Intern. J. Fuzzy Mathematical Archive
Vol. 2, 2013, 70-77
ISSN: 2320 –3242 (P), 2320 –3250 (online)
Published on 4 September 2013
www.researchmathsci.org

International Journal of
Fuzzy Mathematical Archive

The Re-nnd Definite Solutions of the Matrix Equation AXB = C in Minkowski Space M

***D. Krishnaswamy*[*] and *G. Punithavalli*[†]**
[*]Department of Mathematics, [†]Mathematics Wing, DDE
Annamalai University, Annamalainagar, India
[*]e_mail: krishna-swamy2004@yahoo.co.in
[†]e_mail: punithavarman@gmail.com

Received 10 July 2013; accepted 29 July 2013

Abstract. In this paper, we first consider the Matrix equation $AXA^{\sim} = C$, where $A \in C^{n\times m}$, $C \in C^{n\times n}$ and establish necessary and sufficient conditions for the existence of Re-nnd solutions. Further,we determine the necessary and sufficient conditions for the existence of Re-nnd solutions of the equation $AXB = C$ in terms of Minkowski inverses .

Keywords: Re-nnd solutions, Matrix equation, generalized inverse.

AMS Mathematics Subject Classification (2010): 15A09, 15A48

1. Introduction

We shall deal with $C^{n\times m}$ the space of complex *n*-tuples. We shall index the components of a complex vector in C^n from *o* to *n*–1. That is u = $(u_0,u_1,u_2, \dots u_{n-1})$. Let G be the Minkowski metric tensors defined by $Gu = (u_0, -u_1, -u_2, \dots -u_{n-1})$. Clearly the Minkowski metric matrix.

$$G = \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix} \text{ satisfies } G = G^* \text{ and } G^2 = I_n. \tag{1}$$

The Minkowski inner product on C^n is defined by $(u,v) = \langle u, Gv\rangle$ where $\langle .,. \rangle$ denotes the conventional Hilbert (unitary) space inner product. A space with Minkowski inner product is called a Minkowski space denoted as $\mathcal{M}$.

For $A \in C^{m\times n}$, let *R(A), rk(A)* and A^* denote the range space, null space, rank of *A* and the conjugate transpose of A respectively. I_n denotes the unit matrix of order n. For $A \in C^{m\times n}$, $X,Y \in C^n$ using (1) we get.

$(AX,Y) = [AX,GY] = [X,A^*GY] = [X,G[GA^*G]Y] = [X,G A^{\sim} Y] = (X, A^{\sim} Y)$.

The Matrix $A^{\sim} = GA^*G$ is called the Minkowski adjoint of *A* in $\mathcal{M}$, where A^* is the usual Hermitian adjoint of A . It is well known that for $A \in C^{n\times n}$, $rk(AA^*) = rk(A^*A) =$

rk(A) and in general $rk(AA^{\sim}) \neq rk(A^{\sim}A) \neq rk(A)$. Naturally we call a matrix $A \in C^{n\times n}$ is *m*-symmetric in Minkowski space $\mathcal{M}$ if $A^{\sim} = A$. From the definition $A^{\sim} = GA^{*}G$, we have the following equivalence proved in [9].

A is m – symmetric iff AG is Hermitian iff GA is Hermitian. (2)

A is m – symmetric iff $(AX,X) = (X, A^{\sim}X)$, for every $X \in C^n$ (3)

The Hermitian part of X is defined as $H(X) = \frac{1}{2}(X + X^{*})$. We say that X is Re-nnd if $H(X) \geq 0$ and X is Re-pd if $H(X) > 0$. The Hermitian part of X is defined as $H(X) = \frac{1}{2}(GXG + X^{\sim})$ in Minkowski space $\mathcal{M}$. We will say that X is Re-nnd if $H(X) \geq 0$ and X is Re-pd if $H(X) > 0$.

2. Preliminaries

Definition 2.1. For $A \in C^{m\times n}$, A^{g} is said to be a generalized inverse (g-inverse) of A if

$$AA^{g}A = A \tag{4}$$

Definition 2.2. For $A \in C^{m\times n}$, A^{r} is said to be a reflexive g-inverse of A if

$$AA^{r}A = A \text{ and } A^{r}AA^{r} = A^{r} \tag{5}$$

Definition 2.3. For $A \in C^{m\times n}$, the Moore-penrose inverse of A denoted as $A^{\dagger}$ is the unique solution of the equations $AXA = A$, $XAX = X$, AX and XA are Hermitian.

The theory of generalized inverses of a matrix plays a fundamental role in solving matrix equations (refer: 3,10]). By using g-inverse, Re-n.n.d solutions to the matrix equations AXB =C has been studied by many researchers([5,6,8,11-16 ,18]) and for reflexive solutions are determined in [4]. Consistency of matrix equations AXA*=B are discussed involving g-inverses in [2,7,17]. Here, we have a made a similar study, by using Minkowski inverse of a matrix in Minkowski space $\mathcal{M}$. Let us recall the corresponding generalized inverse of a matrix in Minkowski space $\mathcal{M}$ in the following:

Definition 2.4. A^{n} is a right (left) normalized g-inverse of A if $AA^{n}A = A$ and $A^{n}AA^{n} = A^{n}$ and AA^{n} is m-symmetric ($A^{n}A$ is m-symmetric).

Definition 2.5. A^{m} is the Minkowski inverse of A if $AA^{m}A = A$, $A^{m}AA^{m} = A^{m}$, AA^{m} and $A^{m}A$ are *m*-symmetric.

Since the Minkowski inverse A^{m} is also a g-inverse of A, we have the following:

Theorem 2.6([10]). *A* necessary and sufficient condition for the equation $AXB = C$ to have a solution is $AA^{m}CB^{m}B = C$, in which case the general solution is $X = A^{m}CB^{m} + Y - A^{m}AYBB^{m}$, where Y is arbitrary.

3. Results

In [1], equivalent conditions for a block matrix to be nnd are determined by using generalized inverses of a matrix. Here, we shall prove a similar result for a block m-symmetric matrix to be nnd by using Minkowski inverse.

Lemma 3.1. [9] Let $A, B \in C^{m\times n}$ in $\mathcal{M}$, then $N(A^*) \subseteq N(B^*)$ if and only if $N(A^\sim) \subseteq N(B^\sim)$.

Theorem 3.2. Let $M \in C^{(n+m)\times(n+m)}$ be an m-symmetric matrix given by

$$M = \begin{bmatrix} A & B \\ B^\sim & D \end{bmatrix}$$

where $A \in C^{n\times n}$ and $D \in C^{m\times m}$. Then $M \geq 0$ if and only if $A \geq 0$, $AA^m B = B$, $D - B^\sim A^m B \geq 0$.

Proof: Let us partition G in conformity with that of M as G = diag.$\{G_1, G_2\}$, where G_1 and G_2 are metric tensors of order n and m respectively. Since M is m-symmetric, by equation (2) GM and MG are Hermitian block matrices. $M \geq O$ if and only if $GM \geq O$. Now by applying Theorem1 of [1] for GM we have , $M \geq O$ if and only if $GM \geq O$ if and only if $A \geq O$, $AA^m B = B$ and $D - B^\sim A^m B \geq O$. Hence the Theorem.

Next, we give necessary and sufficient conditions for the matrix equation $AX = B$ to have a Re-nnd solution X, where A and B are given matrices of suitable size and presents a possible explicit expression for X in Minkowski space $\mathcal{M}$.

Theorem 3.3. Let $A \in C^{n\times m}$, $B \in C^{n\times m}$. There exists a Re-nnd matrix $X \in C^{m\times m}$ satisfying $AX = B$ if and only if $AA^m B = B$ and $AB^\sim$ is Re-nnd.

Proof: $A \in C^{n\times m}$, $B \in C^{n\times m}$, there exists a Re-nnd Matrix $X \in C^{m\times m}$ satisfying $AX = B$,, $AX = B$ implies that $X = A^m B$. Therefore $AA^m B = B$. Next to show that $AB^\sim$ is Re-nnd.

$$AB^\sim = A(AX)^\sim = A\, X^\sim A^\sim = (A^\sim)^\sim X^\sim A^\sim = (AXA^\sim)^\sim \geq 0.$$

In the other direction let us suppose that $AA^m B = B$ and $AB^\sim$ is Re-nnd. Now to show that $AX = B$ for any Re-nnd matrix $X \in C^{m\times m}$

$$\begin{aligned} AX &= A(X_0 + (I - A^m A)Y(I - A^m A)) \\ &= AX_0 + (AY - AA^m AY)(I - A^m A) \\ &= AX_0 + (AY - AY)(I - A^m A) \\ &= AX_0 \end{aligned}$$

$AX = B$ where X_0 is a solution.

Our main aim is to generalize these results to the equation $AXB = C$ and to present a general form of Re-nnd solutions of it. First we will consider the equation.

$$AXA^\sim = C \tag{6}$$

and find necessary and sufficient conditions for the existence of Re-nnd solutions. The next auxiliary result presents a general form of a solution X of (6) which satisfies $H(X) = 0$.

Lemma 3.4. If $A \in C^{n\times m}$, then $X \in C^{m\times m}$ is a solution of the equation

$$AXA^\sim = 0 \tag{7}$$

which satisfies H(X) = 0 if and only if

$$X = W(I - A^m A) - (I - A^m A)W^\sim \tag{8}$$

for some $W \in C^{m\times m}$.

Proof: Denote r = rank(A). Let us suppose that X is a solution of the equation $AXA^{\sim} = 0$ and H(X) = 0. Using a singular value decomposition of $A = U^{\sim}$ Diag$(D,0)V$, where $U \in C^{n\times n}$, $V \in C^{m\times m}$ are unitary and $D \in C^{r\times r}$ is an invertible matrix. We have that $A^m = V^{\sim}$ Diag$(D^{-1},0)$U and

$$X = V^{\sim} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} V,$$

for some $X_1 \in C^{r\times r}$ and $X_4 \in C^{(m-r)\times(m-r)}$.

From $AXA^{\sim} = 0$ we get that $X_1 = 0$ and by $H(X) = 0$, that $X_3 = -X_2^{\sim}$ and $H(X_4) = 0$. Hence X = $V^{\sim} \begin{bmatrix} 0 & x_2 \\ -x_2^{\sim} & x_4 \end{bmatrix}$ V. Taking into account that $H(X_4) = 0$, for $W = V^{\sim} \begin{bmatrix} I & x_2 \\ 0 & x_{4/2} \end{bmatrix} V$, we have that $X = W(I - A^mA) - (I - A^mA)W^{\sim}$. In the other direction we have to check that for arbitrary $W \in C^{m\times m}$, X defined by $X = W(I - A^mA) - (I - A^mA)W^{\sim}$ is a solution of the equation $AXA^{\sim} = 0$. That is

$$\begin{aligned} AXA^{\sim} &= A[W(I - A^mA) - (I - A^mA)W^{\sim}]A^{\sim} \\ &= A[W - WA^mA - W^{\sim} + A^mAW^{\sim}]A^{\sim} \\ &= AWA^{\sim} - AWA^mAA^{\sim} - AW^{\sim}A^{\sim} + AA^mAW^{\sim}A^{\sim} \\ &= AWA^{\sim} - AWA^{\sim} - AW^{\sim}A^{\sim} + AW^{\sim}A^{\sim} \\ AXA^{\sim} &= 0 \end{aligned}$$

$$\begin{aligned} H(X) &= \frac{1}{2}[GXG + X^{\sim}] \\ &= \frac{1}{2}[G[W(I - A^mA) - (I - A^mA)W^{\sim}]G + [W(I - A^mA) - (I - A^mA)W^{\sim}]]^{\sim} \\ &= \frac{1}{2}[G[W - WA^mA] - [W^{\sim} - A^mAW^{\sim}]G + [W - WA^mA - W^{\sim} + A^mAW^{\sim}]]^{\sim} \\ &= \frac{1}{2}[G[W - W] - [W^{\sim} - W^{\sim}]G + [W - W - W^{\sim} + W^{\sim}]]^{\sim} \end{aligned}$$

$H(X) = 0$.

Theorem 3.5. Let A $\in C^{nxm}$, $C \in C^{nxn}$ be given matrices such that the equation (6) is consistent and let r = rankH(C). There exists a Re-nnd solution of the equation(6) if and only if C is Re-nnd. In this case the general Re-nnd solution is given by

$$X = A^{m^m} C(A^{m^m})^{\sim} + (I - A^mA)UU^{\sim}(I - A^mA)^{\sim} + W(I - A^mA) - (I - A^mA)W^{\sim} \quad (9)$$

with $$A^{m^m} = A^m + (I - A^mA)Z((H(C))^{\frac{1}{2}})^m \quad (10)$$

where A^m, $(H(C))^{\frac{1}{2}})^m$ are arbitrary but fixed Minkowski inverse of A and $(H(C))^{\frac{1}{2}}$ respectively $Z \in C^{mxn}$, $U \in C^{mx(m-r)}$, $w \in C^{mxm}$ are arbitrary matrices.

Proof *:* If X is Re-nnd solution of the equation $AH(X)A^{\sim} = H(C) \geq 0$. In the other direction, if C is Re-nnd, then $X_0 = A^m C(A^m)^{\sim}$ is Re-nnd solution of the equation (6).

$$AX_0A^{\sim} = A[A^mC(A^m)^{\sim}]A^{\sim}$$
$$= AA^mC(A^m)^{\sim}A^{\sim}$$
$$= AA^mC(A^m)^{\sim}A^{\sim}$$
$$AX_0A^{\sim} = C.$$

Let us prove that a representation of the general Re-nnd solution is given by (9). If X is defined by (9), then X is Re-nnd and $AXA^{\sim} = AA^mC(AA^m)^{\sim} = C$. If X is an arbitrary Re-nnd solution of (6), then $H(X)$ is an m-symmetric non-negative definite solution of the equation $AZA^{\sim} = H(C)$, so by Theorem 1 of [7],

$$H(X) = A^{mm}H(C)(A^{m^m})^{\sim} + (I - A^mA)UU^{\sim}(I - A^mA)^{\sim},$$

where A^{mm} is given by (10), for some $Z \in C^{mxn}$ and $U \in C^{mx(m-r)}$.

Note that $H(X) = H((A)^{m^m} C(A^{m^m}))^{\sim} + (I - A^mA)UU^{\sim}(I - A^mA)^{\sim}$,

implying $X = A^{m^m} C(A^{m^m})^{\sim} + (I - A^mA)UU^{\sim}(I - A^mA)^{\sim} + Z$, where $H(Z) = 0$ and $AZA^{\sim} = 0$. Using lemma 3.4, we have that $Z = W(I - A^mA) - (I - A^mA)W^{\sim}$, for some $W \in C^{mxn}$. Hence we get that X has a representation as in (9).

Now let us consider the equation

$$AXB = C \tag{11}$$

where $A \in C^{nxm}$, $B \in C^{mxn}$ and $C \in C^{nxn}$ are given matrices and find necessary and sufficient conditions for the existence of a Re-nnd solution. Without loss of generality we may assume that $n = m$ and the matrices A and B are both non-negative definite.
This follows from the fact that whenever $AXB = C$ is solvable then X is a solution of that equation if and only if X is a solution of the equation $A^{\sim}AXBB^{\sim} = A^{\sim}CB^{\sim}$. Hence from now on, we assume that A and B are non-negative definite matrices from the space C^{nxn}. The following theorem presents necessary and sufficient conditions for the equation $AXB = C$ to have a Re-nnd solution.

Theorem 3.6. Let A, B, $C \in C^{nxn}$ be given matrices such that equation (11) is solvable. Then, there exists a Re-nnd solution of (11) if and only if

$$T = B(A + B)^mC(A + B)^mA \tag{12}$$

is Re-nnd, where $(A + B)^m$ is the Minkowski inverse of A + B. In this case a general Re-nnd solution is given by

$$X = (A+B)^{m^m}(C+Y+Z+W)((A+B)^{m^m})^{\sim}+(I-(A+B)^m(A+B))UU^{\sim}(I-(A+B)^m)(A+B)^{\sim})$$
$$+Q(I-(A+B)^m(A+B)) - (I-(A+B)^m(A+B))Q^{\sim}. \tag{13}$$

where Y, Z, W, are arbitrary solutions of the equations

$$Y(A+B)^mB = C(A+B)^mA,\ A(A+B)^mZ = B(A+B)^mC,\ A(A+B)^mW(A+B)^mB = T, \tag{14}$$

such that $C + Y + Z + W$ is Re-nnd, $(A + B)^{m^m}$ is defined by

$$(A + B)^{m^m} = (A + B)^m + (I - (A+B)^m(A + B))P((H(C + Y + Z + W))^{\frac{1}{2}})^m,$$

where $U \in C^{mx(n-r)}$, $Q \in C^{nxn}$, $P \in C^{nxn}$ are arbitrary, r = rank$(C + Y + Z + W)$.

Proof: Denote by
$E = (A + B)^m B$, $F = C(A + B)^m A$, $K = A(A + B)^m$, $L = B(A + B)^m C$.
Now equations (14) are equivalent to

$$YE = F, KZ = L, KWE = T. \tag{15}$$

Using (4) and the fact E is invertible in $\mathcal{M}$ and $E^m = B^m(A+B)$, we have that

$$\begin{aligned} FE^m E &= C(A+B)^m AB^m (A+B)(A+B)^m B \\ &= C(A+B)^m AB^m B \\ &= C(A+B)^m A \\ &= F \end{aligned}$$

Therefore $FE^m E=F$, which implies that the equation $YE = F$ is consistent. In a similar way, we can prove that the other two equations from(15) are consistent. Furthermore, $T^{\sim} = F^{\sim} E = KL^{\sim}$ is Re-nnd which implies by Theorem 3.4, that the first two equations from(4) have Re-nnd solutions. Now suppose that the equation (11) has a Re-nnd solution X. Then

$$\begin{aligned} H(T) &= H(B(A+B)^m AXB(A+B)^m A) \\ &= (B(A+B)^m A)H(X)(B(A+B)^m A)^{\sim} \geq 0 \end{aligned}$$

Conversely, Let T be Re-nnd. We can check that

$$X_0 = (A+B)^m(C+Y+Z+W)(A+B)^m \tag{16}$$

is a solution of the equation(11) where Y,Z,W are arbitrary solutions of the equations(15). This follows from

$$\begin{aligned} AX_0B &= (A+B)(A+B)^m C(A+B)^m (A+B) \\ &= (A+B) (A+B)^m AA^m CB^m B(A+B)^m (A+B)=AA^m CB^m B \\ AX_0B &= C \end{aligned}$$

Now we have to prove that for some choice of Y,Z,W matrix $C+Y+Z+W$ is Re-nnd which would imply that X_0 is Re-nnd.

Let

$$Y = FE^m – (FE^m)^{\sim} + (E^{m\ \sim} F^{\sim} EE^m +(I–EE^m)^{\sim} (I–EE^m),$$

$$Z = K^mL – (K^m L)^{\sim} +K^m KL^{\sim} (K^m)^{\sim} + (I–K^m K)Q(I–K^m K),$$

$$W = K^m TE^m – (I–K^m K)S–S(I–EE^m),$$

where $Q = (C^{\sim} – K^m T^{\sim} E^m) (C^{\sim} –K^m T^{\sim} E^m)^{\sim}$ and $S= K^m KC^{\sim} +C^{\sim} EE^m$. Obviously Y,Z,W are the solutions of the equations (15) and

$$H(Y) = (E^m)^{\sim} H(T) E^m + (I–EE^m)^{\sim} (I–EE^m),$$

$$H(Z) = K^mH(T) (K^m)^{\sim} +(I–K^mK)H(Q)(I–K^m K)^{\sim},$$

$$H(W) = K^mTE^m +(E^m)^{\sim} T^{\sim} (K^m)^{\sim} –H(C^{\sim} EE^m+K^mKC^{\sim} 2K^m T^{\sim} E^m).$$

Using

$$\begin{aligned} K^m KK^m T^{\sim} E^m &= K^m KK^mKL^{\sim} E^m \\ &= K^mKL^{\sim} E^m \\ &= K^m T^{\sim} E^m, \end{aligned}$$

$$K^m T^{\sim} E^m EE^m = K^mF^{\sim} EE^mEE^m = K^mF^{\sim} EE^m = K^m T^{\sim} E^m,$$

$KC^{\sim}E = KL^{\sim} = T^{\sim}$.

We compute,

$H(C+Y+Z+W) = ((E^m)^{\sim} + K^m)\, H(T)((E^m)^{\sim} + K^m)^{\sim} + [(I-EE^m)^{\sim}(I-K^mK)]D(I-EE^m(I-K^mK^{\sim}],$

where

$$D = \begin{bmatrix} I & C-(E^m)^{\sim}T(K^m)^{\sim} \\ C^{\sim}-K^mT^{\sim}E^m & H(Q) \end{bmatrix}$$

By Theorem 3.2, it follows that D is non-negative definite. So $H(C+Y+Z+W) \geq 0$.

Hence, with such a choice of *Y, Z, W* it can be seen that X_0 defined by (16) is Re-nnd solutions of (11). So we proved the sufficient part of the theorem.

Let X be an arbitrary Re-nnd solutions of (11). It is evident that $Y=AXA$, $Z=BXB$ and $W=BXA$ are solutions of (15), and that $(A+B)X(A+B)=C+Y+Z+W$ is Re-nnd. Now, using Theorem 3.5, we get that X has the representation (13). Let us mention that, if we apply Theorem 3.6 to the equation

$$AX=C \tag{17}$$

We get as corollary for the Theorem 4.4 in [8].

Note that if the equation $AX = C$ is consistent then X is a solution of it if and only if $A^{\sim}AX = A^{\sim}C$. By Theorem 3.6, we get that there exists a Re-nnd solution of the equation $AX = C$ if and only if

$$T = (A^{\sim}A+I)^{-1}A^{\sim}C(A^{\sim}A+I)^{-1}A^{\sim}A$$

is Re-nnd. Note that in this case $(I + A^{\sim}A)$ is invertible matrix. Let us prove that T is Re-nnd if an only if $CA^{\sim}$ is Re-nnd. By

$$(A^{\sim}A+I)^{-1}A^{\sim}A = A^{\sim}A(A^{\sim}A+1)^{-1}$$

we have that

$$T = (A^{\sim}A+I)^{-1}A^{\sim}(CA)^{\sim}((A^{\sim}A+I)^{-1}A^{\sim})^{\sim},$$

That is

$$H(T) = ((A^{\sim}A+I)^{-1}A^{\sim})H(CA^{\sim})((A^{\sim}A+I)^{-1}A^{\sim})^{\sim}$$

From the last equality, $H(CA^{\sim}) \geq 0$ implies that $H(T) \geq 0$.

Now, suppose that $H(T) \geq 0$,then, by the consistence of the equation $AX = C$, it follows that $AA^{\dagger}C = C$ which implies that

$$(A^{\dagger})^{\sim}(A^{\sim}A+I)T((A^{\dagger})^{\sim}(A^{\sim}A+I))^{\sim} = (A^{\dagger})^{\sim}A^{\sim}CA^{\sim}AA^{\dagger} = AA^{\dagger}CA^{\sim} = CA^{\sim}$$

That is $H(CA^{\sim}) = ((A^{\dagger})^{\sim}(A^{\sim}A+I))H(T)((A^{\dagger})^{\sim}(A^{\sim}A+I))^{\sim} \geq 0.$

4. Conclusion

In this paper we consider some special cases and give a complete characterization of the set of Re-nnd solution of $AXA^{\sim} = C$. The necessary and sufficient conditions for the existence of Re-nnd solutions of the equation $AXB = C$ in Minkowski space $\mathcal{M}$ is determined.

REFERENCES

1. A. Albert, Conditions for positive and nonnegative definiteness in terms of pseudoinverses, *SIAM J. Appl. Math.*, 17 (2) (1969), 434-440.
2. J.K. Baksalary, Nonnegative definite and positive definite solutions to the matrix equation $AXA^* = B$, *Linear and Multilinear Algebra*, 16 (1984), 133-139.
3. A. Ben-Israel and T.N.E. Greville, *Generalized Inverses: Theory and Applications*, Wiley-Interscience, New York (1974).
4. D. Cvetkovic-Ilic, The reflexive solutions of the matrix equation $AXB = C$, *Comp. Math. Appl.*, 51(6/7) (2006), 897-902.
5. H.Dai, On the symmetric solutions of linear matrix equations, *Linear Algebra Appl.*, 131 (1990), 1-7.
6. J. Gross, Explicit solutions to the matrix inverse problem $AX = B$, *Linear Algebra Appl.*, 289 (1999), 131-134.
7. J. Gross, Nonnegative definite and positive definite solutions to the matrix equation $AXA^* = B$ -revisited, *Linear Algebra Appl.*, 321(2000), 123-129.
8. C.G. Khatri and S.K.Mitra, Hermitian and nonnegative definite solutions of linear matrix equations, *SIAM J. Appl. Math.*, 31(1976), 579-585.
9. A.R.Meenakshi and D.Krishnaswamy, On Sums of Range Symmetric Matrices in Minkowski Space, *Bull. Malaysian Math. Sc. Soc.*, (Second Series) 25 (2002), 137-148.
10. R. Penrose, A generalized inverse for matrices, *Proc. Cambridge Philos. Soc.*, 51 (1955), 406-413.
11. Z.Y. Peng, An iterative method for the least square symmetric solution of the linear matrix equation $AXB = C$, *Appl. Math. Comp.*, 170(1) (2005), 711-723.
12. Z.Y. Peng and X.Y.Hu, The reflexive and anti-reflexive solutions of the matrix equation $AX = B$, *Linear Algebra Appl.*, 375 (2003), 147-155.
13. Y.X. Peng, X.Y.Hu and L.Zhang, An iteration method for the symmetric solutions and the optimal approximation solution of the matrix equation $AXB = C$, *Appl. Math. Comp.*, 160 (2005), 763-777.
14. Q. Wang and C. Yang, The Re-nonnegative definite solutions to the matrix equation $AXB = C$, comment, *Math. Univ. Carolinae*, 39(1) (1998), 7-13.
15. L. Wu, The Re-positive definite solutions to the matrix inverse problem $AX = B$, *Linear Algebra Appl.*, 174 (1992), 145-151.
16. L. Wu and B. Cain, The Re-nonnegative definite solutions to the matrix inverse problem $AX = B$, *Linear Algebra Appl.*, 236 (1996), 137-146.
17. X. Zhang and M.Y.Cheng, The rank-constrained Hermitian nonnegative definite and positive definite solutions to the matrix equation $AXA^* = B$, *Linear Algebra Appl.*, 370 (2003), 163-174.
18. X. Zhang, Hermitian nonnegative definite and positive definite solutions of the matrix equation $AXB = C$, *Appl. Math. E-Notes*, 4 (2004), 40-47.