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# Solution of Neutrosophic Linear Systems of Equations by Gauss Elimination Method

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***Abstract.*** Neutrosophic sets and numbers are extensions of classical set theory and real numbers, respectively, introduced by Florentin Smarandache in 1995 to address indeterminate, vague, or imprecise information. Neutrosophic numbers are defined by three components: truth-membership, indeterminacy-membership, and falsity-membership degrees, each ranging from 0 to 1. These components encapsulate the degrees of truth, indeterminacy, and falsity associated with a statement or quantity.

Real neutrosophic numbers are a subset of neutrosophic numbers where the truth-membership degree is 1. This simplifies calculations and interpretations, making them closer to classical real numbers while still accounting for indeterminacy and falsity. The Gauss elimination method, when applied to neutrosophic linear equations, involves solving systems of linear equations where the coefficients and constants are represented as real neutrosophic numbers. Adapting traditional Gaussian elimination incorporates the uncertainties and indeterminacies inherent in neutrosophic data, ensuring more robust solutions in uncertain contexts. Overall, neutrosophic sets and numbers, along with their operations and applications such as solving neutrosophic linear equations offer a versatile framework for managing incomplete, imprecise, or contradictory information, making them highly valuable in decision-making and computational sciences.

***Keywords:*** Fuzzy system of equations; Neutrosophic numbers, Gauss-elimination method

***AMS Mathematics Subject Classification (2010):*** 90B05

## 1. Introduction

Atanassov's Intuitionistic Fuzzy Set (IFS) extends the concept of Fuzzy Sets (FS) by introducing additional parameters to better capture uncertainty. While FS primarily relies on a membership function to indicate the degree of an element belonging to a set, IFS incorporates two parameters: a membership degree and a non-membership degree. These parameters must satisfy the condition that their sum is less than or equal to 1. Numerous studies have been conducted on fuzzy and neutrosophic matrix theories (see references).

This extension is especially useful in cases where traditional fuzzy sets cannot fully address uncertainty. Intuitionistic Fuzzy Sets provide a more refined approach, allowing both membership and non-membership (or hesitation) to be explicitly represented. When the sum of the membership and non-membership degrees equals 1 for all elements, the IFS reduces to a classic fuzzy set.

In essence, Intuitionistic Fuzzy Sets are employed when fuzzy sets are insufficient to model uncertainty or when additional information is needed beyond what traditional fuzzy sets can provide. This makes IFS a valuable tool in fields where uncertainty must be handled effectively.

Neutrosophic Sets (NS), introduced by Smarandache, take this advancement further by addressing even more complex forms of uncertainty. NS are defined by three parameters: a truth membership function (t), an indeterminacy membership function (i), and a falsity membership function (f), which represent the degrees of truth, indeterminacy, and falsity.

The strength of Neutrosophic Sets lies in their ability to manage information that is not only uncertain but also incomplete and inconsistent. This makes them particularly useful in real-world situations where traditional fuzzy sets or IFS fall short due to the nature of the available information.

Neutrosophic Sets generalize several existing frameworks, including classical fuzzy sets, interval-valued fuzzy sets, and IFS. Over time, researchers have extended NS into various branches, such as interval neutrosophic sets and generalized neutrosophic soft sets, each addressing specific types of uncertainty in different applications.

Additionally, the development of Neutrosophic Fuzzy Numbers (NFNs) and their associated matrices has further expanded the tools available for managing complex uncertainty across various fields. This ongoing evolution highlights the continuous effort to refine mathematical frameworks to better address real-world problems, where information is often uncertain, incomplete, or inconsistent.

## 2. Neutrosophic number

Samrandche first proposed a concept of **neutrosophic number** which consists of the determinant part and the indeterminate part. It is usually denoted by $N = a + bI$, where $a$ and $b$ are real numbers and $I$ is the indeterminacy such that $I^2 = I, I.0 = 0$ and $\frac{I}{I}$ are undefined. We call $N = a + bI$ as a pure neutrosophic number if $a = 0$. For more details discussions see.

For example, we consider a neutrosophic number $N = 5 + 3I$. If $I \in [0,0.02]$, then it is equivalent to $N \in [5,5.06]$ for $N \geq 5$. This means the determinant part is $5$, whereas the indeterminacy part is $3I$ for $I \in [0,0.02]$, which means the possibility for number $N$ to be a little bigger than $5$.

Note that this number looks like a complex number, but, see that here $I^2 = I$, not $-1$ like a complex number.

The three basic operators defined on neutrosophic numbers $P = p_1 + q_1 I$ and $Q = p_2 + q_2 I$ are as follows:

(i) $P + Q = (p_1 + p_2) + (q_1 + q_2)I$

(ii) $P - Q = (p_1 - p_2) + (q_1 - q_2)I$

(iii) $P \times Q = p_1 p_2 + (p_1 q_2 + q_1 p_2 + q_1 q_2)I$

In real neutrosophic algebra, we denote $K$ as the neutrosophic field over some neutrosophic vector spaces. We call the smallest field generated by $K \cup I$ or $K(I)$ to be the neutrosophic field for it involves the indeterminacy factor in it, where $I$ has the special property that $I^n = I, I + I = I$ and if $t \in K$ be some scalar then $t.I = tI, 0.I = 0$. Thus, we generally denote neutrosophic field $K(I)$ generated by $K \cup I$, i.e. $K(I) = \langle K \cup I \rangle$.

Thus, for different fields of algebra, we can define several types of neutrosophic field generated by the field of neutrosophic vector space.

a) **R** be the field of real numbers, then the neutrosophic field generated by $\langle R \cup I \rangle$ is $R(I)$ and $R \subset R(I)$.
b) **Q** be the field of rational number, then the neutrosophic field generated by $\langle Q \cup I \rangle$ is $Q(I)$ and $Q \subset Q(I)$.
c) **Z** be the field of integers, then the neutrosophic field generated by $\langle Z \cup I \rangle$ is $\mathbf{Z}(I)$.

So several types of neutrosophic numbers are available in the literature. However, many authors are confused about this classification. In this chapter, we will discuss first $\mathbf{I}(I)$, the fuzzy neutrosophic numbers (referred to as FNNs), and then we consider the matrix over real neutrosophic numbers (RNN) $\mathbf{R}(I)$.

## 3. Real neutrosophic matrix

Here we consider the neutrosophic matrix over real numbers based on the work of Smarandache [30]. So it is referred to as a real neutrosophic matrix and is abbreviated by RNM. For details of this matrix see [12,13].

The neutrosophic number over the field of real/complex numbers is defined in the form $a = a_1 + b_1 I$, where $a_1, a_2$ are real or complex numbers and $I$ is the indeterminacy [6].

An RNM is defined as in FNM, i.e. of the form $M = M_1 + M_2 I$ where $M_1$ and $M_2$ are real matrices. The set of real matrices of order $m \times n$ is denoted by $\mathcal{M}_{mn}^{\mathbb{R}}$ and that of order $n \times n$ by $\mathcal{M}_n^{\mathbb{R}}$. The identity RNM of order $n \times n$ is denoted by $U_n$, all diagonal elements are 1 and all other elements are 0.

The null and identity matrices of order $3 \times 3$ are

$$O_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ and \ \ U_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The basic operations on RNMs $M = M_1 + M_2 I$ and $N = N_1 + N_2 I$ are

(i) $M + N = (M_1 + N_1) + (M_2 + N_2)I$

(ii) $M - N = (M_1 - N_1) + (M_2 - N_2)I$

(iii) $MN = (M_1 N_1) + (M_2 N_1 + M_1 N_2 + M_2 N_2)I$. In this case also, $I^n = I^2 = I$ for any positive integer $n$.

Assumed that the order of RNMs is compatible with the appropriate operations.

This matrix looks like a complex matrix, but see that here $I$ represents indeterminacy, not complex $i = \sqrt{-1}$. Also, $I^n = I$ for all positive integer $n$, which is not true for complex numbers.

Let $M = M_1 + M_2 I$ be a RNM, where $M_1, M_2 \in \mathcal{M}_n^{\mathbb{R}}$. Then its determinant is denoted by $det(M)$ or $|M|$ and its value is given by

$$det(M) = det(M_1) + I[det(M_1 + M_2) - det(M_1)].$$

Note that this formula is unlike to the determinant of conventional matrix. But, this definition follows the rules of conventional matrices.

## 4. System of linear equations

Let us consider a system of linear equations

$$MX = N, i.e. (M_1 + M_2 I)(X_1 + X_2 I) = (N_1 + N_2 I) \qquad (3.1)$$

where $M_1, M_2 \in \mathcal{M}_n^{\mathbb{R}}, X_1, X_2, N_1, N_2 \in \mathcal{M}_{n_1}^{\mathbb{R}}$.

The ijth element of M is $m_{ij} = m_{ij}^{(1)} + m_{ij}^{(2)} I$, jth element of X and N are $x_j = x_j^{(1)} + x_j^{(2)} I$ and $n_j = n_j^{(1)} + n_j^{(2)} I$ respectively.

Then the matrices $M_1$ and $M_2$ are $M_1 = (M_{ij}^{(1)})_{n\times n}$ and $M_2 = (M_{ij}^{(2)})_{n\times n}$ respectively. The vectors $X_1, X_2, N_1, N_2$ are

$$X_1 = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ . \\ . \\ . \\ x_n^{(1)} \end{pmatrix}, X_2 = \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \\ . \\ . \\ . \\ x_n^{(2)} \end{pmatrix}, N_1 = \begin{pmatrix} n_1^{(1)} \\ n_2^{(1)} \\ . \\ . \\ . \\ n_n^{(1)} \end{pmatrix}, Q_1 = \begin{pmatrix} n_1^{(2)} \\ n_2^{(2)} \\ . \\ . \\ . \\ n_n^{(2)} \end{pmatrix}.$$

From Eq. (3.1),

$$M_1 X_1 + [(M_1 + M_2)(X_1 + X_2) M_1 X_1] I = N_1 + N_2 I \qquad (3.2)$$

$$M_1 X_1 = N_1 \qquad (3.3)$$

$$(M_1 + M_2)(X_1 + X_2) - M_1 X_1 = N_2, \text{ or } (M_1 + M_2)(X_1 + X_2) = N_1 + N_2 \qquad (3.4)$$

Eq. (3.3), gives the vector $X_1$ and Eq. (3.4) gives $X_1 + X_2$ and the final solution of the Eq. (3.1) is given by $X = X_1 + X_2 I$.

The Gauss-eleiminaion method is used to solve the system of Eqs. (3.3) and (3.4).

## 5. Gauss Elimination method

It is a direct method used to solve systems of linear equations. Its primary goal is to transform the system into an equivalent upper triangular form through a sequence of row operations. Here's a brief introduction to the method:

Let a system of linear equations represented as $MX = N$, where M is the coefficient matrix, X is the vector of unknowns, and N is the vector of constants on the right-hand side.

The augmented matrix is $[M|N]$. We obtain the solution by the following way-

a) Forward Elimination: Reduce the augmented matrix to upper triangular form (also known as row echelon form) by eliminating coefficients below the main diagonal. This is done by subtracting suitable multiples of one equation from another.

b) Back Substitution: Once the augmented matrix is in upper triangular form, solve for each unknown starting from the last row and moving upwards. This involves substituting back the values of the known variables into earlier equations to find subsequent unknowns.

Once back substitution is complete, the resulting matrix $X$ provides the solution to the system of equations $MX = N$. Gauss elimination is a numerical method in numerical linear algebra due to its simplicity and effectiveness in solving systems of equations, including those with many variables and equations.

**Note:** Pivoting: To ensure numerical stability and avoid division by zero or small numbers, pivot elements (the main diagonal elements) are often selected as the largest available in the column being operated on (partial pivoting) or throughout the matrix (complete pivoting).

**General form of the equation**

Let's consider a system of linear equations with neutrosophic coefficients:

$$\begin{aligned}
(a_{11}^{(1)} + a_{11}^{(2)}I)x_1 + (a_{12}^{(1)} + a_{12}^{(2)}I)x_2 + \cdots + (a_{1n}^{(1)} + a_{1n}^{(2)}I)x_n &= b_1^{(1)} + b_1^{(2)}I \\
(a_{21}^{(1)} + a_{21}^{(2)}I)x_1 + (a_{22}^{(1)} + a_{22}^{(2)}I)x_2 + \cdots + (a_{2n}^{(1)} + a_{2n}^{(2)}I)x_n &= b_2^{(1)} + b_2^{(2)}I \\
&\vdots \\
(a_{m1}^{(1)} + a_{m1}^{(2)}I)x_1 + (a_{m2}^{(1)} + a_{m2}^{(2)}I)x_2 + \cdots + (a_{mn}^{(1)} + a_{mn}^{(2)}I)x_n &= b_m^{(1)} + b_m^{(2)}I
\end{aligned} \tag{3.5}$$

where $a_{ij}^{(1)}, a_{ij}^{(2)}, b_i^{(1)}, b_i^{(2)}$ are real numbers and I represents the indeterminate component. The Eq. (3.5) can be written in matrix form

$$\begin{bmatrix} a_{11}^{(1)} + a_{11}^{(2)}I & a_{12}^{(1)} + a_{12}^{(2)}I & \dots & a_{1n}^{(1)} + a_{1n}^{(2)}I & \\ a_{21}^{(1)} + a_{21}^{(2)}I & a_{22}^{(1)} + a_{22}^{(2)}I & \dots & a_{2n}^{(1)} + a_{2n}^{(2)}I & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1}^{(1)} + a_{m1}^{(2)}I & a_{m2}^{(1)} + a_{m2}^{(2)}I & \dots & a_{mn}^{(1)} + a_{mn}^{(2)}I & \end{bmatrix} \begin{bmatrix} x_1^{(1)} + x_1^{(2)}I \\ x_2^{(1)} + x_2^{(2)}I \\ \vdots \\ x_m^{(1)} + x_m^{(2)}I \end{bmatrix} = \begin{bmatrix} b_1^{(1)} + b_1^{(2)}I \\ b_2^{(1)} + b_2^{(2)}I \\ \vdots \\ b_m^{(1)} + b_m^{(2)}I \end{bmatrix}$$

$$\text{i.e.} \quad (A_1 + A_2I)(X_1 + X_2I) = (B_1 + B_2I)$$

From this equation, we get two equations (3.3) and (3.4). Solving these two equations by Gauss elimination, we get $X_1$ and $X_1 + X_2$. The final solution of the Eq.(3.1) is $X = X_1 + X_2I$

## 6. Illustration

Let us consider the following Neutrosophic system of equations:

$$\left.\begin{array}{ccccccc} (0+2I)x_1 & + & (1+3I)x_2 & + & (5+I)x_3 & = & 3+2I \\ (4+6I)x_1 & + & (2+4I)x_2 & + & (0+7I)x_3 & = & 4+I \\ (5+2I)x_1 & + & (0+4I)x_2 & + & (0+3I)x_3 & = & 5+3I \end{array}\right\} \tag{6.1}$$

By Gauss elimination method.

The given equation is written in matrix form

$$\begin{bmatrix} (0+2I) & (1+3I) & (5+I) \\ (4+6I) & (2+4I) & (0+7I) \\ (5+2I) & (0+4I) & (0+3I) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3+2I \\ 4+I \\ 5+3I \end{bmatrix}$$

That is

$$\begin{bmatrix} (0+2I) & (1+3I) & (5+I) \\ (4+6I) & (2+4I) & (0+7I) \\ (5+2I) & (0+4I) & (0+3I) \end{bmatrix} \begin{bmatrix} x_1^{(1)} + x_1^{(2)}I \\ x_2^{(1)} + x_2^{(2)}I \\ x_3^{(1)} + x_3^{(2)}I \end{bmatrix} = \begin{bmatrix} 3+2I \\ 4+I \\ 5+3I \end{bmatrix}$$

That is,

$$(M_1 + M_2 I)(X_1 + X_2 I) = (N_1 + N_2 I) \tag{6.2}$$

$$\text{i.e.} \quad (MX) = N \tag{6.3}$$

where

$$M_1 = \begin{bmatrix} 0 & 1 & 5 \\ 4 & 2 & 0 \\ 5 & 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 2 & 3 & 1 \\ 6 & 4 & 7 \\ 2 & 4 & 3 \end{bmatrix}, X_1 = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} X_2 = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix}, N_1 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, N_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

The real system of equations are

$$M_1 X_1 = N_1 \tag{6.4}$$

$$(M_1 + M_2)(X_1 + X_2) = N_1 + N_2 \tag{6.5}$$

From equation (6.4),

$$\begin{bmatrix} 0 & 1 & 5 \\ 4 & 2 & 0 \\ 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

By Gauss Elimination method,

$$\begin{bmatrix} 0 & 1 & 5 & | & 3 \\ 4 & 2 & 0 & | & 4 \\ 5 & 0 & 0 & | & 5 \end{bmatrix}$$

We multiply 3rd row by $(-4/5)$ and add to the 2nd row. Then the augmented matrix becomes,

$$\begin{bmatrix} 0 & 1 & 5 & | & 3 \\ 0 & 2 & 0 & | & 0 \\ 5 & 0 & 0 & | & 5 \end{bmatrix}$$

Again we multiply 2nd row by $(-1/2)$ and add to the 1st row. Then the augmented matrix becomes,

$$\begin{bmatrix} 0 & 0 & 5 & | & 3 \\ 0 & 2 & 0 & | & 0 \\ 5 & 0 & 0 & | & 5 \end{bmatrix}$$

Therefore,

$$5x_1^{(1)} = 5$$
$$2x_2^{(1)} = 0$$
$$5x_3^{(1)} = 3$$

i.e., $X_1 = \begin{bmatrix} 1 \\ 0 \\ 3/5 \end{bmatrix}$

From equation (6.5),

$$\begin{bmatrix} 2 & 4 & 6 \\ 10 & 6 & 7 \\ 7 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1^{(1)} + x_1^{(2)} \\ x_2^{(1)} + x_2^{(2)} \\ x_3^{(1)} + x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 8 \end{bmatrix}$$

By Gauss Elimination method,

$$\begin{bmatrix} 2 & 4 & 6 & | & 5 \\ 10 & 6 & 7 & | & 5 \\ 7 & 4 & 3 & | & 8 \end{bmatrix}$$

We multiply 1st row by $(-10/2)$ and $(-7/2)$ successively and add to the 2nd row and 3rd row respectively. Then the augmented matrix becomes,

$$\begin{bmatrix} 2 & 4 & 6 & | & 5 \\ 0 & 14 & 23 & | & 20 \\ 0 & 10 & 18 & | & 19/2 \end{bmatrix}$$

We multiply 2nd row by $(-10/14)$ and add to the 3rd row. Then the augmented matrix becomes,

$$\begin{bmatrix} 2 & 4 & 6 & | & 5 \\ 0 & 14 & 23 & | & 20 \\ 0 & 0 & 11/7 & | & -67/14 \end{bmatrix}$$

Therefore,

$$x_3^{(1)} + x_3^{(2)} = -67/22$$
$$i.e, x_3^{(2)} = -491/110$$

$$14(x_2^{(1)} + x_2^{(2)}) + 23(x_3^{(1)} + x_3^{(2)}) = 20$$
$$i.e., \quad x_2^{(1)} + x_2^{(2)} = 283/44$$
$$i.e., \quad x_2^{(2)} = 283/44$$

$$2(x_1^{(1)} + x_1^{(2)}) + 4(x_2^{(1)} + x_2^{(2)}) + 6(x_3^{(1)} + x_3^{(2)}) = 5$$
$$i.e., \quad x_1^{(1)} + x_1^{(2)} = -27/22$$
$$i.e., \quad x_1^{(2)} = -49/22$$

i.e, $X_2 = \begin{bmatrix} -491/110 \\ 283/44 \\ -49/22 \end{bmatrix}$

Then the final solution is

$$X = \begin{bmatrix} x_1^{(1)} + x_1^{(2)} I \\ x_2^{(1)} + x_2^{(2)} I \\ x_3^{(1)} + x_3^{(2)} I \end{bmatrix} = \begin{bmatrix} 1 - 49/22I \\ 283/44I \\ 3/5 - 401/110I \end{bmatrix}$$

**Verification of the solution**

The given equation (6.1) can be written as

$$M_1 X_1 + [(\mathrm{M}_1 + M_2)(X_1 + X_2) - M_1 X_1] I$$

$$= \begin{bmatrix} 0 & 1 & 5 \\ 4 & 2 & 0 \\ 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3/5 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 \\ 10 & 6 & 7 \\ 7 & 4 & 3 \end{bmatrix} \begin{bmatrix} -27/22 \\ 283/44 \\ -67/22 \end{bmatrix} I - \begin{bmatrix} 0 & 1 & 5 \\ 4 & 2 & 0 \\ 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3/5 \end{bmatrix} I$$

$$= \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + \begin{pmatrix} 5 \\ 5 \\ 8 \end{pmatrix} I - \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} I = \begin{bmatrix} 3 + 2I \\ 4 + I \\ 5 + 3I \end{bmatrix}$$

**Lemma 1.** *If $M_1$ and $M_1 + M_2$ are non-singular then the solution of Neutrosophic system of equations is unique.*

**Proof:** Assume that $X_1^{(1)}, X_2^{(1)}$ be two solutions of

$$M_1 X_1 = N_1$$

Therefore,

$$M_1 X_1^{(1)} = N_1$$

and

$$M_1 X_2^{(1)} = N_1$$

Now,

$$M_1 (X_1^{(1)} - X_2^{(1)}) = 0$$

Let, $X = X_1^{(1)} - X_2^{(1)}$.

So, $M_1 X = 0$.

Since, $M_1$ is non-singular then the homogeneous system $M_1 X = 0$ has only solution $X = 0$.

Hence, $X_1^{(1)} - X_2^{(1)} = 0$

i.e, $X_1^{(1)} = X_2^{(1)}$.

Then, the solution $X_1$ is unique.

Similarly, let $X_1^{(1)} + X_2^{(1)}$, $X_1^{(2)} + X_2^{(2)}$ be two solutions of

$$(M_1 + M_2)(X_1 + X_2) = N_1 + N_2$$

Therefore,

$$(M_1 + M_2)(X_1^{(1)} + X_2^{(1)}) = N_1 + N_2 \qquad (6.6)$$

and

$$(M_1 + M_2)(X_1^{(2)} + X_2^{(2)}) = N_1 + N_2 \qquad (6.7)$$

Subtract (6.7) from (6.6), we have

$$(M_1 + M_2)((X_1^{(1)} + X_2^{(1)}) - (X_1^{(2)} + X_2^{(2)})) = 0 \qquad (6.8)$$

Let, $X_1 + X_2 = (X_1^{(1)} + X_2^{(1)}) - (X_1^{(2)} + X_2^{(2)})$
So, $(M_1 + M_2)(X_1 + X_2) = 0$.
Since, $M_1 + M_2$ is non-singular then the homogeneous system
$(M_1 + M_2)(X_1 + X_2) = 0$ has only solution $(X_1 + X_2) = 0$.
Hence, $(X_1^{(1)} + X_2^{(1)}) - (X_1^{(2)} + X_2^{(2)})$
i.e, $X_1^{(1)} + X_2^{(1)} = X_1^{(2)} + X_2^{(2)}$
Then, the solution $X_1 + X_2$ is unique.
Therefore, the solution of the Neotrosophic system of equation is unique.

**Lemma 2.** *If $M_1$ and $M_1 + M_2$ are non-singular. Then $M_2$ may or may not be non-singular.*
**Proof:** Let $M_1$ and $M_1 + M_2$ be non-singular, i.e. $|M_1| = 0$ and $|M_1 + M_2| = 0$
Let $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $M_1 + M_2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$
Here $M_1$ and $M_1 + M_2$ are non-singular.
In this case,

$$M_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} |M_2| = 4 - 1 = 3 \neq 0$$

So, $M_2$ is also non-singular.
Let $M_1 = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ and $M_1 + M_2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$
Here $M_1$ and $M_1 + M_2$ are non-singular.
In this case, $M_2 = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} |M_2| = 0 - 0 = 0$.
So, $M_2$ is singular.

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**Authors' Contributions.** It is the author's full contribution.

## REFERENCES

1. A.K.Adak, M.Bhowmik and M.Pal, Some properties of generalized intuitionistic fuzzy nilpotent matrices over distributive lattice, *Fuzzy Information and Engineering*, 4(4) (2012) 371-387.
2. K.T.Atanassov, Intuitionistic fuzzy sets, VII ITKR's Session, Sofia, Bulgarian (1983).
3. M.Bhowmik and M.Pal, Some results on intuitionistic fuzzy matrices and intuitionistic circulant fuzzy matrices, *International Journal of Mathematical Sciences*, 7(1-2) (2008) 177-192.
4. M.Dhar, S.Broumi and F.Smarandache, A note on square neutrosophic fuzzy matrices. *Neutrosophic Sets and Systems*, 3 (2014)37-41.
5. I. Deli and S.Broumi, Neutrosophic soft matrices and NSM-decision making, *Journal of Intelligent and Fuzzy Systems*, 28(5) (2015) 2233-2241.
6. V. W. B. Kandasamy and F. Smarandache, Some Neutrosophic Algebraic Structures

and Neutrosophic Algebraic Structures, Hexis, Phoenix, Arizona, 2006.
7. K.H.Kim and F.W.Roush, Generalized fuzzy matrices, *Fuzzy Sets and Systems*, 4 (1980) 293-315.
8. R.Mahapatra, S.Samanta, M.Pal and Q.Xin, Link prediction in social networks by neutrosophic graph, *International Journal of Computational Intelligence Systems*, 13(1) (2020) 1699-1713. DOI: 10.2991/ijcis.d.201015.002
9. R.Mahapatra, S.Samanta and M.Pal, Generalized neutrosophic planar graphs and its application, *Journal of Applied Mathematics and Computing* 65 (2021) 693-712. DOI: 10.1007/s12190-020-01411-x
10. R.Mahapatra, S.Samanta and M.Pal, Edge colouring of neutrosophic graphs and its application in detection of phishing website, *Discrete Dynamics in Nature and Society* 2022, 1149724. DOI: 10.1155/2022/1149724
11. R.Mahapatra, S.Samanta and M.Pal, Detecting influential node in a network using neutrosophic graph and its application, *Soft Computing* 27(14) (2023) 9247-9260. DOI: 10.1007/s00500-023-08234-5
12. M. Pal, *Recent Developments of Fuzzy Matrix Theory and Applications*, Springer, Switzerland, 2024. DOI: 10.1007/978-3-031-56936-4
13. M. Pal, Neutrosophic Matrix and Neutrosophic Fuzzy Matrix. In: Recent Developments of Fuzzy Matrix Theory and Applications. Springer, Cham. https://doi.org/10.1007/978-3-031-56936-4_10
14. M.Pal, S.K.Khan and A.K.Shyamal, Intuitionistic fuzzy matrices, *Notes on Intuitionistic Fuzzy Sets*, 8(2) (2002) 51-62.
15. M. Pal, Interval-valued fuzzy matrices with interval-valued rows and columns, *Fuzzy Information and Engineering*, 7(3) (2015) 335-368.
16. M. Pal, Fuzzy matrices with fuzzy rows and columns, *Journal of Intelligent and Fuzzy Systems*, 30(1) (2016) 561-573. DOI: 10.3233/IFS-151780
17. M. Pal and S. Mondal, Bipolar fuzzy matrices, *Soft Computing*, 23(20) (2019) 9885-9897. DOI: 10.1007/s00500-019-03912-9
18. M. Pal, Intuitionistic fuzzy matrices with uncertain rows and columns and their application in decision making problem, *Journal of Multiple-Valued Logic and Soft Computing*, 35 (2020) 281-306.
19. M.Palanikumar, C.Jana, B.Sarkar and M.Pal, q-rung logarithmic Pythagorean neutrosophic vague normal aggregating operators and their applications in agricultural robotics, *AIMS Mathematics*, 8(12) (2023) 30209-30243.
DOI: 10.3934/math.20231544
20. J.J.Peng, and J. Wang, Multi-valued neutrosophic sets and its application in multi-criteria decision-making problems, *Neutrosophic Sets and Systems*, 10 (2015) 3-17.
21. A. A. Salama and S. A. Al-Blowi, Neutrosophic set and neutrosophic topological spaces, *IOSR Journal of Math.*, 3 (4) (2012) 31-35.
22. A. K. Shyamal and M. Pal, Two new operators on fuzzy matrices, *Journal of Applied Mathematics and Computing*, 15 (2004) 91-107. https://doi.org/10.1007/BF02935748
23. A. K. Shyamal and M. Pal, Interval-valued fuzzy matrices, *The Journal of Fuzzy Mathematics*, 14(3) (2006) 583-604.
24. A. K. Shyamal, M. Pal, Triangular fuzzy matrices, *Iranian Journal of Fuzzy Systems*, 4(1) (2007) 75-87.
25. F. Smarandache, Neutrosophy: neutrosophic probability, set, and logic: analytic

synthesis & synthetic analysis. Rehoboth: American Research Press, (1998).

26. F. Smarandache, A unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic. Rehoboth: American Research Press, (1999).
27. F. Smarandache, A unifying field in logics: neutrosophic logics. *Multiple Valued Logic*, 8(3) (2002) 385-438.
28. F. Smarandache, Neutrosophic set, a generalization of intuitionistic fuzzy sets, *International Journal of Pure and Applied Mathematics*, 24 (2005) 287-297.
29. F. Smarandache, Neutrosophic set- a generalization of intuitionistic fuzzy set. Granular Computing, *IEEE, International Conference*, (2006) 38-42. doi:10.1109/GRC.2006.1635754.
30. F. Smarandache, Neutrosophic set-a generalization of intuitonistic fuzzy set, *Journal of Defence Resources Management*, 1(1) (2010) 107-116.
31. H. Wang, F.Smarandache, Y.Q.Zhang and R.Sunderraman, Interval neutrosophic sets and logic: theory and applications in computing. Arizona, Hexis (2005).
32. H. Wang et al., Single valued neutrosophic sets, Proc. of 10th Int. Conf. on Fuzzy Theory and Technology, Salt Lake City, Utah, July 21-26 (2005).
33. H. Wang, F. Smarandache, Y.Q. Zhang and R. Sunderraman, Single valued neutrosophic sets, *Multispace and Multistructure*, 4 (2010) 410-413.
34. J.J. Wang and X.E. Li, TODIM method with multi-valued neutrosophic sets, *Control and Decision*, 30 (2015) 1139-1142. (in Chinese)
35. J. Ye, Similarity measures between interval neutrosophic sets and their applications in multi-criteria decision-making, *Journal of Intelligent and Fuzzy Systems*, 26 (2014) 165-172.
36. J. Ye, Hesitant interval neutrosophic linguistic set and its application in multiple attribute decision making, *Int. J. Mach. Learn. Cybern.* 10 (2019) 667-678.