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An Extended Result on Sub-compatible and Sub-sequential Continuous Maps in Fuzzy Metric Space

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Abstract. The principal motive of this paper is to establish a common fixed point theorem for six self-maps in a fuzzy metric space using the concepts of sub−compatibility and sub-sequential continuity. Our results extend and generalise several known results of fixed point theory in different spaces.

Keywords: Fixed point, fuzzy metric space, sub compatibility, sub sequential continuity, common fixed point theorem

AMS Mathematics Subject Classification (2010): 46B85, 55M20

1. Introduction

The foundations of fuzzy set theory and fuzzy mathematics were laid down by Zadeh [25] in 1965 by the introduction of the notion of fuzzy sets. The theory of fuzzy sets has vast applications in applied sciences and engineering, such as neural network theory, stability theory, mathematical programming, genetics, nervous systems, image processing, control theory etc. to name a few. The theory of fixed points is one of the basic tools for handling physical formulations. This has led to the development and fuzzification of several concepts of analysis and topology. In 1975, Kramosil and Michalek [12] introduced the concept of a fuzzy metric space by generalizing the concept of a probabilistic metric space to the fuzzy situation. The concept of Kramosil and Michalek of a fuzzy metric space was later modified by George and Veeramani [5] in 1994. In 1988, Grabeic [6] followed the concept of Kramosil and Michalek and obtained the fuzzy version of Banach's fixed point theorem. Using the notion of weak commuting property, Sessa [18] improved commutative conditions in fixed point theorems. Jungck [10] introduced the concept of compatibility in metric spaces. The concept of compatibility in fuzzy metric space was proposed by Mishra et al. [13]. In 2006, Jungck and Rhodes [11] introduced the concept of weakly compatible maps which was a more generalized concept than compatible maps. The concept of compatibility in fuzzy metric spaces was brought forward by Singh and Chauhan [19]. Popa [15] proved some fixed point theorems for weakly compatible non-continuous mappings using implicit relations. Imdad [7] extended his work by using implicit relations for coincidence commuting property. Singh and Jain [20] extended the results of Popa [15] for fuzzy metric spaces. In 2007, Jain and Singh [8] used the concept of compatible maps of type (A) and proved a fixed point theorem for six self maps in a fuzzy metric space.

In 2007 itself, Jain et. al [9] using the concept of compatible maps of type (β), proved a fixed point theorem in fuzzy metric space. Singh et al. [21, 22] in 2010 and 2011 proved the fixed point theorems in fuzzy metric space and menger space using the concepts of semi-compatibility, weak compatibility and compatibility of type (β) respectively. The notions of sub−compatible maps and sub−sequential continuity, which are weaker than occasionally weak compatibility and reciprocal continuity respectively, were introduced by Bouhadjera and Godet –Thobie [3] in 2009 and proved a common fixed point theorem. Ranjeth Kumar et al. [16] introduced the concepts of sub−compatibility and sub−sequential continuity in $2-$ metric spaces and proved a common fixed point theorem. Singh et al. [24] in 2011 used the concepts of sub−compatibility and sub−sequential continuity in fuzzy metric spaces and proved a common fixed point theorem. Ali et al. [1,2] in 2015 and 2016 proved the fixed point theorems in fuzzy metric space and $G-$ metric space using the concepts of sub–compatibility and sub–sequential continuity respectively.

In this paper, we prove a common fixed point theorem for six self maps in a fuzzy metric space using the concepts of sub-compatibility and sub-sequential continuity. The established result generalize, extend, unify and fuzzify several existing fixed point results in metric space and fuzzy metric space. For the sake of completeness we recall some definitions and results in the next section.

2. Preliminaries

Definition 2.1. A $t-norm$ or more precisely triangular norm $*$ is a binary operation defined on $[0,1]$ such that for all $a,b,c,d \in [0,1]$, following conditions are satisfied:

(1) $a * 1 = 1$;
(2) $a * b = b * a$;
(3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$;
(4) $a * (b * c) = (a * b) * c$.

Definition 2.2. The $3-tuple$ $(X, \mathcal{M}, *)$ is called a fuzzy metric space if X is an arbitrary non – empty set, $*$ is a continuous $t-norm$ and $\mathcal{M}$ is a fuzzy set in $X^2 \times (0,\infty)$ satisfying the following conditions, for all $x,y,z \in X$ and $s,t > 0$:

(1) $\mathcal{M}(x,y,0) > 0$;
(2) $\mathcal{M}(x,y,t) = 1$ for all $t > 0$, iff $x = y$;
(3) $\mathcal{M}(x,y,t) = \mathcal{M}(y,x,t)$;
(4) $\mathcal{M}(x,y,t) * \mathcal{M}(y,z,s) \leq \mathcal{M}(x,z,t+s)$;
(5) $\mathcal{M}(x,y,\ .) : (0,\infty) \to [0,1]$ is continuous.

Example 2.1. Let (X,d) be a metric space. Define $a * b = \min(a,b)$, and

$$\mathcal{M}(x,y,t) = \frac{t}{t + d(x,y)}$$

induced by the metric d is often called the standard fuzzy metric.

Definition 2.3. A sequence $\{x_n\}$ in a fuzzy metric space $(X, \mathcal{M}, *)$ is said to be a Cauchy sequence if, for each $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{M}\,(x_n, x_m, t) > 1 - \varepsilon \text{ for all } n, m \geq n_0.$$

A sequence $\{x_n\}$ in a fuzzy metric space $(X, \mathcal{M}, *)$ is said to be convergent to $x \in X$ if there exists $n_0 \in \mathbb{N}$ such that $\lim_{n\to\infty} \mathcal{M}\,(x_n, x, t) > 1 - \varepsilon$ for all $t > 0$ & $n \geq n_0$. A fuzzy metric space $(X, \mathcal{M}, *)$ is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 2.4. Two self-mappings A and B of a fuzzy metric space $(X, \mathcal{M}, *)$ are said to be weakly commuting if $\mathcal{M}\,(ABz, BAz, t) \geq \mathcal{M}\,(Az, Bz, t)$ for all $z \in X$ and $t > 0$.

Definition 2.5. A pair (A, B) of self mappings of a fuzzy metric space $(X, \mathcal{M}, *)$ is said to be reciprocal continuous if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} AB\,x_n = Ax \text{ and } \lim_{n\to\infty} BA\,x_n = Bx$$

whenever $\lim_{n\to\infty} A\,x_n = \lim_{n\to\infty} B\,x_n = x$ for some $x \in X$. If A and B are both continuous then they are obviously reciprocally continuous but the converse is not necessarily true.

Definition 2.6. Let A and B be mappings from a fuzzy metric space $(X, \mathcal{M}, *)$ into itself. Then the mappings are said to be compatible if

$$\lim_{n\to\infty} \mathcal{M}(ABx_n, BAx_n, t) = 1, \text{ for all } t > 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \in X$$

Definition 2.7. If A and B are two self mappings of a fuzzy metric space $(X, \mathcal{M}, *)$, then a point $x \in X$ is called the coincidence point of A and B if and only if $Ax = Bx$.

Definition 2.8. Two self mappings A and B of a fuzzy metric space $(X, \mathcal{M}, *)$ are said to be weakly compatible or coincidently commuting if they commute at their coincidence points, i e, if $ABx = BAx$ whenever $Ax = Bx$ for some $x \in X$.

Remark 2.1. It can be easily verified that compatible mappings are also weakly compatible but the converse is not necessarily true.

Definition 2.9. Two self mappings A and B of a fuzzy metric space $(X, \mathcal{M}, *)$ are said to be occasionally weakly compatible if and only if there exists a point $x \in X$ which is the coincidence point of A and B at which A and B commute.

Definition 2.10. A pair (A, B) of self mappings of a fuzzy metric space $(X, \mathcal{M}, *)$ is said to be semi-compatible if there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} AB\,x_n = Bx$ whenever $\lim_{n\to\infty} A\,x_n = \lim_{n\to\infty} B\,x_n = x$ for some $x \in X$.

Definition 2.11. Two self mappings A and B of a fuzzy metric space $(X, \mathcal{M}, *)$ are said to be sub compatible if there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} A x_n = \lim_{n\to\infty} B x_n = x, x \in X$ and satisfy $\lim_{n\to\infty} \mathcal{M}(ABx_n, BAx_n, t) = 1$

Remark 2.2. From the above definitions it is obvious that occasionally weakly compatible mappings are sub compatible. However, in general, the converse is not true.

Definition 2.12. Two self mappings A and B of a fuzzy metric space $(X, \mathcal{M}, *)$ are said to be sub sequentially continuous if and only if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} A x_n = \lim_{n\to\infty} B x_n = x, x \in X$ and satisfy $\lim_{n\to\infty} A Bx_n = At$ and $\lim_{n\to\infty} BA x_n = Bt$.

Remark 2.3. If two self mappings A and B are continuous or reciprocally continuous then they are sub sequentially continuous also. However, in general, the converse is not true.

Lemma 2.1. Let $\{x_n\}$ be a sequence in a fuzzy metric space$(X, \mathcal{M}, *)$. If there exists a number$k$, $0 < k < 1$, such that $\mathcal{M}(x_n, x_{n+1}, kt) \geq \mathcal{M}(x_{n-1}, x_n, t)$for all $t > 0$. Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.2. If for all $x, y \in X$, $t > 0$ and $0 < k < 1$, $\mathcal{M}(x, y, kt) \geq \mathcal{M}(x, y, t)$, then $x = y$.

Proof: Suppose that there exists $0 < k < 1$ such that $\mathcal{M}(x, y, kt) \geq \mathcal{M}(x, y, t)$ for all $x, y \in X$ and $t > 0$. Then $\mathcal{M}(x, y, t) \geq \mathcal{M}\left(x, y, \frac{t}{k}\right)$, and so
$\mathcal{M}(x, y, t) \geq \mathcal{M}\left(x, y, \frac{t}{k^n}\right)$ for positive integer n.
Taking limit as $n \to \infty$,
$\mathcal{M}(x, y, t) \geq 1$ and hence $x = y$.

Ali et al. [1] proved the following result a fixed point theorem in fuzzy metric spaces using sub−compatibility and sub−sequential continuity

Theorem 2.1. Let A, B, S *and* T be four self maps of a fuzzy metric space $(X, \mathcal{M}, *)$ with continuous $t - norm$ defined by $t * t \geq t$ for all $t \in [0,1]$. If the pairs (A, S) and (B, T) are sub-compatible and sub-sequentially continuous and

(1) A and S have a coincidence point;
(2) B and T have a coincidence point;
(3) $\mathcal{M}(Ax, By, kt) \geq \mathcal{M}(Sx, Ty, t) * \mathcal{M}(By, Sx, t) * \mathcal{M}(Ax, Ty, t) *$

$$\mathcal{M}(Sx, Ax, t) * \frac{a\,\mathcal{M}(Ax, By, t) + b\,\mathcal{M}(Ax, Ty, t)}{a\,\mathcal{M}(By, Ty, t) + b}$$
$$* \frac{c\,\mathcal{M}(Sx, By, t) + d\,\mathcal{M}(Sx, Ty, t)}{c\,\mathcal{M}(By, Ty, t) + d} * \frac{e\,\mathcal{M}(Ax, Ty, t) + f\,\mathcal{M}(Sx, Ax, t)}{e\,\mathcal{M}(Sx, Ty, t) + f}$$

for all $x, y \in X$ and $t > 0$, where $k \in (0,1)$ and $a, b, c, d, e, f \geq 0$ with $a\,\&\,b$, $c\,\&\,d$ and $e\,\&\,f$ cannot be simultaneously 0. Then A, B, S and T have a unique common fixed point in X.
We are now extending Ali et al. [1] work as the following result.

3. The main results

Theorem 3.1. Let P, Q, S, T, A and B be six self maps of a fuzzy metric space $(X, \mathcal{M}, *)$ with continuous $t-norm$ defined by $t * t \geq t$ for all $t \in [0,1]$. If the pairs (P, AB) and (Q, ST) are sub-compatible and sub-sequentially continuous and

(1) P and AB have a coincidence point;

(2) Q and ST have a coincidence point;

(3) $$\mathcal{M}(Px, Qy, kt) \geq \mathcal{M}(ABx, STy, t) * \mathcal{M}(Qy, ABx, t) * \mathcal{M}(Px, STy, t) * \mathcal{M}(ABx, Px, t) * \frac{a\,\mathcal{M}(Px, Qy, t) + b\,\mathcal{M}(Px, STy, t)}{a\,\mathcal{M}(Qy, STy, t) + b} * \frac{c\,\mathcal{M}(ABx, Qy, t) + d\,\mathcal{M}(ABx, STy, t)}{c\,\mathcal{M}(Qy, STy, t) + d} * \frac{e\,\mathcal{M}(Px, STy, t) + f\,\mathcal{M}(ABx, Px, t)}{e\,\mathcal{M}(ABx, STy, t) + f} * \frac{g\,\mathcal{M}(Px, Qy, t) + h\,\mathcal{M}(Px, STy, t)}{g\,\mathcal{M}(ABx, Qy, t) + h\,\mathcal{M}(ABx, STy, t)} * \mathcal{M}(Qy, STy, t) * \frac{a\,\mathcal{M}(Px, STy, t) + b\,\mathcal{M}(Px, Qy, t) + c\,\mathcal{M}(ABx, Px, t)}{a\,\mathcal{M}(ABx, STy, t) + b\,\mathcal{M}(ABx, Qy, t) + c} * \mathcal{M}(Px, Qy, t) * \frac{d\,\mathcal{M}(Px, Qy, t) + e\,\mathcal{M}(ABx, STy, t) + f\,\mathcal{M}(Px, STy, t)}{d\,\mathcal{M}(Qy, STy, t) + e + f}$$

for all $x, y \in X$ and $t > 0$, where $k \in (0,1)$ and $a, b, c, d, e, f, g, h \geq 0$ with $a\,\&\,b$, $c\,\&\,d$, $e\,\&\,f$, $g\,\&\,h$, $a, b\,\&\,c$ and $d, e\,\&\,f$ cannot be simultaneously 0.

Then P, Q, S, T, A and B have a unique common fixed point in X.

Proof: Since the pairs (P, AB) and (Q, ST) are sub−compatible and sub−sequentially continuous, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} P\,x_n = \lim_{n\to\infty} AB\,x_n = z$, where $z \in X$ and satisfy

$$\mathcal{M}(PABx_n, ABPx_n, t) = \mathcal{M}(Pz, ABz, t) = 1$$

and $\lim_{n\to\infty} Q\,y_n = \lim_{n\to\infty} ST\,y_n = z'$, where $z' \in X$ and which satisfy

$$\mathcal{M}(QSTy_n, STQy_n, t) = \mathcal{M}(Qz', STz', t) = 1$$

Therefore, $Pz = ABz$ and $Qz' = STz'$, that is, z is a coincidence point of P and AB and z' is a coincidence point of Q and ST. Now, we prove that $z = z'$. Putting $x = x_n$ and $y = y_n$ in inequality (3), we get

$$\mathcal{M}(Px_n, Qy_n, kt) \geq \mathcal{M}(ABx_n, STy_n, t) * \mathcal{M}(Qy_n, ABx_n, t) * \mathcal{M}(Px_n, STy_n, t) * \mathcal{M}(ABx_n, Px_n, t) * \frac{a\,\mathcal{M}(Px_n, Qy_n, t) + b\,\mathcal{M}(Px_n, STy_n, t)}{a\,\mathcal{M}(Qy_n, STy_n, t) + b} * \frac{c\,\mathcal{M}(ABx_n, Qy_n, t) + d\,\mathcal{M}(ABx_n, STy_n, t)}{c\,\mathcal{M}(Qy_n, STy_n, t) + d}$$

$$* \frac{e\,\mathcal{M}\,(Px_n, STy_n, t) + f\,\mathcal{M}\,(ABx_n, Px_n, t)}{e\,\mathcal{M}\,(ABx_n, STy_n, t) + f} * \frac{g\,\mathcal{M}(Px_n, Qy_n, t) + h\,\mathcal{M}(Px_n, STy_n, t)}{g\,\mathcal{M}(ABx_n, Qy_n, t) + h\,\mathcal{M}(ABx_n, STy_n, t)} * \mathcal{M}\,(Qy_n, STy_n, t)$$

$$* \frac{a\,\mathcal{M}(Px_n, STy_n, t) + b\,\mathcal{M}\,(Px_n, Qy_n, t) + c\,\mathcal{M}(ABx_n, Px_n, t)}{a\,\mathcal{M}(ABx_n, STy_n, t) + b\,\mathcal{M}(ABx_n, Qy_n, t) + c}$$

$$* \mathcal{M}\,(Px_n, Qy_n, t) * \frac{d\,\mathcal{M}\,(Px_n, Qy_n, t) + e\,\mathcal{M}(ABx_n, STy_n, t) + f\,\mathcal{M}(Px_n, STy_n, t)}{d\,\mathcal{M}\,(Qy_n, STy_n, t) + e + f}$$

Taking limit $n \to \infty$ in the above, we obtain

$$\mathcal{M}\,(z, z', kt) \geq \mathcal{M}(z, z', t) * \mathcal{M}(z', z, t) * \mathcal{M}(z, z', t) * \mathcal{M}(z, z, t) * \frac{a\,\mathcal{M}\,(z, z', t) + b\,\mathcal{M}\,(z, z', t)}{a\,\mathcal{M}\,(z', z', t) + b} * \frac{c\,\mathcal{M}\,(z, z', t) + d\,\mathcal{M}\,(z, z', t)}{c\,\mathcal{M}\,(z', z', t) + d} * \frac{e\,\mathcal{M}\,(z, z', t) + f\,\mathcal{M}\,(z, z, t)}{e\,\mathcal{M}\,(z, z', t) + f} * \frac{g\,\mathcal{M}(z, z', t) + h\,\mathcal{M}(z, z', t)}{g\,\mathcal{M}(z, z', t) + h\,\mathcal{M}(z, z', t)} * \mathcal{M}\,(z', z', t) * \frac{a\,\mathcal{M}(z, z', t) + b\,\mathcal{M}\,(z, z', t) + c\,\mathcal{M}(z, z, t)}{a\,\mathcal{M}(z, z', t) + b\,\mathcal{M}(z, z', t) + c} * \mathcal{M}\,(z, z', t) * \frac{d\,\mathcal{M}\,(z, z', t) + e\,\mathcal{M}(z, z', t) + f\,\mathcal{M}(z, z', t)}{d\,\mathcal{M}\,(z', z', t) + e + f}$$

$$\Longrightarrow \mathcal{M}\,(z, z', kt) \geq \mathcal{M}(z, z', t) * \mathcal{M}(z', z, t) * \mathcal{M}(z, z', t) * 1 * \mathcal{M}\,(z, z', t) * \mathcal{M}\,(z, z', t) * 1 * 1 * 1 * 1 * \mathcal{M}\,(z, z', t) * \mathcal{M}\,(z, z', t)$$

$$\Longrightarrow \mathcal{M}\,(z, z', kt) \geq \mathcal{M}\,(z, z', t) * \mathcal{M}\,(z', z, t) * \mathcal{M}\,(z, z', t) * \mathcal{M}\,(z, z', t) * \mathcal{M}\,(z, z', t) * \mathcal{M}\,(z, z', t) * \mathcal{M}\,(z, z', t)$$

$$\Longrightarrow \mathcal{M}\,(z, z', kt) \geq \mathcal{M}\,(z, z', t)$$

Therefore, $z = z'$.

Again, we claim that $Pz = z$.

Putting $x = z$ and $y = y_n$ in inequality (3), we get

$$\mathcal{M}\,(Pz, Qy_n, kt) \geq \mathcal{M}(ABz, STy_n, t) * \mathcal{M}(Qy_n, ABz, t) * \mathcal{M}(Pz, STy_n, t) * \mathcal{M}(ABz, Pz, t)$$

$$* \frac{a\,\mathcal{M}\,(Pz, Qy_n, t) + b\,\mathcal{M}\,(Pz, STy_n, t)}{a\,\mathcal{M}\,(Qy_n, STy_n, t) + b} * \frac{c\,\mathcal{M}\,(ABz, Qy_n, t) + d\,\mathcal{M}\,(ABz, STy_n, t)}{c\,\mathcal{M}\,(Qy_n, STy_n, t) + d}$$

$$* \frac{e\,\mathcal{M}\,(Pz, STy_n, t) + f\,\mathcal{M}\,(ABz, Pz, t)}{e\,\mathcal{M}\,(ABz, STy_n, t) + f} * \frac{g\,\mathcal{M}(Pz, Qy_n, t) + h\,\mathcal{M}(Pz, STy_n, t)}{g\,\mathcal{M}(ABz, Qy_n, t) + h\,\mathcal{M}(ABz, STy_n, t)}$$

$$* \mathcal{M}\,(Qy_n, STy_n, t) * \frac{a\,\mathcal{M}(Pz, STy_n, t) + b\,\mathcal{M}\,(Pz, Qy_n, t) + c\,\mathcal{M}(ABz, Pz, t)}{a\,\mathcal{M}(ABz, STy_n, t) + b\,\mathcal{M}(ABz, Qy_n, t) + c}$$

$$* \mathcal{M}\,(Pz, Qy_n, t) * \frac{d\,\mathcal{M}\,(Pz, Qy_n, t) + e\,\mathcal{M}(ABz, STy_n, t) + f\,\mathcal{M}(Pz, STy_n, t)}{d\,\mathcal{M}\,(Qy_n, STy_n, t) + e + f}$$

Taking limit $n \to \infty$ in the above, we obtain

$$\mathcal{M}\,(Pz, z', kt) \geq \mathcal{M}(Pz, z', t) * \mathcal{M}(z', Pz, t) * \mathcal{M}(Pz, z', t) * \mathcal{M}(Pz, Pz, t)$$

$$* \frac{a\,\mathcal{M}\,(Pz, z', t) + b\,\mathcal{M}\,(Pz, z', t)}{a\,\mathcal{M}\,(z', z', t) + b} * \frac{c\,\mathcal{M}\,(Pz, z', t) + d\,\mathcal{M}\,(Pz, z', t)}{c\,\mathcal{M}\,(z', z', t) + d}$$

$$* \frac{e\,\mathcal{M}\,(Pz, z', t) + f\,\mathcal{M}\,(Pz, Pz, t)}{e\,\mathcal{M}\,(Pz, z', t) + f} * \frac{g\,\mathcal{M}(Pz, z', t) + h\,\mathcal{M}(Pz, z', t)}{g\,\mathcal{M}(Pz, z', t) + h\,\mathcal{M}(Pz, z', t)}$$

$$* \mathcal{M}\,(z', z', t) * \frac{a\,\mathcal{M}(Pz, z', t) + b\,\mathcal{M}\,(Pz, z', t) + c\,\mathcal{M}(Pz, Pz, t)}{a\,\mathcal{M}(Pz, z', t) + b\,\mathcal{M}(Pz, z', t) + c}$$

$$* \mathcal{M}\,(Pz, z', t) * \frac{d\,\mathcal{M}\,(Pz, z', t) + e\,\mathcal{M}(Pz, z', t) + f\,\mathcal{M}(Pz, z', t)}{d\,\mathcal{M}\,(z', z', t) + e + f}$$

$$\Longrightarrow \mathcal{M}\,(Pz, z', kt)$$

$$\geq \mathcal{M}\,(Pz, z', t) * \mathcal{M}\,(z', Pz, t) * \mathcal{M}\,(Pz, z', t) * 1 * \mathcal{M}\,(Pz, z', t)$$

$$* \mathcal{M}\,(Pz, z', t) * 1 * 1 * 1 * 1 * \mathcal{M}\,(Pz, z', t) * \mathcal{M}\,(Pz, z', t)$$

$$\Longrightarrow \mathcal{M}\,(Pz, z', kt) \geq \mathcal{M}\,(Pz, z', t) * \mathcal{M}\,(z', Pz, t) * \mathcal{M}\,(Pz, z', t) * \mathcal{M}\,(Pz, z', t)$$

$$* \mathcal{M}\,(Pz, z', t) * \mathcal{M}\,(Pz, z', t) * \mathcal{M}\,(Pz, z', t)$$

$$\Longrightarrow \mathcal{M}\,(Pz, z', kt) \geq \mathcal{M}\,(Pz, z', t)$$

Therefore, $Pz = z' = z$. Next, we claim that $Qz = z$. Putting $x = z$ & $y = z$ in inequality (3), we get $\mathcal{M}\,(Pz, Qz, kt) \geq \mathcal{M}(ABz, STz, t) * \mathcal{M}(Qz, ABz, t) * \mathcal{M}(Pz, STz, t) * \mathcal{M}(ABz, Pz, t)$

$$* \frac{a\,\mathcal{M}\,(Pz, Qz, t) + b\,\mathcal{M}\,(Pz, STz, t)}{a\,\mathcal{M}\,(Qz, STz, t) + b} * \frac{c\,\mathcal{M}\,(ABz, Qz, t) + d\,\mathcal{M}\,(ABz, STz, t)}{c\,\mathcal{M}\,(Qz, STz, t) + d}$$

$$* \frac{e\,\mathcal{M}\,(Pz, STz, t) + f\,\mathcal{M}\,(ABz, Pz, t)}{e\,\mathcal{M}\,(ABz, STz, t) + f} * \frac{g\,\mathcal{M}(Pz, Qz, t) + h\,\mathcal{M}(Pz, STz, t)}{g\,\mathcal{M}(ABz, Qz, t) + h\,\mathcal{M}(ABz, STz, t)}$$

$$* \mathcal{M}\,(Qz, STz, t) * \frac{a\,\mathcal{M}(Pz, STz, t) + b\,\mathcal{M}\,(Pz, Qz, t) + c\,\mathcal{M}(ABz, Pz, t)}{a\,\mathcal{M}(ABz, STz, t) + b\,\mathcal{M}(ABz, Qz, t) + c}$$

$$* \mathcal{M}\,(Pz, Qz, t) * \frac{d\,\mathcal{M}\,(Pz, Qz, t) + e\,\mathcal{M}(ABz, STz, t) + f\,\mathcal{M}(Pz, STz, t)}{d\,\mathcal{M}\,(Qz, STz, t) + e + f}$$

Taking limit $n \rightarrow \infty$ in the above, we obtain

$$\mathcal{M}(z,Qz,kt) \geq \mathcal{M}(z,z,t) * \mathcal{M}(Qz,z,t) * \mathcal{M}(z,z,t) * \mathcal{M}(z,z,t) * \frac{a\,\mathcal{M}(z,Qz,t) + b\,\mathcal{M}(z,z,t)}{a\,\mathcal{M}(Qz,z,t) + b} * \frac{c\,\mathcal{M}(z,Qz,t) + d\,\mathcal{M}(z,z,t)}{c\,\mathcal{M}(Qz,z,t) + d} * \frac{e\,\mathcal{M}(z,z,t) + f\,\mathcal{M}(z,z,t)}{e\,\mathcal{M}(z,z,t) + f} * \frac{g\,\mathcal{M}(z,Qz,t) + h\,\mathcal{M}(z,z,t)}{g\,\mathcal{M}(z,Qz,t) + h\,\mathcal{M}(z,z,t)} * \mathcal{M}(Qz,z,t) * \frac{a\,\mathcal{M}(z,z,t) + b\,\mathcal{M}(z,Qz,t) + c\,\mathcal{M}(z,z,t)}{a\,\mathcal{M}(z,z,t) + b\,\mathcal{M}(z,Qz,t) + c} * \mathcal{M}(z,Qz,t) * \frac{d\,\mathcal{M}(z,Qz,t) + e\,\mathcal{M}(z,z,t) + f\,\mathcal{M}(z,z,t)}{d\,\mathcal{M}(Qz,z,t) + e + f}$$

$$\Longrightarrow \mathcal{M}(z,Qz,kt) \geq 1 * \mathcal{M}(Qz,z,t) * 1 * 1 * 1 * 1 * 1 * 1 * \mathcal{M}(Qz,z,t) * 1 * \mathcal{M}(z,Qz,t) * 1$$

$$\Longrightarrow \mathcal{M}(z,Qz,kt) \geq \mathcal{M}(Qz,z,t) * \mathcal{M}(Qz,z,t) * \mathcal{M}(z,Qz,t)$$

$$\Longrightarrow \mathcal{M}(z,Qz,kt) \geq \mathcal{M}(Qz,z,t)$$

Therefore, $z = Qz = STz$. Hence we have $z = Pz = Qz = STz = ABz$, that is, z is a common fixed point of P, Q, S, T, A and B.

Uniqueness: Let w be another common fixed point of P, Q, S, T, A and B. Then

$$Pw = Qw = ABw = STw = w$$

Putting $x = z$ and $y = w$ in inequality (3), we get

$$\mathcal{M}(Pz,Qw,kt) \geq \mathcal{M}(ABz,STw,t) * \mathcal{M}(Qw,ABz,t) * \mathcal{M}(Pz,STw,t) * \mathcal{M}(ABz,Pz,t) * \frac{a\,\mathcal{M}(Pz,Qw,t) + b\,\mathcal{M}(Pz,STw,t)}{a\,\mathcal{M}(Qw,STw,t) + b} * \frac{c\,\mathcal{M}(ABz,Qw,t) + d\,\mathcal{M}(ABz,STw,t)}{c\,\mathcal{M}(Qw,STw,t) + d} * \frac{e\,\mathcal{M}(Pz,STw,t) + f\,\mathcal{M}(ABz,Pz,t)}{e\,\mathcal{M}(ABz,STw,t) + f} * \frac{g\,\mathcal{M}(Pz,Qw,t) + h\,\mathcal{M}(Pz,STw,t)}{g\,\mathcal{M}(ABz,Qw,t) + h\,\mathcal{M}(ABz,STw,t)} * \mathcal{M}(Qw,STw,t) * \frac{a\,\mathcal{M}(Pz,STw,t) + b\,\mathcal{M}(Pz,Qw,t) + c\,\mathcal{M}(ABz,Pz,t)}{a\,\mathcal{M}(ABz,STw,t) + b\,\mathcal{M}(ABz,Qw,t) + c} * \mathcal{M}(Pz,Qw,t) * \frac{d\,\mathcal{M}(Pz,Qw,t) + e\,\mathcal{M}(ABz,STw,t) + f\,\mathcal{M}(Pz,STw,t)}{d\,\mathcal{M}(Qw,STw,t) + e + f}$$

Taking limit $n \rightarrow \infty$ in the above, we obtain

$$\mathcal{M}(z,w,kt) \geq \mathcal{M}(z,w,t) * \mathcal{M}(w,z,t) * \mathcal{M}(z,w,t) * \mathcal{M}(z,z,t) * \frac{a\,\mathcal{M}(z,w,t) + b\,\mathcal{M}(z,w,t)}{a\,\mathcal{M}(w,w,t) + b} * \frac{c\,\mathcal{M}(z,w,t) + d\,\mathcal{M}(z,w,t)}{c\,\mathcal{M}(w,w,t) + d} * \frac{e\,\mathcal{M}(z,w,t) + f\,\mathcal{M}(z,z,t)}{e\,\mathcal{M}(z,w,t) + f} * \frac{g\,\mathcal{M}(z,w,t) + h\,\mathcal{M}(z,w,t)}{g\,\mathcal{M}(z,w,t) + h\,\mathcal{M}(z,w,t)}$$

$$* \mathcal{M}(w,w,t) * \frac{a\,\mathcal{M}(z,w,t) + b\,\mathcal{M}(z,w,t) + c\,\mathcal{M}(z,z,t)}{a\,\mathcal{M}(z,w,t) + b\,\mathcal{M}(z,w,t) + c}$$

$$* \mathcal{M}(z,w,t) * \frac{d\,\mathcal{M}(z,w,t) + e\,\mathcal{M}(z,w,t) + f\,\mathcal{M}(z,w,t)}{d\,\mathcal{M}(w,w,t) + e + f}$$

$$\Rightarrow \mathcal{M}(z,w,kt) \geq \mathcal{M}(z,w,t) * \mathcal{M}(w,z,t) * \mathcal{M}(z,w,t) * 1 * \mathcal{M}(z,w,t) * \mathcal{M}(z,w,t) * 1 * 1 * 1 * 1 * \mathcal{M}(z,w,t) * \mathcal{M}(z,w,t)$$

$$\Rightarrow \mathcal{M}(z,w,kt) \geq \mathcal{M}(z,w,t) * \mathcal{M}(w,z,t) * \mathcal{M}(z,w,t) * \mathcal{M}(z,w,t) * \mathcal{M}(z,w,t) * \mathcal{M}(z,w,t) * \mathcal{M}(z,w,t)$$

$$\Rightarrow \mathcal{M}(z,w,kt) \geq \mathcal{M}(z,w,t)$$

Therefore, $z = w$. Hence z is a unique common fixed point of A, B, S and T.

4. Conclusion

In this paper, we have used the concepts of sub−compatibility and sub−sequential continuity to prove a common fixed point theorem for six self-maps in a fuzzy metric space. Our results generalize, extend, unify and fuzzify several existing fixed point results in metric space and fuzzy metric space. This can be further extended by increasing the number of self-maps with a new class of inequality.

REFERENCES

1. S.S.Ali, J.Jain and M.F.Khan, A fixed point theorem in fuzzy metric spaces using sub–compatibility and sub–sequential continuity. *International Journal of Current Engineering and Technology*, 5(6) (2015) 40-45.
2. S.S.Ali, J.Jain, P.I.Sanodia and S.Jain, Extension of sub–compatible and sub-sequential continuous maps in G-metric space, *International Journal of Current Engineering and Technology*,6(6) (2016) 2017-2019.
3. Bouhadjera and Godet-Thobie. Common fixed point theorems for pairs of sub compatible maps. *ArXiv: 0906.3159*, 1 [math. F.A.] (2009) 1 – 16.
4. Y.J.Cho, S.Sedghi and N.Shobe, Generalized fixed point theorems for compatible mappings with some types in fuzzy metric spaces, *Chaos, Solitons and Fractals*, 39 (2009) 2233-2244.
5. A.George and P.Veeramani, On some results in fuzzy metric spaces, Fuzzy *Sets and Systems*, 64 (1994) 395-399.
6. M.Grabiec, Fixed points in fuzzy metric spaces, *Fuzzy Sets and Systems*, 27 (1988) 385-389.
7. M.Imdad, S.Kumar and M.S.Khan, Remarks on some fixed point theorems satisfying an implicit relation, *Radovi Mathematics*, 11 (2002) 135-143.
8. A.Jain and B.Singh, A fixed point theorem for compatible mappings of type (A) in fuzzy metric space, *Acta Ciencia Indica*, XXXIII M(2) (2007) 339-346.

9. A.Jain, M.Sharma and B.Singh, Fixed point theorem using compatibility of type (β) in fuzzy metric spaces, *Chh. Journal of Science and Technology*, 3 & 4 (2007) 53-62.
10. G.Jungck, Compatible mappings and common fixed points, *International Journal of Mathematics and Mathematical Sciences*, 9 (4) (1986) 771-779.
11. Jungck and Rhoades, Fixed point theorems for occasionally weakly compatible mappings, *Fixed Point Theory*, 7 (2) (2006) 287-296.
12. I.Kramosil and J.Michalek, Fuzzy metric and statistical metric spaces, *Kybernetica*, 11 (1975) 326-334.
13. S.N.Mishra, N.Sharma and S.I.Singh, Common fixed points of maps on fuzzy metric space, *International Journal of Mathematics and Mathematical Sciences*, 17 (1994) 253-258.
14. V.Pant, Contractive conditions and common fixed points in fuzzy metric space, *Journal of Fuzzy Mathematics*, 14 (2) (2006) 267-272.
15. V.Popa, Some fixed point theorems for weakly compatible mappings, *Randovi Mathematics*, 10 (2001) 245-252.
16. S.Ranjeth Kumar, Loganathan and M.Peer Mohamed. Common fixed point theorems for sub compatible and sub sequentially continuous maps in 2 –metric spaces, *Inter. Math. Forum*, 7(24) (2012) 1187-1200.
17. D.Regan and M.Abbas. Necessary and sufficient conditions for common fixed point theorems in fuzzy metric spaces, *Demonstratio Mathematica*, 52 (4) (2009) 879-892.
18. S.Sessa. On weak commutativity condition of mapping in fixed point consideration. *Publ. Inst. Math. (Beograd) N.S.*, 32(46) (1982) 149-153.
19. B.Singh and M.S.Chauhan, Common fixed points of compatible maps in fuzzy metric spaces, *Fuzzy Sets and System*, 115 (2000) 471 - 475.
20. B.Singh and Jain, Semi-compatibility and fixed point theorems in fuzzy metric spaces using implicit relation, *International Journal of Math. and Mathematical Sciences*, 16 (2005) 2617 - 2629.
21. B.Singh, A.Jain and A.K.Govery. Compatibility of type (β) and fixed point theorem in fuzzy metric spaces, *Applied Mathematical Sciences*, 5 (11) (2011) 517 – 528.
22. B.Singh, A.Jain and B.Lodha, On common fixed point theorems for semi compatible mappings in Menger spaces, *Commentationes Mathematicae*, 50 (2) (2010) 127 – 139.
23. B.Singh, A.Jain and A.A.Masoodi. Semi compatibility, weak compatibility and fixed point theorem in fuzzy metric spaces, *International Mathematical Forum*, 5 (61) (2010) 3041 – 3051.
24. B.Singh, A.Jain and A.A.Wani. Sub compatibility and fixed point theorem in fuzzy metric spaces, *International Journal of Mathematical Analysis*, 5 (27) (2011) 1301 – 1308.
25. L.Zadeh, Fuzzy sets, *Inform and Control*, 8 (1965) 338−353.