}^{(2)}I & . & . & l_{nn}^{(1)}+l_{nn}^{(2)}I \end{pmatrix}$$

$$=\begin{pmatrix} 1+0I & u_{12}^{(1)}+u_{12}^{(2)}I & u_{13}^{(1)}+u_{13}^{(2)}I & . & . & u_{1n}^{(1)}+u_{1n}^{(2)}I \\ 0+0I & 1+0I & u_{23}^{(1)}+u_{23}^{(2)}I & . & . & u_{2n}^{(1)}+u_{2n}^{(2)}I \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0+0I & 0+0I & 0+0I & . & . & 1+0I \end{pmatrix}$$

After multiplying RHS of the above equation, compairing both sides, we get

$l_{i1}^{(1)}=p_{i1}^{(1)}$, $i=1,2,\dots,n$

$l_{i1}^{(2)}=p_{i1}^{(2)}, i=1,2,\dots,n$

$$l_{11}^{(1)}u_{1j}^{(1)}=p_{1j}^{(1)}, j=2,3,\dots,n$$

$$l_{11}^{(2)}u_{1j}^{(1)}+(l_{11}^{(1)}+l_{11}^{(2)})u_{1j}^{(2)}=p_{1j}^{(2)}, j=2,3,\dots,n$$

$l_{i1}^{(1)}u_{12}^{(1)}+l_{i2}^{(1)}=p_{i2}^{(1)}$, $i=2,3,\dots,n$

$$l_{i1}^{(2)}u_{12}^{(1)}+(l_{i1}^{(1)}+l_{i1}^{(2)})u_{12}^{(2)}+l_{i2}^{(2)}=p_{i2}^{(2)}, i=2,3,\dots,n$$

$$l_{21}^{(1)}u_{1j}^{(1)}+l_{22}^{(1)}u_{2j}^{(1)}=p_{2j}^{(1)}, j=3,4,\dots,n$$

$$l_{21}^{(2)}u_{1j}^{(1)}+(l_{21}^{(1)}+l_{21}^{(2)})u_{1j}^{(2)}+l_{22}^{(2)}u_{2j}^{(1)}+(l_{22}^{(1)}+l_{22}^{(2)})u_{2j}^{(2)}=p_{2j}^{(2)}, j=3,4,\dots,n$$

$$l_{i1}^{(1)}u_{13}^{(1)}+l_{i2}^{(1)}u_{23}^{(1)}+l_{i3}^{(1)}=p_{i3}^{(1)}, i=3,4,\dots,n$$

$$l_{i1}^{(2)}u_{13}^{(1)}+(l_{i1}^{(1)}+l_{i2}^{(2)})u_{13}^{(2)}+l_{i2}^{(2)}u_{23}^{(1)}+(l_{i2}^{(1)}+l_{i2}^{(2)})u_{23}^{(2)}+l_{i3}^{(2)}=p_{i3}^{(2)}, i=3,4,\dots,n$$

....

$\sum_{i,j=1}^{n-1}\left[l_{nj}^{(1)}u_{in}^{(1)}\right]+l_{nn}^{(1)}=p_{nn}^{(1)}$

$\sum_{i,j=1}^{n-1}\left[l_{nj}^{(2)}u_{in}^{(1)}+(l_{nj}^{(1)}+l_{nj}^{(2)})u_{in}^{(2)}\right]+l_{nn}^{(2)}=p_{nn}^{(2)}$.

From these above equations , we have to calculate all $l_{ij}^{(1)}$,$l_{ij}^{(2)}$,$u_{ij}^{(1)}$ and $u_{ij}^{(2)}$.

Thereafter , we can calculate L and U by the equations $L=L_1+L_2I$ and $U=U_1+U_2I$ respectively. Again, we have to calculate L^{-1} and $U^{-1}$by using the equations $L^{-1}=L_1^{-1}+[(L_1+L_2)^{-1}-L_1^{-1}]I$ and $U^{-1}=U_1^{-1}+[(U_1+U_2)^{-1}-U_1^{-1}]I$ respectively.

From the equation $LZ=Q$ gives Z and again from $UY=Z$, we have Y.

Thus the final solution is $Y=Y_1+Y_2I$.

6. Illustration

Example 1. Let us consider the neutrosophic system of equations

$$\begin{pmatrix}2+4I & 3-I\\ 4-2I & -5+7I\end{pmatrix}\begin{pmatrix}y_1^{(1)}+y_1^{(2)}I\\ y_2^{(1)}+y_2^{(2)}I\end{pmatrix}=\begin{pmatrix}8+2I\\ -6+12I\end{pmatrix}$$

In this problem,

$$P_1=\begin{pmatrix}2 & 3\\ 4 & -5\end{pmatrix},\ P_2=\begin{pmatrix}4 & -1\\ -2 & 7\end{pmatrix},\ Q_1=\begin{pmatrix}8\\ -6\end{pmatrix},\ Q_2=\begin{pmatrix}4\\ 14\end{pmatrix},$$

$$Y_1=\begin{pmatrix}y_1^{(1)}\\ y_2^{(1)}\end{pmatrix},\ Y_2=\begin{pmatrix}y_1^{(2)}\\ y_2^{(2)}\end{pmatrix}.$$

Let $P=LU$, where $L=\begin{pmatrix}l_{11} & 0\\ l_{21} & l_{22}\end{pmatrix}$ and $U=\begin{pmatrix}u_{11} & u_{12}\\ 0 & u_{22}\end{pmatrix}$.

The ijth element of L is $l_{ij}=l_{ij}^{(1)}+l_{ij}^{(2)}I$. The ijth element of U is $u_{ij}=u_{ij}^{(1)}+u_{ij}^{(2)}I$ for $i\neq j$ and $u_{ij}=1+0I$ for $i=j$.

Now,

$$P=\begin{pmatrix}2+4I & 3-I\\ 4-2I & -5+7I\end{pmatrix}=LU=\begin{pmatrix}l_{11}^{(1)}+l_{11}^{(2)}I & 0+0I\\ l_{21}^{(1)}+l_{21}^{(2)}I & l_{21}^{(1)}+l_{22}^{(2)}I\end{pmatrix}\begin{pmatrix}1+0I & u_{12}^{(1)}+u_{12}^{(2)}I\\ 0+0I & 1+0I\end{pmatrix}$$

$$=\begin{pmatrix}l_{11}^{(1)}+l_{11}^{(2)}I & l_{11}^{(1)}u_{12}^{(1)}+l_{11}^{(2)}u_{12}^{(1)}I+l_{11}^{(1)}u_{12}^{(2)}I+l_{11}^{(2)}u_{12}^{(2)}I\\ l_{21}^{(1)}+l_{21}^{(2)}I & l_{21}^{(1)}u_{12}^{(1)}+l_{21}^{(2)}u_{12}^{(1)}I+l_{21}^{(1)}u_{12}^{(2)}I+l_{21}^{(2)}u_{12}^{(2)}I+l_{22}^{(1)}+l_{22}^{(2)}I\end{pmatrix}$$

Compairing both sides of the above , we get

$l_{11}^{(1)}=2, l_{11}^{(2)}=4, l_{21}^{(1)}=4, l_{21}^{(2)}=-2, l_{22}^{(1)}=-11, l_{22}^{(2)}=\frac{37}{3}, u_{12}^{(1)}=\frac{3}{2}, u_{12}^{(2)}=-\frac{7}{6}$

Therefore,

$$L = L_1 + L_2 I = \begin{pmatrix} 2 & 0 \\ 4 & -11 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ -2 & \frac{37}{3} \end{pmatrix} I$$

and,

$$U = U_1 + U_2 I = \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{7}{6} \\ 0 & 0 \end{pmatrix} I.$$

Now, we compute $L_1 + L_2$ and $U_1 + U_2$.
So,

$$L_1 + L_2 = \begin{pmatrix} 6 & 0 \\ 2 & \frac{4}{3} \end{pmatrix} \text{ and}$$

$$U_1 + U_2 = \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix}.$$

Now, we compute L^{-1} and U^{-1}. So,

$$\begin{aligned} L^{-1} &= L_1^{-1} + [(L_1 + L_2)^{-1} - L_1^{-1}] I \\ &= \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{2}{11} & -\frac{1}{11} \end{pmatrix} + \left[\begin{pmatrix} \frac{1}{6} & 0 \\ -\frac{1}{4} & \frac{3}{4} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{2}{11} & -\frac{1}{11} \end{pmatrix} \right] I \\ &= \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{2}{11} & -\frac{1}{11} \end{pmatrix} + \begin{pmatrix} -\frac{1}{3} & 0 \\ -\frac{19}{44} & \frac{37}{44} \end{pmatrix} I \\ &= \begin{pmatrix} \frac{1}{2} - \frac{1}{3} I & 0 + 0I \\ \frac{2}{11} - \frac{19}{44} I & -\frac{1}{11} + \frac{37}{44} I \end{pmatrix} \end{aligned}$$

and,

$$\begin{aligned} U^{-1} &= U_1^{-1} + [(U_1 + U_2)^{-1} - U_1^{-1}] I \\ &= \begin{pmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{pmatrix} + \left[\begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{pmatrix} \right] I \\ &= \begin{pmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{7}{6} \\ 0 & 0 \end{pmatrix} I \end{aligned}$$

$$= \begin{pmatrix} 1+0I & -\frac{3}{2}+\frac{7}{6}I \\ 0+0I & 1+0I \end{pmatrix}$$

Now, from the equation $LZ = Q$,we have

$$Z = L^{-1}Q$$
$$= \begin{pmatrix} \frac{1}{2}-\frac{1}{3}I & 0+0I \\ \frac{2}{11}-\frac{19}{44}I & -\frac{1}{11}+\frac{37}{44}I \end{pmatrix}\begin{pmatrix} 8+2I \\ -6+12I \end{pmatrix}$$
$$= \begin{pmatrix} 4-\frac{7}{3}I \\ 2-4I \end{pmatrix}$$

Again, from the equation $UY = Z$,we have

$$Y = U^{-1}Z$$
$$= \begin{pmatrix} 1+0I & -\frac{3}{2}+\frac{7}{6}I \\ 0+0I & 1+0I \end{pmatrix}\begin{pmatrix} 4-\frac{7}{3}I \\ 2-4I \end{pmatrix}$$
$$= \begin{pmatrix} 1+\frac{4}{3}I \\ 2-4I \end{pmatrix}$$

Thus, the final solution is

$$Y = \begin{pmatrix} 1+\frac{4}{3}I \\ 2-4I \end{pmatrix}.$$

Example 2: Let us consider the neutrosophic system of equations

$$\begin{pmatrix} 2+I & 1+3I & 3+2I \\ 4+2I & 3+3I & 1+3I \\ 1+4I & 3+2I & 3+4I \end{pmatrix}\begin{pmatrix} y_1^{(1)}+y_1^{(2)}I \\ y_2^{(1)}+y_2^{(2)}I \\ y_3^{(1)}+y_3^{(2)}I \end{pmatrix} = \begin{pmatrix} 3+10I \\ 2+5I \\ 4+I \end{pmatrix}$$

In this problem,

$$P_1 = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 3 & 1 \\ 1 & 3 & 3 \end{pmatrix},\ P_2 = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 3 & 3 \\ 4 & 2 & 4 \end{pmatrix},\ Q_1 = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix},\ Q_2 = \begin{pmatrix} 10 \\ 5 \\ 1 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} y_1^{(1)} \\ y_2^{(1)} \\ y_3^{(1)} \end{pmatrix}, \ Y_2 = \begin{pmatrix} y_1^{(2)} \\ y_2^{(2)} \\ y_3^{(2)} \end{pmatrix}.$$

Let $P = LU$, where $L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$ and $U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$.

The ijth element of L is $l_{ij} = l_{ij}^{(1)} + l_{ij}^{(2)} I$. The ijth element of U are $u_{ij} = u_{ij}^{(1)} + u_{ij}^{(2)} I$ for $i \neq j$ and $u_{ij} = 1 + 0I$ for $i = j$.

Now ,

$$P = \begin{pmatrix} 2+I & 1+3I & 3+2I \\ 4+2I & 3+3I & 1+3I \\ 1+4I & 3+2I & 3+4I \end{pmatrix} = LU$$

$$= \begin{pmatrix} l_{11}^{(1)} + l_{11}^{(2)} I & 0+0I & 0+0I \\ l_{21}^{(1)} + l_{21}^{(2)} I & l_{22}^{(1)} + l_{22}^{(2)} I & 0+0I \\ l_{31}^{(1)} + l_{31}^{(2)} I & l_{32}^{(1)} + l_{32}^{(2)} I & l_{33}^{(1)} + l_{33}^{(2)} I \end{pmatrix} \begin{pmatrix} 1+0I & u_{12}^{(1)} + u_{12}^{(2)} & u_{13}^{(1)} + u_{13}^{(2)} \\ 0+0I & 1+0I & u_{23}^{(1)} + u_{23}^{(2)} \\ 0+0I & 0+0I & 1+0I \end{pmatrix}$$

After multiplying RHS of the above equation, compairing both sides, we get $l_{11}^{(1)} = 2, l_{11}^{(2)} = 1, l_{21}^{(1)} = 4, l_{21}^{(2)} = 2, l_{31}^{(1)} = 1, l_{31}^{(1)} = 4 l_{22}^{(1)} = 1, l_{22}^{(2)} = -3, l_{32}^{(1)} = \frac{5}{2}, l_{32}^{(1)} = -\frac{25}{6}, l_{33}^{(1)} = 14, l_{33}^{(1)} = -\frac{31}{3}, u_{12}^{(1)} = \frac{1}{2}, u_{12}^{(2)} = \frac{5}{6}, u_{13}^{(1)} = \frac{3}{2}, u_{13}^{(2)} = 2, u_{23}^{(1)} = -5, u_{23}^{(2)} = 8.$

Therefore,

$$L = L_1 + L_2 I$$
$$= \begin{pmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & \frac{5}{2} & 14 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 4 & -\frac{25}{6} & -\frac{31}{3} \end{pmatrix} I$$

and

$$U = U_1 + U_2 I$$
$$= \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix} I.$$

Now, we compute $L_1 + L_2$ and $U_1 + U_2$.

So,

$$L_1+L_2=\begin{pmatrix}3 & 0 & 0\\ 6 & -2 & 0\\ 5 & -\frac{5}{3} & \frac{11}{3}\end{pmatrix}$$

and,

$$U_1+U_2=\begin{pmatrix}1 & \frac{4}{3} & \frac{5}{3}\\ 0 & 1 & 3\\ 0 & 0 & 1\end{pmatrix}.$$

Now, we compute L^{-1} and U^{-1}. So,

$$L^{-1}=L_1^{-1}+[(L_1+L_2)^{-1}-L_1^{-1}]I$$

$$=\begin{pmatrix}\frac{1}{2} & 0 & 0\\ -2 & 1 & 0\\ \frac{9}{28} & -\frac{5}{28} & \frac{1}{14}\end{pmatrix}+\left[\begin{pmatrix}\frac{1}{3} & 0 & 0\\ 1 & -\frac{1}{2} & 0\\ 0 & -\frac{5}{22} & \frac{3}{11}\end{pmatrix}-\begin{pmatrix}\frac{1}{2} & 0 & 0\\ -2 & 1 & 0\\ \frac{9}{28} & -\frac{5}{28} & \frac{1}{14}\end{pmatrix}\right]I$$

$$=\begin{pmatrix}\frac{1}{2} & 0 & 0\\ -2 & 1 & 0\\ \frac{9}{28} & -\frac{5}{28} & \frac{1}{14}\end{pmatrix}+\begin{pmatrix}-\frac{1}{6} & 0 & 0\\ 3 & -\frac{3}{2} & 0\\ -\frac{9}{28} & -\frac{15}{308} & \frac{31}{154}\end{pmatrix}I$$

$$=\begin{pmatrix}\frac{1}{2}-\frac{1}{6}I & 0+0I & 0+0I\\ -2+3I & 1-\frac{3}{2}I & 0+0I\\ \frac{9}{28}-\frac{9}{28}I & -\frac{5}{28}-\frac{15}{308}I & \frac{1}{14}-\frac{31}{154}I\end{pmatrix}$$

and

$$U^{-1}=U_1^{-1}+[(U_1+U_2)^{-1}-U_1^{-1}]I$$

$$=\begin{pmatrix}1 & -\frac{1}{2} & -4\\ 0 & 1 & 5\\ 0 & 0 & 1\end{pmatrix}+\left[\begin{pmatrix}1 & -\frac{4}{3} & \frac{7}{3}\\ 0 & 1 & -3\\ 0 & 0 & 1\end{pmatrix}-\begin{pmatrix}1 & -\frac{1}{2} & -4\\ 0 & 1 & 5\\ 0 & 0 & 1\end{pmatrix}\right]I$$

$$=\begin{pmatrix}1 & -\frac{1}{2} & -4\\ 0 & 1 & 5\\ 0 & 0 & 1\end{pmatrix}+\begin{pmatrix}0 & -\frac{5}{6} & \frac{19}{3}\\ 0 & 0 & -8\\ 0 & 0 & 0\end{pmatrix}I$$

$$=\begin{pmatrix}1+0I & -\frac{1}{2}-\frac{5}{6}I & -4+\frac{19}{3}I\\ 0+0I & 1+0I & 5-8I\\ 0+0I & 0+0I & 0+0I\end{pmatrix}$$

Now, from the equation $LZ = Q$,we have

$$Z = L^{-1}Q$$

$$= \begin{pmatrix} \frac{1}{2}-\frac{1}{6}I & 0+0I & 0+0I \\ -2+3I & 1-\frac{3}{2}I & 0+0I \\ \frac{9}{28}-\frac{9}{28}I & -\frac{5}{28}-\frac{15}{308}I & \frac{1}{14}-\frac{31}{154}I \end{pmatrix} \begin{pmatrix} 3+10I \\ 2+5I \\ 4+I \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2}+\frac{17}{6}I \\ -4+\frac{27}{2}I \\ \frac{25}{28}-\frac{345}{308}I \end{pmatrix}$$

Again, from the equation $UY = Z$,we have

$$Y = U^{-1}Z$$

$$= \begin{pmatrix} 1+0I & -\frac{1}{2}-\frac{5}{6}I & -4+\frac{19}{3}I \\ 0+0I & 1+0I & 5-8I \\ 0+0I & 0+0I & 0+0I \end{pmatrix} \begin{pmatrix} \frac{3}{2}+\frac{17}{6}I \\ -4+\frac{27}{2}I \\ \frac{25}{28}-\frac{345}{308}I \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{14}-\frac{2031}{231}I \\ \frac{13}{28}+\frac{2993}{308}I \\ \frac{25}{28}-\frac{345}{308}I \end{pmatrix}$$

Thus, the final solution is

$$Y = \begin{pmatrix} -\frac{1}{14}-\frac{2031}{231}I \\ \frac{13}{28}+\frac{2993}{308}I \\ \frac{25}{28}-\frac{345}{308}I \end{pmatrix}.$$

7. Conclusion

In conclusion, the exploration of neutrosophic numbers and real neutrosophic matrices offers valuable insights into handling uncertainty and vagueness within mathematical frameworks. The operations defined on real neutrosophic matrices, such as addition, multiplication, and scalar multiplication, extend traditional linear algebra to accommodate the nuanced representation of uncertainty through truth-membership, indeterminacy-membership, and falsity-membership degrees.

The study of inverse operations on real neutrosophic matrices underscores the complexity and significance of managing uncertainty in matrix computations. Techniques for computing the inverse under various conditions contribute to the robustness of neutrosophic algebra, enabling practitioners to address real-world problems where precise information may be lacking or incomplete.

Moreover, the application of LU decomposition to solve systems of real neutrosophic linear algebraic equations demonstrates practical methodologies for extracting meaningful solutions from matrices with neutrosophic entries. By decomposing matrices into lower and upper triangular forms and employing forward and backward substitution, LU decomposition provides an efficient computational approach to resolve systems affected by uncertainty.

Overall, the methodologies and techniques discussed in this paper illustrate the potential of neutrosophic mathematics in enhancing decision-making processes and modeling scenarios where uncertainty is inherent. Future research could further explore advanced applications and refine computational algorithms to broaden the practical utility of neutrosophic algebra across diverse fields, from engineering to economics, ensuring robust solutions in the face of uncertain data.

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