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Row-Average-Max-Norm of Fuzzy Matrix

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Abstract. Fuzzy matrices play a vital role in handling different models in an uncertain environment. In this paper, we have defined the row-average-max norm of fuzzy matrices. We also investigated some properties and lemma of the row-average-max norm of the fuzzy matrix.

Keywords: Norm of a fuzzy matrix, row-average-max norm of fuzzy matrix, properties of row-average-max norm of fuzzy matrix

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1. Introduction

An analysis of linear algebra is one of the most popular and fascinating areas in the last few decades due to its interdependency with other applied and pure areas. Measuring the length of vectors is an essential analysis in different theoretical development aspects in many potential applications. For this purpose, norm functions are considered on a vector space.

A norm on a real vector space V is a function $\|.\|: V \to R$ satisfying

1. $\|u\| > 0$ for any non-zero $u \in V$.
2. $\|ru\| = |r|\|u\|$ for any $r \in R$ and $u \in V$.
3. $\|u + v\| \leq \|u\| + \|v\|$ for any $u, v \in V$

In general, the norm signifies the measure of the size of the vector u where equation (1) requires the size to be positive, equation (2) requires the size to be scaled as the vector is scaled, and equation (3) is known as the triangle inequality having its origin in the notion of distance in R^3. The equation (2) is called the homogeneous condition and this condition ensures that the norm of the zero vector in V is 0; thiscondition is often included in the definition of a norm.

A familiar example of norms on R^n are the l_p norms, where $1 \leq p \leq \infty$ defined by

$l_p(u) = \{\sum_{j=1}^{n} |u|_j{}^p\}^{\frac{1}{p}}$ if $1 \leq p < \infty$ and
$l_p(u) = max_{1 \leq j \leq n} |u_j|$ if $p = \infty$
for any $u = (u_1, u_2, \ldots\ldots, u_n)' \in R^n$.

It is worthy to remember that if one defines an l_p function on R^n as defined above with $0 < p < 1$, then it does not satisfy the triangle inequality, hence is not a norm.

Suppose the norm on a real vector space V is given. In that case, numerous aspects can be formulated, such as one can compare the norms of vectors, discussing the convergence of the sequence of vectors, studying limits and continuity of transformations, and considering approximation problems such as finding the nearest element in a subset or a subspace of V to a given vector. These problems arise naturally in analysis, numerical analysis, differential equations, Markov chains etc.

The norm determines the "size" of a matrix that is necessarily related to how many rows or columns the matrix contains. The norm of a square matrix M is a non-negative real number denoted by $\|M\|$. There are several different ways of defining a matrix norm, but they all share the following properties:

1. $\|M\| \geq 0$ for any square matrix M.
2. $\|M\| = 0$ iff the matrix $M = 0$.
3. $\|KM\| = |K| \|M\|$ for any scaler K.
4. $\|M + N\| \leq \|M\| + \|N\|$ for any square matrix M, N.
5. $\|MN\| \leq \|M\| \|N\|$.

Fuzzy matrix norm:

Like vector norm and matrix norm, the norm of a fuzzy matrix is also a function $\|.\| : M_n(F) \to [0,1]$ which satisfies the following properties

1. $\|M\| \geq 0$ for any square matrix M.
2. $\|M\| = 0$ iff the fuzzy matrix $M = 0$.
3. $\|KM\| = |K|\|M\|$ for any scalar $K \in [0,1]$.
4. $\|M + N\| \leq \|M\| + \|N\|$ for any two fuzzy matrices M and N.
5. $\|MN\| \leq \|M\|\|N\|$ for any fuzzy matrix M and N .

In this project paper, we have defined different types of norms on fuzzy matrices.

1.1. Motivation

To analyze different geometrical and analytical structures, norms on a vector space could be employed. The choice of utilizing the norm decides the convergence of a sequence in an infinite dimensional vector space. This phenomena leads to many interesting questions and research in analysis and functional analysis.

In a finite-dimensional vector space V, two norms $\|.\|_1$ and $\|.\|_2$ are said to be equivalent if there exist two positive constants such that $a\|v\|_1 \leq \|v\|_2 \leq \|v\|_1$ for all $v \in V$.

To prove the convergence concerning one norm for a given sequence is easier than the other. In an application such as numerical analysis, one would like to use a norm that can determine convergence efficiently. Hence, it is a good idea to know different norms.

Secondly, in some cases, a specific norm may be needed to deal with a certain problem. For instance, if one travels in Manhattan and wants to measure the distance from a location marked as the origin $(0,0)$ to a destination marked as (x, y) on the map, one may use the l_2 norm of (x, y), which measures the straight line distance between two points, or one may need to use the l_1 norm of v, which measures the distance for a taxi cab to drive from $(0,0)$ to (x, y). The l_1 norm is sometimes referred to as the taxi cab norm for this reason.

In approximation theory, solutions of a problem can vary with different problems. For example, if W is a subspace of R^n and v does not belongs to, then for $1 < p < \infty$ there is a unique $u_0 \in W$ such that $\|v - u_0\| \leq \|v - u\|$ for all $u \in W$, but the uniqueness condition may fail if $p = 1$ or ∞. To see a concrete example let $v = (1,0)$ and $W = \{(0, y): y \in R\}$. Then for all $y \in [-1,1]$ we have $1 - \|v - (0, y)\| \leq \|v - w\|$ for all $\in W$. For some problems, having a unique approximation is good, but for others it may be better to have many so that one of them can be chosen to satisfy additional conditions.

1.2. Literature review

It is well known that matrices play a vital role in several areas including mathematics, physics, statistics, engineering, social sciences. An ample number of methods have been reported in several journals as well as in books. But our real-life problems including social science, medical science, environment etc. do not always involve crisp data. Furthermore, owing to various types of uncertainties present in our daily life problems we cannot successfully use traditional classical matrices. Nowadays, probability, fuzzy sets, intuitionistic fuzzy sets, vague sets and rough sets are used as mathematical tools for dealing with uncertainties. Fuzzy matrices arise in many applications, one of which is as adjacency matrices of fuzzy relations and fuzzy relational equations have important applications in pattern classification and in handing fuzziness in knowledge-based systems.

First-time Fuzzy matrices were introduced by Thomason [44], where they discussed on the convergence of powers of a fuzzy matrix. Ragab et al. [34, 35] presented some properties of the min-max composition of fuzzy matrices. Hashimoto [18, 19] studied the canonical form of a transitive fuzzy matrix. Hemashina et al. [20] Investigated iterates of fuzzy circulant matrices. Powers and nilpotent conditions of matrices over a distributive lattice are considered by Tan [43]. After that Pal, Bhowmik, Adak, Shyamal, Mondal have done a lot of work on fuzzy, intuitionistic fuzzy, interval-valued fuzzy, etc. matrices [1-12, 26-33, 37-41].

The elements of a fuzzy matrix lie in the closed interval$[0,1]$. Although every matrix, in general, is not a fuzzy matrix still we can see that all fuzzy matrices. We see the fuzzy interval, i.e., the unit interval is a subset of reals. Thus, a matrix, in general, is not a fuzzy matrix since the unit interval $[0,1]$ is contained in the set of reals. The big question arises when it comes to the addition of two fuzzy matrices M and N and getting the sum of them to be fuzzy matrix. The answer in general is not possible for the sum of two fuzzy matrices may turn out to be a matrix which is not a fuzzy matrix. If we add above two fuzzy matrices M and N then all entries in $M + N$ will not lie in $[0,1]$, hence $M + N$ is only just a matrix and not a fuzzy matrix.

Henceforth, the max or min operations could be defined in the case of fuzzy matrices. Therefore, under the max or min operation, the resultant matrix is again a fuzzy matrix. In general, to add two matrices we use max operation.

It is evident that the product of two fuzzy matrices under usual matrix multiplication is not a fuzzy matrix. So, we need to define a compatible operation analogous to the product so that the product again happens to be a fuzzy matrix. However even for this new operation if the product MN is to be defined we need the number of columns of M to be equal to the number of rows of N. The two types of operation are called max-min operation and min-max operation.

In [23], we introduced max-norm and square-max norm, row and column max-average nom of fuzzy matrices and some properties of these two norms. In this paper, we have defined row-average-max norm with some properties

Definition 1. *An $n \times n$ fuzzy matrix A is called reflexive iff $a_{ii} = 1$ for all i=1,2,...,n. It is called α -reflexive iff $a_{ii} \geq \alpha$ for all i=1,2,...,n where $\alpha \in [0,1]$. It is called weakly reflexive iff $a_{ii} \geq a_{ij}$ for all i,j=1,2,...,n. An $n \times n$ fuzzy matrix A is called irreflexive iff $a_{ii} = 0$ for all i=1,2,...,n.*

Definition 2. *An $n \times n$ fuzzy matrix S is called symmetric iff $s_{ij} = s_{ji}$ for all i,j=1,2,...,n. It is called antisymmetric iff $S \wedge S' \leq I_n$ where I_n is the usual unit matrix.*

Note that the condition $S \wedge S' \leq I_n$, means that $s_{ij} \wedge s_{ji} = 0$ for all $i \neq j$ and $s_{ii} \leq 1$ for all i. So if $S_{ij} = 1$ then $s_{ji} = 0$, which the crisp case.

Definition 3. *An $n \times n$ fuzzy matrix N is called nilpotent iff $N^n = 0$ (the zero matrix). If $N^m = 0$ and $N^{m-1} \neq 0$; $1 \leq m \leq n$ then N is called nilpotent of degree m. An $n \times n$ fuzzy matrix E is called idempotent iff $E^2 = E$. It is called transitive iff $E^2 \leq E$. It is called compact iff $E^2 \geq E$.*

Definition 4. *A triangular fuzzy matrix of order $m \times n$ is defined as $A = (a_{ij})_{m \times n}$ where $a_{ij} = \langle m_{ij}, \alpha_{ij}, \beta_{ij} \rangle$ is the ij^{th} element of A, m_{ij} is the mean value of a_{ij} and α_{ij}, β_{ij} are left and right spread of a_{ij} respectively.*

2. Preliminaries

In this section different types of matrix norm and fuzzy matrix norm are discussed.

2.1. Matrix norm

Definition 5. *(The maximum absolute column sum). Simply we sum the absolute values down each column and then take the biggest answer. (A useful reminder is that "1" is a tall, thin character and a column is a tall, thin quantity.)*

$$\|P\|_1 = \max_{1 \leq j \leq n} \left(\sum_{i=1}^{n} | p_{ij} | \right).$$

Definition 6. *The infinity norm of a square matrix is the maximum of the absolute row sum. Simply we sum the absolute values along each row and then take the biggest answer. The infinity norm of a matrix A is defined by*

$$\|P\|_\infty = \max_{1 \leq i \leq n} \left(\sum_{j=1}^{n} | p_{ij} | \right).$$

Definition 7. *The Euclidean norm of a square matrix is the square root of the sum of all the squares of the elements. This is similar to ordinary "Pythagorean" length where the size of a vector is found by taking the square root of the sum of the squares of all the elements. The Euclidean norm of a matrix A is defined by*

$$\|P\|_E = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(p_{ij})^2}.$$

Any definition you can define of which satisfies the five condition mentioned at the beginning of this section is a definition of a norm. There are many many possibilities, but the three given above are among the most commonly used.

2.2. Norm of fuzzy matrix

Definition 8. ***Max norm*** *(Maity [23]): Max norm of a fuzzy matrix* $A \in M_n(F)$ *is denoted by* $\|A\|_M$ *which gives the maximum element of the fuzzy matrix and it is defined by*

$$\|A\|_M = \bigvee_{i,j=1}^{n} a_{ij}$$

Definition 9. *(Maity [23]): Square-max norm of a fuzzy matrix A is denoted by* $\|A\|_{SM}$ *and define by* $\|A\|_{SM} = (\bigvee_{i,j=1}^{n} a_{ij})^2 = (\|A\|_M)^2$.

In this norm at first we will find the maximum element of the fuzzy matrix and then square it.

Definition 10. ***Row-max-average Norm*** *(Maity [24])**:** Row-max-average norm of a fuzzy matrix A is denoted by* $\|A\|_{RMA}$ *and define by* $\|A\|_{RMA} = \frac{1}{n}\sum_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij})$

Here, at first we find maximum element in each row. Then we determine the average of the maximum element.

Definition 11. ***Column-max-average norm*** *(Maity [24])**:** The Column-max-average norm of a fuzzy matrix* A *is denoted by* $\|A\|_{CMA}$ *and define by* $\|A\|_{CMA} = \frac{1}{n}\sum_{j=1}^{n}(\bigvee_{i=1}^{n} a_{ij})$.

Here we find maximum element in each column and then average of the maximum elements.

Definition 12. ***Pseudo norm on fuzzy matrix*** *(Maity [24])**:** A norm of a fuzzy matrix is called pseudo norm of a fuzzy matrix if it fulfill the following conditions*

1. $\|A\| \geq 0$ for any fuzzy matrix A.
2. if $A = 0$ then $\|A\| = 0$.
3. $\|kA\| = |k|\,\|A\|$ for any scaler $k \in [0,1]$.

4. $\|A+B\| \leq \|A\|+\|B\|$ for any two fuzzy matrix A and B.

5. $\|AB\| \leq \|A\|\|B\|$ for any two fuzzy matrix A and B.

Definition 13. ***Max-min Norm*** *(Maity [24]):* *Max-Min norm of a fuzzy matrix A is denoted by* $\|A\|_{MM}$ *and define by* $\|A\|_{MM} = \bigwedge_{i=1}^{n} (\bigvee_{j=1}^{n} a_{ij})$

Here, first we find the maximum element in each row and then minimum of the maximum elements.

Definition 13. ***Column-average-max Norm*** *(Samanta and Maity [36]): Colum-average-max norm of a fuzzy matrix A is denoted by* $\|A\|_{CAM}$ *and defined by* $\|A\|_{CAM} = \bigvee_{j=1}^{n} \left(\frac{1}{n}\sum_{i=1}^{n} a_{ij}\right)$.

Here, firstly we find the average value in each row and then find maximum of these average values.

2.3. Addition and multiplication of fuzzy matrices

We have used the operator $\oplus$ for addition of fuzzy matrices and used the operator $\otimes$ for multiplication of fuzzy matrices. This two operators are define by the following way.

If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & & b_{nn} \end{bmatrix}$

Then $A \oplus B = \begin{bmatrix} \frac{a_{11}+b_{11}}{2} & \frac{a_{12}+b_{12}}{2} & \cdots & \frac{a_{1n}+b_{1n}}{2} \\ \frac{a_{21}+b_{21}}{2} & \frac{a_{22}+b_{22}}{2} & \cdots & \frac{a_{2n}+b_{2n}}{2} \\ \vdots & \vdots & & \vdots \\ \frac{a_{n1}+b_{n1}}{2} & \frac{a_{n2}+b_{n2}}{2} & \cdots & \frac{a_{nn}+b_{nn}}{2} \end{bmatrix}$ and

$$A \otimes B = \begin{bmatrix} \wedge\{a_{11}, b_{11}\} & \wedge\{a_{12}, b_{12}\} & \cdots & \wedge\{a_{1n}, b_{1n}\} \\ \wedge\{a_{21}, b_{21}\} & \wedge\{a_{22}, b_{22}\} & \cdots & \wedge\{a_{2n}, b_{2n}\} \\ \vdots & \vdots & & \vdots \\ \wedge\{a_{n1}, b_{n1}\} & \wedge\{a_{n2}, b_{n2}\} & \cdots & \wedge\{a_{nn}, b_{nn}\} \end{bmatrix}$$

In this type of multiplication, fuzzy matrices will be of same order.

Example 1. If $A = \begin{bmatrix} 0.4 & 0.3 & 0.6 \\ 0.7 & 0.2 & 0.5 \\ 0.3 & 0.4 & 0.1 \end{bmatrix}$ and $B = \begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.1 & 0.6 & 0.3 \\ 0.3 & 0.6 & 0.5 \end{bmatrix}$

Then $A \oplus B = \begin{bmatrix} 0.3 & 0.4 & 0.5 \\ 0.4 & 0.4 & 0.4 \\ 0.3 & 0.5 & 0.3 \end{bmatrix}$ and $A \otimes B = \begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.1 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix}$

3. Row-average-max-norm (RAM)

Here we will define a new type of norm called Row-Average-Max norm. We will use new type of operators of fuzzy matrices for this norm. Here, at first, we will determine the average of the elements in each row. Then we will find the maximum element of this average. Row-Average-Max Norm of a fuzzy matrix C is denoted by $\|C\|_{RAM}$ and defined by

$$\|C\|_{RAM} = \bigvee_{i=1}^{n} \frac{1}{n}(\sum_{j=1}^{n} c_{ij})$$

Lemma 1. *All the conditions of norm are satisfied by*

$$\|C\|_{RAM} = \bigvee_{i=1}^{n} \frac{1}{n}(\sum_{j=1}^{n} c_{ij})$$

Proof: Let us consider

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ c_{n1} & c_{n2} & & c_{nn} \end{bmatrix} \text{ and } D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ d_{n1} & d_{n2} & & d_{nn} \end{bmatrix}$$

$\|C\|_{RAM} = \vee_{i=1}^{n} \frac{1}{n}(\sum_{j=1}^{n} c_{ij})$ and $\|D\|_{RAM} = \vee_{i=1}^{n} \frac{1}{n}(\sum_{j=1}^{n} d_{ij})$

(i) As all $c_{ij} \geq 0$ so according to the definition of Row-Average-Max norm obviously $\|C\|_{RAM} \geq 0$.

Now $\|C\|_{RAM} = 0 \Leftrightarrow \vee_{i=1}^{n} \frac{1}{n}(\sum_{j=1}^{n} c_{ij}) = 0$

$\Leftrightarrow \sum_{j=1}^{n} c_{ij} = 0$ for all $i = 1,2,\ldots..n$

$\Leftrightarrow c_{i1} = c_{i2} = \cdots = c_{in,}$ for all $i = 1,2,\ldots..,n$

$\Leftrightarrow c_{ij} = 0$ for all $i,j = 1,2,\ldots.,n$

$\Leftrightarrow C = 0$

So, $\|C\|_{RAM} = 0$ iff $C = 0$

(ii) Here we define a new type of scalar multiplication as follows

$$\alpha c_{ij} = \begin{cases} |\alpha| if |\alpha| \leq \|C\|_{RAM} \\ \|C\|_{RAM} if |\alpha| > \|C\|_{RAM} \end{cases}$$

So, if $|\alpha| \leq \|C\|_{RAM}$ *then* $\|\alpha C\|_{RAM} = |\alpha| = |\alpha|\|C\|_{RAM}$

and if $|\alpha| > \|C\|_{RAM}$ *then* $\|\alpha C\|_{RAM} = \|C\|_{RAM} = |\alpha|\|C\|_{RAM.}$

Therefore $\|\alpha C\|_{RAM} = |\alpha|\|C\|_{RAM}$ for all $\alpha \epsilon [0,1]$.

(iii) Now,

$$C \oplus D = \begin{bmatrix} \frac{c_{11}+d_{11}}{2} & \frac{c_{12}+d_{12}}{2} & \cdots & \frac{c_{1n}+d_{1n}}{2} \\ \frac{c_{21}+d_{21}}{2} & \frac{c_{22}+d_{22}}{2} & \cdots & \frac{c_{2n}+d_{2n}}{2} \\ \vdots & \vdots & & \vdots \\ \frac{c_{n1}+d_{n1}}{2} & \frac{c_{n2}+d_{n2}}{2} & \cdots & \frac{c_{nn}+d_{nn}}{2} \end{bmatrix}$$

Then $\|C \oplus D\|_{RAM}$

$= \vee_{i=1}^{n} \left[\frac{\frac{c_{i1}+d_{i1}}{2} + \frac{c_{i2}+d_{i2}}{2} + \ldots\ldots \frac{c_{in}+d_{in}}{2}}{n}\right]$

$= \vee_{i=1}^{n} \left[\frac{c_{i1}+d_{i1}+c_{i2}+d_{i2}+\cdots\cdots+c_{in}+d_{in}}{2n}\right]$

$= \vee_{i=1}^{n} \left[\frac{\sum_{j=1}^{n} c_{ij} + \sum_{j=1}^{n} d_{ij}}{2n}\right]$

$\leq \frac{1}{2}[\vee_{i=1}^{n} \frac{1}{n} \sum_{j=1}^{n} c_{ij} + \vee_{i=1}^{n} \frac{1}{n} \sum_{j=1}^{n} d_{ij}]$

$= \frac{\|C\|_{RAM} + \|D\|_{RAM}}{2}$

$= \|C\|_{RAM} \oplus \|D\|_{RAM}$

So, $\|C \oplus D\|_{RAM} \leq \|C\|_{RAM} \oplus \|D\|_{RAM}$

(iv) $C \otimes D = \begin{bmatrix} \wedge\{c_{11}, d_{11}\} & \wedge\{c_{12}, d_{12}\} & \cdots & \wedge\{c_{1n}, d_{1n}\} \\ \wedge\{c_{21}, d_{21}\} & \wedge\{c_{22}, d_{22}\} & \cdots & \wedge\{c_{2n}, d_{2n}\} \\ \vdots & \vdots & & \vdots \\ \wedge\{c_{n1}, d_{n1}\} & \wedge\{c_{n2}, d_{n2}\} & \cdots & \wedge\{c_{nn}, d_{nn}\} \end{bmatrix}$

Now, $\wedge\{c_{ij}, d_{ij}\} \leq c_{ij} and\ d_{ij} for all\ i, j.$

$So,\ \sum_{j=1}^{n}\{\wedge(c_{ij}, d_{ij})\} \leq \sum_{j=1}^{n} c_{ij} and \sum_{j=1}^{n} d_{ij} for all\ i.$

$\Rightarrow \vee_{i=1}^{n} \frac{1}{n} \sum_{j=1}^{n}\{\wedge(c_{ij}, d_{ij})\} \leq \vee_{i=1}^{n} \frac{1}{n} \sum_{j=1}^{n} c_{ij} and \vee_{i=1}^{n} \frac{1}{n} \sum_{j=1}^{n} d_{ij}.$

Therefore $\|C \otimes D\|_{RAM} \leq \|C\|_{RAM} \otimes \|D\|_{RAM}$

Hence, all the conditions of norm are satisfied by Row-Average-Max.

3.1 Properties of row-average-max norm

Properties 1. *If P and Q are two fuzzy matrices then*

$$\|(P \oplus Q)^T\|_{RAM} \leq \|P^T\|_{RAM} \oplus \|Q^T\|_{RAM}.$$

Proof: Let us consider

$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ p_{n1} & p_{n2} & & p_{nn} \end{bmatrix}$ and $Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ q_{n1} & q_{n2} & & q_{nn} \end{bmatrix}$

Then $P \oplus Q = \begin{bmatrix} \frac{p_{11}+q_{11}}{2} & \frac{p_{12}+q_{12}}{2} & \cdots & \frac{p_{1n}+q_{1n}}{2} \\ \frac{p_{21}+q_{21}}{2} & \frac{p_{22}+q_{22}}{2} & \cdots & \frac{p_{2n}+q_{2n}}{2} \\ \vdots & \vdots & & \vdots \\ \frac{p_{n1}+q_{n1}}{2} & \frac{p_{n2}+q_{n2}}{2} & \cdots & \frac{p_{nn}+q_{nn}}{2} \end{bmatrix}$ and

$(P \oplus Q)^T = \begin{bmatrix} \frac{p_{11}+q_{11}}{2} & \frac{p_{21}+q_{21}}{2} & \cdots & \frac{p_{n1}+q_{n1}}{2} \\ \frac{p_{12}+q_{12}}{2} & \frac{p_{22}+q_{22}}{2} & \cdots & \frac{p_{n2}+q_{n2}}{2} \\ \vdots & \vdots & & \vdots \\ \frac{p_{1n}+q_{1n}}{2} & \frac{p_{2n}+q_{2n}}{2} & \cdots & \frac{p_{nn}+q_{nn}}{2} \end{bmatrix}$

$So, \|(P \oplus Q)^T\|_{RAM} = \vee_{i=1}^{n} \frac{1}{n} (\sum_{i=1}^{n} \frac{p_{ji}+q_{ji}}{2}) \leq \vee_{i=1}^{n} \frac{1}{2n} (\sum_{j=1}^{n} p_{ji} + \sum_{j=1}^{n} q_{ji})$

$\leq \frac{\vee_{i=1}^{n} \frac{1}{n}(\sum_{i=1}^{n} p_{ji}) + \vee_{i=1}^{n} \frac{1}{n}(\sum_{j=1}^{n} q_{ji})}{2}$

$= \frac{\|P^T\|_{RAM} + \|Q^T\|_{RAM}}{2}$

$= \|P^T\|_{RAM} \oplus \|Q^T\|_{RAM}$

So, $\|(P \oplus Q)^T\|_{RAM} \leq \|P^T\|_{RAM} \oplus \|Q^T\|_{RAM}.$

Example 2.

Let $P = \begin{bmatrix} 0.5 & 0.3 & 0.7 \\ 0.2 & 0.8 & 0.4 \\ 0.7 & 0.1 & 0.3 \end{bmatrix}$ and $Q = \begin{bmatrix} 0.6 & 0.2 & 0.5 \\ 0.3 & 0.7 & 0.4 \\ 0.5 & 0.4 & 0.6 \end{bmatrix}$

Then $P \oplus Q = \begin{bmatrix} 0.55 & 0.25 & 0.6 \\ 0.25 & 0.75 & 0.4 \\ 0.6 & 0.25 & 0.45 \end{bmatrix}$

$(P \oplus Q)^T = \begin{bmatrix} 0.55 & 0.25 & 0.6 \\ 0.25 & 0.75 & 0.25 \\ 0.6 & 0.4 & 0.45 \end{bmatrix}$

$So, \|(P \oplus Q)^T\|_{RAM} = 0.483$

Now, $P^T = \begin{bmatrix} 0.5 & 0.2 & 0.7 \\ 0.3 & 0.8 & 0.1 \\ 0.7 & 0.4 & 0.3 \end{bmatrix}$ and $Q^T = \begin{bmatrix} 0.6 & 0.3 & 0.5 \\ 0.2 & 0.7 & 0.4 \\ 0.5 & 0.4 & 0.6 \end{bmatrix}$

Here, $\|P^T\|_{RAM} = 0.467$, $\|Q^T\|_{RAM} = 0.5$

Then $\|P^T\|_{RAM} \oplus \|Q^T\|_{RAM} = 0.483$

So, $\|(P \oplus Q)^T\|_{RAM} \leq \|P^T\|_{RAM} \oplus \|Q^T\|_{RAM}$

Properties 2. *If P and Q are two fuzzy matrices and* $P \leq Q$ *then* $\|P\|_{RAM} \leq \|Q\|_{RAM}$

Proof: As $P \leq Q$, so $p_{ij} \leq q_{ij}$ for all i, j.

This implies, $\sum_{j=1}^{n} p_{ij} \leq \sum_{j=1}^{n} q_{ij}$ for all i.

$\Rightarrow \frac{1}{n}(\sum_{j=1}^{n} p_{ij}) \leq \frac{1}{n}(\sum_{j=1}^{n} q_{ij})$, for all i.

$\Rightarrow \vee_{i=1}^{n} \frac{1}{n}(\sum_{j=1}^{n} p_{ij}) \leq \vee_{i=1}^{n} \frac{1}{n}(\sum_{j=1}^{n} q_{ij})$

$\Rightarrow \|P\|_{RAM} \leq \|Q\|_{RAM}.$

Example 3.

Let $P = \begin{bmatrix} 0.5 & 0.3 & 0.1 \\ 0.4 & 0.6 & 0.4 \\ 0.3 & 0.2 & 0.1 \end{bmatrix}$ and $Q = \begin{bmatrix} 0.6 & 0.4 & 0.2 \\ 0.5 & 0.7 & 0.5 \\ 0.4 & 0.3 & 0.2 \end{bmatrix}$

$\|P\|_{RAM} = 0.467 \text{ and } \|Q\|_{RAM} = 0.567$

Therefore, if $P \leq Q$ then $\|P\|_{RAM} \leq \|Q\|_{RAM}$

Properties 3. *If P,Q and R are three fuzzy matrices and* $P \leq Q$ *then*

$$\|P \otimes R\|_{RAM} \leq \|Q \otimes R\|_{RAM} \text{hold}.$$

Proof: Let us consider

$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ p_{n1} & p_{n2} & & p_{nn} \end{bmatrix}, Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ q_{n1} & q_{n2} & & q_{nn} \end{bmatrix}$

and $R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ r_{n1} & r_{n2} & & r_{nn} \end{bmatrix}$

Then $P \otimes R = \begin{bmatrix} \wedge\{p_{11}, r_{11}\} & \wedge\{p_{12}, r_{12}\} & \cdots & \wedge\{p_{1n}, r_{1n}\} \\ \wedge\{p_{21}, r_{21}\} & \wedge\{p_{22}, r_{22}\} & \cdots & \wedge\{p_{2n}, r_{2n}\} \\ \vdots & \vdots & \cdots & \vdots \\ \wedge\{p_{n1}, r_{n1}\} & \wedge\{p_{n2}, r_{n2}\} & & \wedge\{p_{nn}, r_{nn}\} \end{bmatrix}$ and

$$Q \otimes R = \begin{bmatrix} \wedge\{q_{11}, r_{11}\} & \wedge\{q_{12}, r_{12}\} & \cdots & \wedge\{q_{1n}, r_{1n}\} \\ \wedge\{q_{21}, r_{21}\} & \wedge\{q_{22}, r_{22}\} & \cdots & \wedge\{q_{2n}, r_{2n}\} \\ \vdots & \vdots & \cdots & \vdots \\ \wedge\{q_{n1}, r_{n1}\} & \wedge\{q_{n2}, r_{n2}\} & & \wedge\{q_{nn}, r_{nn}\} \end{bmatrix}$$

$\|P \otimes R\|_{RAM} = \vee_{i=1}^{n} \frac{1}{n}\left[\sum_{j=1}^{n}\{\wedge(p_{ij}, r_{ij})\}\right]$

and $\|Q \otimes R\|_{RAM} = \vee_{i=1}^{n} \frac{1}{n}\left[\sum_{j=1}^{n}\{\wedge(q_{ij}, r_{ij})\}\right]$

Now, $P \leq Q$ *implies* $p_{ij} \leq q_{ij}$ *for all* i, j.

$\Rightarrow \wedge\{p_{ij}, r_{ij}\} \leq \wedge\{q_{ij}, r_{ij}\} for all i, j.$

$\Rightarrow \sum_{j=1}^{n}\{\wedge(p_{ij}, r_{ij})\} \leq \sum_{j=1}^{n}\{\wedge(q_{ij}, r_{ij})\}$ *for all* i.

$\Rightarrow \vee_{i=1}^{n} \frac{1}{n}\left[\sum_{j=1}^{n}\{\wedge(p_{ij}, r_{ij})\}\right] \leq \vee_{i=1}^{n} \frac{1}{n}\left[\sum_{j=1}^{n}\{\wedge(q_{ij}, r_{ij})\}\right]$

$\Rightarrow \|P \otimes R\|_{RAM} \leq \|Q \otimes R\|_{RAM}$

So, if $P \leq Q$ then $\|P \otimes R\|_{RAM} \leq \|Q \otimes R\|_{RAM}$.

Example 4.

Let $P = \begin{bmatrix} 0.5 & 0.3 & 0.6 \\ 0.2 & 0.5 & 0.4 \\ 0.6 & 0.1 & 0.4 \end{bmatrix}$, $Q = \begin{bmatrix} 0.6 & 0.4 & 0.5 \\ 0.3 & 0.7 & 0.6 \\ 0.8 & 0.4 & 0.6 \end{bmatrix}$ and $R = \begin{bmatrix} 0.3 & 0.9 & 0.4 \\ 0.1 & 0.8 & 0.2 \\ 0.7 & 0.2 & 0.3 \end{bmatrix}$

Now, $P \otimes R = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.1 & 0.5 & 0.2 \\ 0.6 & 0.1 & 0.3 \end{bmatrix}$ and $Q \otimes R = \begin{bmatrix} 0.3 & 0.4 & 0.4 \\ 0.1 & 0.7 & 0.2 \\ 0.7 & 0.2 & 0.3 \end{bmatrix}$

Then $\|P \otimes R\|_{RAM} = 0.333$, $\|Q \otimes R\|_{RAM} = 0.4$

So, $\|P \otimes R\|_{RAM} \leq \|Q \otimes R\|_{RAM}$

Hence if $P \leq Q$, then $\|P \otimes R\|_{RAM} \leq \|Q \otimes R\|_{RAM}$

Properties 4. *If* P *and* Q *are two fuzzy matrices, then* $\|P \oplus Q\|_{RAM} = \|Q \oplus P\|_{RAM}$.

Proof: Let us consider

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ p_{n1} & p_{n2} & & p_{nn} \end{bmatrix} \text{and } Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ q_{n1} & q_{n2} & & q_{nn} \end{bmatrix}$$

Then
$$P \oplus Q = \begin{bmatrix} \frac{p_{11}+q_{11}}{2} & \frac{p_{12}+q_{12}}{2} & \cdots & \frac{p_{1n}+q_{1n}}{2} \\ \frac{p_{21}+q_{21}}{2} & \frac{p_{22}+q_{22}}{2} & \cdots & \frac{p_{2n}+q_{2n}}{2} \\ \vdots & \vdots & & \vdots \\ \frac{p_{n1}+q_{n1}}{2} & \frac{p_{n2}+q_{n2}}{2} & \cdots & \frac{p_{nn}+q_{nn}}{2} \end{bmatrix}$$

Now, $\|P \oplus Q\|_{RAM} = \vee_{i=1}^{n}\left[\frac{\frac{p_{i1}+q_{i1}}{2} + \frac{p_{i2}+q_{i2}}{2} + \ldots\ldots \frac{p_{in}+q_{in}}{2}}{n}\right]$

$= \vee_{i=1}^{n}\left[\frac{1}{n}\left\{\frac{q_{i1}+p_{i1}}{2} + \frac{q_{i2}+p_{i2}}{2} + \cdots \frac{q_{in}+p_{in}}{2}\right\}\right]$

$= \|Q \oplus P\|_{RAM}$

Hence $\|P \oplus Q\|_{RAM} = \|Q \oplus P\|_{RAM}$.

Example 5. Let $P = \begin{bmatrix} 0.6 & 0.4 & 0.2 \\ 0.5 & 0.7 & 0.3 \\ 0.8 & 0.5 & 0.4 \end{bmatrix}$ and $Q = \begin{bmatrix} 0.4 & 0.6 & 0.2 \\ 0.7 & 0.3 & 0.5 \\ 0.5 & 0.8 & 0.4 \end{bmatrix}$

Now, $P \oplus \mathrm{Q} = \begin{bmatrix} 0.5 & 0.5 & 0.2 \\ 0.6 & 0.5 & 0.4 \\ 0.65 & 0.65 & 0.4 \end{bmatrix}$ and $Q \oplus \mathrm{P} = \begin{bmatrix} 0.5 & 0.5 & 0.2 \\ 0.6 & 0.5 & 0.4 \\ 0.65 & 0.65 & 0.4 \end{bmatrix}$

Then, $\|P \oplus Q\|_{RAM} = 0.567$ $and \|Q \oplus P\|_{RAM} = 0.567$

So, $\|P \oplus Q\|_{RAM} = \|Q \oplus P\|_{RAM}$.

Properties 5. *If* P *and* Q *are two fuzzy matrices, then* $\|P \otimes Q\|_{RAM} = \|Q \otimes P\|_{RAM}$.

Proof: Let us consider

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ p_{n1} & p_{n2} & & p_{nn} \end{bmatrix} \text{ and } Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ q_{n1} & q_{n2} & & q_{nn} \end{bmatrix}$$

Then $P \otimes Q = \begin{bmatrix} \wedge\{p_{11}, q_{11}\} & \wedge\{p_{12}, q_{12}\} & \cdots & \wedge\{p_{1n}, q_{1n}\} \\ \wedge\{p_{21}, q_{21}\} & \wedge\{p_{22}, q_{22}\} & \cdots & \wedge\{p_{2n}, q_{2n}\} \\ \vdots & \vdots & \cdots & \vdots \\ \wedge\{p_{n1}, q_{n1}\} & \wedge\{p_{n2}, q_{n2}\} & & \wedge\{p_{nn}, q_{nn}\} \end{bmatrix}$

Now, $\|P \otimes Q\|_{RAM} = \vee_{\mathrm{i}=1}^{\mathrm{n}}[\frac{1}{\mathrm{n}}\sum_{\mathrm{j}=1}^{\mathrm{n}}\{\wedge(\mathrm{p_{ij}}, \mathrm{q_{ij}})\}]$

$= \vee_{i=1}^{n}\left[\frac{1}{n}\sum_{j=1}^{n}\{\wedge(q_{ij}, p_{ij})\}\right]$

$= \|Q \otimes P\|_{RAM}$

So, $\|P \otimes Q\|_{RAM} = \|Q \otimes P\|_{RAM}$.

Example 6. Let $P = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.5 & 0.6 & 0.2 \\ 0.7 & 0.4 & 0.8 \end{bmatrix}$ and $Q = \begin{bmatrix} 0.4 & 0.6 & 0.2 \\ 0.7 & 0.3 & 0.5 \\ 0.5 & 0.8 & 0.4 \end{bmatrix}$

Now, $P \otimes \mathrm{Q} = \begin{bmatrix} 0.4 & 0.3 & 0.1 \\ 0.5 & 0.3 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$ and $Q \otimes \mathrm{P} = \begin{bmatrix} 0.4 & 0.3 & 0.1 \\ 0.5 & 0.3 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$

Then $\|P \otimes Q\|_{RAM} = 0.433$ and $\|Q \otimes P\|_{RAM} = 0.433$

So, $\|P \otimes Q\|_{RAM} = \|Q \otimes P\|_{RAM}$.

Definition 14. *Define a mapping* $d: M_n(F) \times M_n(F) \to [0,1]$ *as* $d(P, Q) = \|P \oplus Q\|_{RAM}$ *for all* P, Q *in* $M_n(F)$.

Proposition 1. *The above mapping d satisfies the following condition for all* P, Q, R *in* $M_n(F)$.

(i) $d(P, Q) \geq 0$ *and* $d(P, Q) = 0$ *iff* $P = Q = 0$.

(ii) $d(P, Q) = d(Q, P)$.

(iii) $d(P, Q) \leq d(P, R) + d(Q, R)$ *for all* P, Q, R *in* $M_n(F)$.

Proof: (i) $(P, Q) = \|P \oplus \mathrm{Q}\|_{RAM} \geq 0$. [by first condition of norm]

Again, $\mathrm{d(P, Q)} = 0 \Leftrightarrow \|P \oplus \mathrm{Q}\|_{RAM} = 0$

$\Leftrightarrow \|P\|_{RAM} \oplus \|Q\|_{RAM} = 0$

$\Leftrightarrow \frac{\|P\|_{RAM} + \|Q\|_{RAM}}{2} = 0$

$\Leftrightarrow \|P\|_{RAM} + \|Q\|_{RAM} = 0$
$\Leftrightarrow \mathrm{P} + \mathrm{Q} = 0$
$\Leftrightarrow \mathrm{P} = 0$ and $\mathrm{Q} = 0$

(ii) $d(P,Q) = \|P \oplus Q\|_{RAM} = \|Q \oplus P\|_{RAM} = d(Q,P)$ [by properties 4]

(iii)$d(P,Q) = \|P \oplus Q\|_{RAM} \leq \|P \oplus Q\|_{RAM} \oplus \|R\|_{RAM}$
$= \|P \oplus \mathrm{Q} \oplus \mathrm{R}\|_{RAM}$
$= \|P \oplus \mathrm{Q} \oplus \mathrm{R} \oplus \mathrm{R}\|_{RAM}$
$= \|(P \oplus \mathrm{R}) + (Q \oplus \mathrm{R})\|_{RAM}$
$= \|P \oplus \mathrm{R}\|_{RAM} + \|Q \oplus \mathrm{R}\|_{RAM}$
$= d(P,R) + d(Q,R)$

So, $d(P,Q) \leq d(P,R) + d(Q,R) forall\ P,Q,R\ in\ M_n(F)$.

Example 7. Let $P = \begin{bmatrix} 0.5 & 0.2 & 0.4 \\ 0.6 & 0.3 & 0.7 \\ 0.3 & 0.8 & 0.1 \end{bmatrix}$, $Q = \begin{bmatrix} 0.8 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.5 \\ 0.6 & 0.2 & 0.6 \end{bmatrix}$ and $R = \begin{bmatrix} 0.4 & 0.6 & 0.3 \\ 0.3 & 0.5 & 0.4 \\ 0.7 & 0.2 & 0.5 \end{bmatrix}$

Now, $P \oplus \mathrm{Q} = \begin{bmatrix} 0.65 & 0.15 & 0.35 \\ 0.45 & 0.35 & 0.6 \\ 0.45 & 0.5 & 0.35 \end{bmatrix}$. Then $\|P \oplus Q\|_{RAM} = 0.467$

$P \oplus \mathrm{R} = \begin{bmatrix} 0.45 & 0.4 & 0.35 \\ 0.45 & 0.4 & 0.55 \\ 0.5 & 0.5 & 0.3 \end{bmatrix}$ and $Q \oplus \mathrm{R} = \begin{bmatrix} 0.6 & 0.35 & 0.3 \\ 0.3 & 0.45 & 0.45 \\ 0.65 & 0.2 & 0.55 \end{bmatrix}$

Then $\|P \oplus \mathrm{R}\|_{RAM} = 0.467$ and $\|Q \oplus R\|_{RAM} = 0.467$

(i) $d(P,Q) = \|P \oplus Q\|_{RAM} = 0.467 > 0$

(ii) $Q \oplus \mathrm{P} = \begin{bmatrix} 0.65 & 0.15 & 0.35 \\ 0.45 & 0.35 & 0.6 \\ 0.45 & 0.5 & 0.35 \end{bmatrix}$. Then $\|Q \oplus P\|_{RAM} = 0.467$

So, $d(P,Q) = d(Q,P)$.

(iii) Here $d(P,R) + d(Q,R) = \|P \oplus R\|_{RAM} + \|Q \oplus R\|_{RAM} = 0.934$

So, $d(P,Q) \leq d(P,R) + d(Q,R)$

Definition 15. *Define a mapping:*$d': M_n(F) \times M_n(F) \to [0,1]$ *as* $d'(P,Q) = Min\{\|P\|_{RAM}, \|Q\|_{RAM}\}$, *for all* P,Q *in* $M_n(F)$.

Proposition 2. *The above mapping* d' *satisfies the following condition for all* P,Q,R *in*$M_n(F)$.

(i) $d'(P,Q) \geq 0$ *and* $d'(P,Q) = 0 iff P = 0$ *or* $Q = 0$ *or both* $P = Q = 0$.

(ii) $d'(P,Q) = d'(Q,P)$.

Proof: (i) $d'(P,Q) = Min\{\|\mathrm{P}\|_{\mathrm{RAM}}, \|Q\|_{RAM}\} \geq 0$ as $\|P\|_{RAM} \geq 0$ and $\|Q\|_{RAM} \geq 0$.

Now, $d'(P,Q) = Min\{\|\mathrm{P}\|_{\mathrm{RAM}}, \|Q\|_{RAM}\} = 0$
$\Rightarrow \|\mathrm{P}\|_{\mathrm{RAM}} = 0$ or$\|Q\|_{RAM} = 0$ or both $\|\mathrm{P}\|_{\mathrm{RAM}} = \|Q\|_{RAM} = 0$
$\Rightarrow$ either $P = 0$ or $Q = 0$ or both $P = Q = 0$

$(ii) d'(P,Q) = Min\{\|\mathrm{P}\|_{\mathrm{RAM}}, \|\mathrm{Q}\|_{\mathrm{RAM}}\} = \{\mathrm{Min}\|Q\|_{RAM}, \|P\|_{RAM}\} = d'(Q,P)$

So, $d'(P,Q) = d'(Q,P)$.

Proposition 3. *If* $P,Q \in M_n(F)$ *and* $P \leq Q$ *then* $d'(P,R) \leq d'(Q,R)$ *for all*$R \in M_n(F)$.

Proof: Since, $P \leq Q$, so $\|P\|_{RAM} \leq \|Q\|_{RAM}$.
Now, $d'(P,R) = Min\{\|P\|_{RAM}, \|R\|_{RAM}\}$ and $d'(Q,R) = Min\{\|Q\|_{RAM}, \|R\|_{RAM}\}$.
Case-1:
If $\|P\|_{RAM} \leq \|Q\|_{RAM} \leq \|R\|_{RAM}$ then
$d'(P,R) = \|P\|_{RAM} \leq \|Q\|_{RAM} = d'(Q,R)$.
That is, $d'(P,R) \leq d'(Q,R)$.
Case-2:
If $\|R\|_{RAM} \leq \|P\|_{RAM} \leq \|Q\|_{RAM}$ then
$d'(P,R) = \|R\|_{RAM} = d'(Q,R)$.
Case-3:
If $\|P\|_{RAM} \leq \|R\|_{RAM} \leq \|Q\|_{RAM}$ then $d'(P,R) = \|P\|_{RAM}$ and $d'(Q,R) = \|R\|_{RAM}$.
So, $d'(P,R) \leq d'(Q,R)$.
Therefore, $d'(P,R) \leq d'(Q,R)$ for all $R \in M_n(F)$.

3.2. Algorithm

Input: A fuzzy matrix of order $n \times n$
Output: A real number in [0,1]

```
Max=0;
for i=1 to n
    ans=0;
    for j=1 to n
    ans=ans+a[i][j];
      end for
    ans =ans/n;
       if max<ans then max=ans;
end for
return max;
```

4. Conclusion

In this paper, two types of operators on fuzzy matrices have been used for our further analysis. By employing these, we have defined row-average-max norm with some important properties. The coffee industry of India is the sixth largest producer of coffee in the world. India coffee is said to be the finest coffee grown in the shade rather than direct sunlight anywhere in the world. Coffee cultivators in Kodai Hills faced different types of issues including Labour shortage, Storage problem, Low margin of profit, Monsoon failures and unseasonal rains etc. Fuzzy matrices are used to analysis the problems encountered by the coffee cultivators in Kodai Hills. Hence, in these aspects norm of fuzzy matrices will help in this case.

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