Intern. J. Fuzzy Mathematical Archive
Vol. 2, 2013, 58-69
ISSN: 2320 –3242 (P), 2320 –3250 (online)
Published on 2 September 2013
www.researchmathsci.org

International Journal of
Fuzzy Mathematical
Archive

Convergence of Maxarithmetic Mean-Minarithmetic Mean Powers of Intuitionistic Fuzzy Matrices

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Received 16 August 2013; accepted 28 August 2013

Abstract. Intuitionistic fuzzy relations on finite universes can be represent by intuitionistic fuzzy matrices and the limiting behavior of the power matrices depends on the algebraic operation employed on the matrices. In this paper, the power of intuitionistic fuzzy matrices with maxarithmetic mean-minarithmetic mean operation have been studied. Here it is shown that the power of intuitionistic fuzzy matrices with the said operations are always convergent.

Keywords: Intuitionistic fuzzy matrix, intuitionistic fuzzy graph, convergence of intuitionistic fuzzy matrix, eigen intuitionistic fuzzy set.

AMS Mathematics Subject Classification (2010): 08A72, 15B15

1. Introduction

Intuitionistic fuzzy matrices (IFMs) have been proposed to represent intuitionistic fuzzy relations on finite universes where relationships between elements are more or less vague. Let X and Y be two universes. It is well known that an intuitionistic fuzzy relation ρ on $X \times Y$ can be presented by an IFM (say R). Furthermore, the composition of finite intuitionistic fuzzy relations can be represented as a product of IFMs. The powers of a finite intuitionistic fuzzy relation defined as $R^m = R^{m-1} \circ R$, have a m-th power representation of R with "$\circ$" operation. The power of an IFM play a crucial role to determine the transitive closure of the underlying IFM.

Thomason's paper [12] published in 1977 was the first to explain the behavior of powers of a fuzzy matrix. He showed that the max-min power of a fuzzy matrix either converge to an idempotent matrix or to oscillate with a finite period. Latter, a number of works on power convergence of fuzzy matrix were published [3, 5, 6, 8, 13, 14]. The behavior of max-product power of a fuzzy matrix quite different from the case with the max-min fuzzy matrices [4]. It is well known that the max-product operation is one of the max-archimedean t-norms. Recent development in this regard has been extended to the infinite products of a finite number of fuzzy matrices [7]. Then Bhowmik and Pal [2] shown that the powers of IFMs are either converge or oscillate with a finite period under the max-min operation. Latter, Pradhan and Pal [9,10] studied the intuitionistic fuzzy linear transfomations and derive the g-inverse of block intuitionistic fuzzy matrices.

Here we take this type of operation for the product of two IFMs, as the sum of the membership and non-membership values of an element of IFMs lies in $[0,1]$. If we take only max-arithmetic mean operation to get the product of two IFMs, then it may arise that the sum of membership and non-membership values of an element be greater than 1.

Let $A=[a_{ij}]=[\langle a_{ij\mu},a_{ij\nu}\rangle]$ be an $n\times m$ intuitionistic fuzzy matrix with $a_{ij\mu}\in[0,1]$ and $a_{ij\nu}\in[0,1]$ such that, $0\le a_{ij\mu}+a_{ij\nu}\le 1$. For the sake of convenience, we denote a_{ij} by $[A]_{ij}$, that is, $[A]_{ij}=\langle a_{ij\mu},a_{ij\nu}\rangle$. Then the maxarithmetic mean-minarithmetic mean operation ``$\circ$'' between two IFM $A=[a_{ij}]_{n\times m}$ and $B=[b_{ij}]_{m\times n}$ can be stated as

$$[A\circ B]_{ij}=\langle \max_{1\le t\le m}\{\frac{a_{it\mu}+b_{tj\mu}}{2}\}, \min_{1\le t\le m}\{\frac{a_{it\nu}+b_{tj\nu}}{2}\}\rangle,\ \textit{for all}\ 1\le i,j\le n.$$

In this paper, we are interested in the behavior of maxarithmetic mean-minarithmetic mean power of an IFM. The product of IFMs under the operation defined above is

(i) non-commutative, that is, $A\circ B\ne B\circ A$,

(ii) non-associative, that is, $(A\circ B)\circ C\ne A\circ(B\circ C)$,

(iii) $A\circ I\ne A$, where I is the intuitionistic fuzzy identity matrix.

(iv) $A\circ 0\ne 0$, where 0 is the intuitionistic fuzzy null matrix.

It is also proved that, maxarithmetic mean-minarithmetic mean powers of an IFM converge and the limit matrix of the power sequence has some special form.

2. Preliminaries

In this section, some elementary aspects that are necessary for this paper are introduced. In fuzzy matrix, the elements of a matrix are the membership degrees only, but in an intuitionistic fuzzy matrix the membership degree and non-membership degree both are represented, which is defined as follows.

Definition 2.1. (Intuitionistic fuzzy matrices) An intuitionistic fuzzy matrix A of order $m\times n$ is defined as $A=[x_{ij},\langle a_{ij\mu},a_{ij\nu}\rangle]_{m\times n}$ where $a_{ij\mu}$, $a_{ij\nu}$ are called membership and non-membership values of x_{ij} in A, which maintains the condition $0\le a_{ij\mu}+a_{ij\nu}\le 1$. For simplicity, we write $A=[a_{ij}]_{m\times n}$, where $a_{ij}=\langle a_{ij\mu},a_{ij\nu}\rangle$.

In arithmetic operations, only the values of $a_{ij\mu}$ and $a_{ij\nu}$ are needed so from here we only consider the values of $a_{ij}=\langle a_{ij\mu},a_{ij\nu}\rangle$. All elements of an IFM are the members of $\langle F\rangle=\{\langle a,b\rangle:0\le a+b\le 1\}$.

To compute the m-th power of an IFM we take the help of the weight of a path of length m of an intuitionistic fuzzy graph $G=(\mu,\nu,V,E)$, where V is the vertex set, E is the edge set, μ and ν represent the membership and the non-membership values of both the vertices and edges.

Definition 2.2. (Intuitionistic fuzzy graph) A graph $G=(\mu,\nu,V,E)$ is said to be maxmin intuitionistic fuzzy graph if

(i) $V=\{v_1,v_1,\ldots,v_n\}$ such that, $\mu_1:V\to[0,1]$ and $\nu_1:V\to[0,1]$, denote the degree of membership and the degree of non-membership value of the element $v_i\in V$ respectively and $0\le\mu_1(v_i)+\nu_1(v_i)\le 1$, for every $v_i\in V$, and

(ii) $E\subseteq V\times V$ where $\mu_2:V\times V\to[0,1]$ and $\nu_2:V\times V\to[0,1]$ are such that $\mu_2(v_i,v_j)\le max\{\mu_1(v_i),\mu_1(v_j)\}$ and $\nu_2(v_i,v_j)\ge min\{\nu_1(v_i),\nu_1(v_j)\}$, denotes the membership and non-membership values of an edge $(v_i,v_j)\in E$ respectively, where, $0\le\mu_2(v_i,v_j)+\nu_2(v_i,v_j)\le 1$, for every $(v_i,v_j)\in E$.

The IFG $G=(\mu,\nu,V,E)$ is said to be complete if $\mu_2(v_i,v_j)=max\{\mu_1(v_i),\mu_1(v_j)\}$ and $\nu_2(v_i,v_j)=min\{\nu_1(v_i),\nu_1(v_j)\}$ for all $v_i,v_j\in V$.

The example of an intuitionistic fuzzy graph with four vertices is given in the Figure 1.

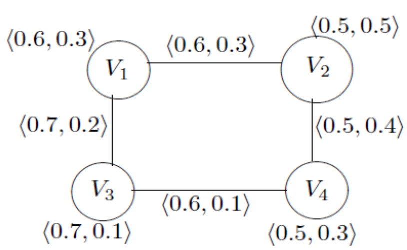

Figure 1: Intuitionistic fuzzy graph

Definition 2.3. *A path* P *in an IFG* $G=(\mu,\nu,V,E)$ *is said to be a* m*-path if it is a sequence of* $(m+1)$ *distinct vertices* $v_0,v_1,\ldots,v_m$ *of the vertex set* V .

As the maxarithmetic mean-minarithmetic mean operation is non-associative so we define the powers A^k of A by $A^k=(A^{k-1})\circ A$ for $k=2,3,\ldots,n$ and in that case $(A^{k-1})\circ A$ not equal to $A\circ(A^{k-1})$.

The directed intuitionistic fuzzy graph corresponding to the IFM A, is defined by $G=(\mu,\nu,V,E)$ with the vertex set $V=\{1,2,\ldots,n\}$ and the set of edges $E=\{(i,j)\in V\times V\,|\,1\le i,j\le n\}$. A path $P(i_0,i_1,\ldots,i_k)$ is a sequence of k edges $(i_0,i_1),(i_1,i_2),\ldots,(i_{k-1},i_k)$. The weight of the path $P(i_0,i_1,\ldots,i_k)$ is denoted by $w(P(i_0,i_1,\ldots,i_k))$ or simply by $w(P)$, is defined by

$$w(P(i_0, i_1, \ldots, i_k)) = \left\langle w_\mu(P), w_\nu(P) \right\rangle$$

$$= \left\langle \frac{a_{i_0 i_1 \mu} + a_{i_1 i_2 \mu} + 2a_{i_2 i_3 \mu} + \ldots + 2^{k-2} a_{i_{k-1} i_k \mu}}{2^{k-1}}, \frac{a_{i_0 i_1 \nu} + a_{i_1 i_2 \nu} + 2a_{i_2 i_3 \nu} + \ldots + 2^{k-2} a_{i_{k-1} i_k \nu}}{2^{k-1}} \right\rangle.$$

A path $P(i_0, i_1, \ldots, i_k)$ is called a critical path from vertex i_0 to i_k if $w(P) = \langle 1,0 \rangle$, that is, $\langle a_{i_0 i_1 \mu}, a_{i_0 i_1 \nu} \rangle = \langle a_{i_1 i_2 \mu}, a_{i_1 i_2 \nu} \rangle = \ldots = \langle a_{i_{k-1} i_k \mu}, a_{i_{k-1} i_k \nu} \rangle = \langle 1,0 \rangle$. A circuit C of length k is a path $P(i_0, i_1, \ldots, i_k)$ with $i_k = i_0$, where $i_1, i_2, \ldots, i_{k-1}$ are distinct. A circuit C with $w(C) = \langle 1,0 \rangle$ is called a critical circuit and vertices on critical circuit are called critical vertices.

Comparison between intuitionistic fuzzy matrices have an important role in our work, which is defined below.

Definition 2.4. (Dominance of IFM)

Let $A, B \in F_{m \times n}$ such that $A = (\langle a_{ij\mu}, a_{ij\nu} \rangle)$ and $B = (\langle b_{ij\mu}, b_{ij\nu} \rangle)$, then we write $A \le B$ if, $a_{ij\mu} \le b_{ij\mu}$ and $a_{ij\nu} \ge b_{ij\nu}$ for all i, j, and we say that A is dominated by B or B dominates A. A and B are said to be comparable, if either $A \le B$ or $B \le A$.

3. Main result

In this section, it is proved that the maxarithmetic mean-minarithmetic mean powers of an IFM converge. It is also shown that, the limit IFM has the feature that all elements of each column are identical. We denote the s-th path from the vertex i to the vertex j of length k by $(P_k^s(i, j))$.

Theorem 3.1. *Let A be a square IFM of order n. Then the ij-th element of the m-th power of A is,* $[A^m]_{ij} = \langle \max_s \{w_\mu(P_m^s(i, j))\}, \min_s \{w_\nu(P_m^s(i, j))\} \rangle$.

Proof: Let $W = \langle \max_s \{w_\mu(P_k^s(i, j))\}, \min_s \{w_\nu(P_k^s(i, j))\} \rangle$. We proceed by induction on m.

The assertion is true for $m = 1$. Assume that the assertion is true for $m = k - 1$. Choose $1 \le s \le n$ such that,

$$\left\langle \frac{[A^{k-1}]_{is\mu} + [A]_{sj\mu}}{2}, \frac{[A^{k-1}]_{is\nu} + [A]_{sj\nu}}{2} \right\rangle = \left\langle \max_{1 \le t \le n} \frac{[A^{k-1}]_{it\mu} + [A]_{tj\mu}}{2}, \min_{1 \le t \le n} \frac{[A^{k-1}]_{it\nu} + [A]_{tj\nu}}{2} \right\rangle$$

$$= [A^k]_{ij}.$$

By induction hypothesis, there are paths of length $(k-1)$, $P(i_0 = i, i_1, \ldots, i_{k-1} = s)$ such that,

$[A^{k-1}]_{is} = \langle w_\mu(P^s_{k-1}(i,s)), w_\nu(P^s_{k-1}(i,s))\rangle$

$$= \langle \frac{a_{i_0 i_1 \mu} + a_{i_1 i_2 \mu} + 2a_{i_2 i_3 \mu} + \ldots + 2^{k-3} a_{i_{k-2} i_{k-1} \mu}}{2^{k-2}}, \frac{a_{i_0 i_1 \nu} + a_{i_1 i_2 \nu} + 2a_{i_2 i_3 \nu} + \ldots + 2^{k-3} a_{i_{k-2} i_{k-1} \nu}}{2^{k-2}} \rangle.$$

Let $P(i_0 = i, i_1, \ldots, i_{k-1} = s, i_k = j)$ be any path of length k from the vertex i to the vertex j with

$$\langle w_\mu(P^s_k(i,j)), w_\nu(P^s_k(i,j))\rangle = \langle \frac{a_{i_0 i_1 \mu} + a_{i_1 i_2 \mu} + 2a_{i_2 i_3 \mu} + \ldots + 2^{k-2} a_{i_{k-1} i_k \mu}}{2^{k-1}}, \frac{a_{i_0 i_1 \nu} + a_{i_1 i_2 \nu} + 2a_{i_2 i_3 \nu} + \ldots + 2^{k-2} a_{i_{k-1} i_k \nu}}{2^{k-1}} \rangle$$

$$= \langle \frac{a_{i_0 i_1 \mu} + a_{i_1 i_2 \mu} + 2a_{i_2 i_3 \mu} + \ldots + 2^{k-3} a_{i_{k-2} i_{k-1} \mu}}{2^{k-1}} + \frac{a_{i_{k-1} i_k \mu}}{2}, \frac{a_{i_0 i_1 \nu} + a_{i_1 i_2 \nu} + 2a_{i_2 i_3 \nu} + \ldots + 2^{k-3} a_{i_{k-2} i_{k-1} \nu}}{2^{k-1}} + \frac{a_{i_{k-1} i_k \nu}}{2} \rangle = [A^k]_{ij}.$$

This implies, $[A^k]_{ij} \le W.$ (1)

Now, let $P(i_0 = i, i_1, \ldots, i_k = j)$ be given arbitrary path of length k. Then taking $P(i_0 = i, i_1, \ldots, i_{k-1})$ as any path of length $(k-1)$ we get,

$$\langle w_\mu(P^s_m(i,j)), w_\nu(P^s_m(i,j))\rangle = \langle \frac{a_{i_0 i_1 \mu} + a_{i_1 i_2 \mu} + 2a_{i_2 i_3 \mu} + \ldots + 2^{k-2} a_{i_{k-1} i_k \mu}}{2^{k-1}}, \frac{a_{i_0 i_1 \nu} + a_{i_1 i_2 \nu} + 2a_{i_2 i_3 \nu} + \ldots + 2^{k-2} a_{i_{k-1} i_k \nu}}{2^{k-1}} \rangle$$

$$= \langle \frac{a_{i_0 i_1 \mu} + a_{i_1 i_2 \mu} + 2a_{i_2 i_3 \mu} + \ldots + 2^{k-3} a_{i_{k-2} i_{k-1} \mu}}{2^{k-1}} + \frac{a_{i_{k-1} i_k \mu}}{2}, \frac{a_{i_0 i_1 \nu} + a_{i_1 i_2 \nu} + 2a_{i_2 i_3 \nu} + \ldots + 2^{k-3} a_{i_{k-2} i_{k-1} \nu}}{2^{k-1}} + \frac{a_{i_{k-1} i_k \nu}}{2} \rangle$$

$$\le \frac{[A]_{i i_{k-1}} + [A]_{i_{k-1} i_k}}{2} \le [A^k]_{ij}.$$

This implies, $[A^k]_{ij} \ge W.$ (2)

By (1) and (2), the only possibility is, $[A^k]_{ij} = W$.

Hence the assertion is true for $m = k$ also. That is, the assertion is true for any integer *m*.

Theorem 3.2. *Let A be an $n \times n$ IFM. Then the power sequence of A converges to its limit matrix. That is,* $\lim_{m\to\infty} A^m = \mathring{A}$. *Also For each* $1 \le j \le n$, $\mathring{A}_{rj} = \mathring{A}_{sj}$, *for all* $1 \le r, s \le n$.

Proof: (First Part) Let $1 \le r, s \le n$ be fixed and $P(i_0 = r, i_1, \ldots, i_{m-1}, i_m = s)$ be any given path of length m . We remove the vertex i_1 from the above paths $P^s_m(r,s)$ and to form the corresponding path $P(i_0 = r, i_2, \ldots, i_{m-1}, i_m = s)$ of length $(m-1)$ from the vertex r to the vertex s. Then,

$$w(P_m^s(r,s)) = \langle w_\mu(P_m^s(r,s)), w_\nu(P_m^s(r,s))\rangle$$
$$= \langle \frac{a_{i_0i_1\mu} + a_{i_1i_2\mu} + 2a_{i_2i_3\mu} + \ldots + 2^{m-2} a_{i_{m-1}i_m\mu}}{2^{m-1}}, \frac{a_{i_0i_1\nu} + a_{i_1i_2\nu} + 2a_{i_2i_3\nu} + \ldots + 2^{m-2} a_{i_{m-1}i_m\nu}}{2^{m-1}}\rangle$$

and

$$w(P_{m-1}^s(r,s)) = \langle w_\mu(P_{m-1}^s(r,s)), w_\nu(P_{m-1}^s(r,s))\rangle$$
$$= \langle \frac{a_{i_0i_2\mu} + a_{i_2i_3\mu} + 2a_{i_3i_4\mu} + \ldots + 2^{m-3} a_{i_{m-1}i_m\mu}}{2^{m-2}}, \frac{a_{i_0i_2\nu} + a_{i_2i_3\nu} + 2a_{i_3i_4\nu} + \ldots + 2^{m-3} a_{i_{m-1}i_m\nu}}{2^{m-2}}\rangle$$
$$= \langle \frac{a_{i_0i_2\mu}}{2^{m-2}} + \frac{2a_{i_2i_3\mu} + 2^2 a_{i_3i_4\mu} + \ldots + 2^{m-2} a_{i_{m-1}i_m\mu}}{2^{m-1}}, \frac{a_{i_0i_2\mu}}{2^{m-2}} + \frac{2a_{i_2i_3\nu} + 2^2 a_{i_3i_4\nu} + \ldots + 2^{m-2} a_{i_{m-1}i_m\nu}}{2^{m-1}}\rangle.$$

From the above two equalities we can obtain,

$$w(P_m) = w(P_{m-1}) + \frac{\langle a_{i_0i_1\mu}, a_{i_0i_1\nu}\rangle}{2^{m-1}} + \frac{\langle a_{i_1i_2\mu}, a_{i_1i_2\nu}\rangle}{2^{m-1}} - \frac{\langle a_{i_0i_2\mu}, a_{i_0i_2\nu}\rangle}{2^{m-2}}.$$

Now, as $a_{ij\mu} \in [0,1]$ and $a_{ij\nu} \in [0,1]$ for all $1 \le i, j \le n$ and $w(P_{m-1}^s(r,s)) \le [A^{m-1}]_{rs}$ (by Theorem 3.1). It implies, $w(P_m^s(r,s)) \le [A^{m-1}]_{rs} + \frac{\langle 1,0\rangle}{2^{m-2}}$.

Since, $P_m^s(r,s)$ is any arbitrary path with length m, this leads the following inequality

$$[A^m]_{rs} \le [A^{m-1}]_{rs} + \frac{\langle 1,0\rangle}{2^{m-2}}. \quad (3)$$

On the other hand, let $P_{m-1}^s(r,s) = (i_0 = r, i_2, \ldots, i_{m-1}, i_m = s)$ be any path of length $(m-1)$ from the vertex r to the vertex s. Choose $1 \le i_1 \le n$. Let $P_m^s(r,s) = (i_0 = r, i_1, \ldots, i_{m-1}, i_m = s)$, is the corresponding path of length m from the vertex r to the vertex s and

$$w(P_{m-1}^s(r,s)) = w(P_m^s(r,s)) - \frac{\langle a_{i_0i_1\mu}, a_{i_0i_1\nu}\rangle}{2^{m-1}} - \frac{\langle a_{i_1i_2\mu}, a_{i_1i_2\nu}\rangle}{2^{m-1}} + \frac{\langle a_{i_0i_2\mu}, a_{i_0i_2\nu}\rangle}{2^{m-2}}$$
$$\le wP_m^s(r,s)) + \frac{\langle 1,0\rangle}{2^{m-2}}.$$

This implies, $[A^{m-1}]_{rs} \le [A^m]_{rs} + \frac{\langle 1,0\rangle}{2^{m-2}}$. (4)

From (3) and (4), we obtain $|[A^m]_{rs} - [A^{m-1}]_{rs}| \le \frac{\langle 1,0\rangle}{2^{m-2}}$.

Let N be a fixed natural number and for all $n \ge N$,

$$\begin{aligned}|[A^n]_{rs}-[A^N]_{rs}| &\le |[A^n]_{rs}-[A^{n-1}]_{rs}|+|[A^{n-1}]_{rs}-[A^{n-2}]_{rs}|+\ldots+|[A^{N+1}]_{rs}-[A^N]_{rs}|\\ &\le \frac{\langle 1,0\rangle}{2^{m-2}}+\frac{\langle 1,0\rangle}{2^{m-3}}+\ldots+\frac{\langle 1,0\rangle}{2^{N-1}}\\ &\le \frac{\langle 1,0\rangle}{2^{N-2}}.\end{aligned}$$

Therefore, the sequence $\{[A^m]_{rs}\}$ is a Cauchy sequence and hence $\{[A^m]_{rs}\}$ converge. That imply, $\lim_{m\to\infty} A^m = Å$.

(Second Part) Let $P_m(rj) = (i_0 = r, i_1, \ldots, i_{m-1}, i_m = j)$ be any path of length m from the vertex r to the vertex j and $P_m(sj) = (i_0 = s, i_1, \ldots, i_{m-1}, i_m = j)$ be any path of length of m from the vertex s to the vertex j. Then,

$$w(P_m(rj)) = \left\langle \frac{a_{ri_1\mu}+a_{i_1i_2\mu}+2a_{i_2i_3\mu}+\ldots+2^{m-2}a_{i_{m-1}j\mu}}{2^{m-1}}, \frac{a_{ri_1\nu}+a_{i_1i_2\nu}+2a_{i_2i_3\nu}+\ldots+2^{m-2}a_{i_{m-1}j\nu}}{2^{m-1}}\right\rangle, \textit{and}$$

$$w(P_m(sj)) = \left\langle \frac{a_{si_1\mu}+a_{i_1i_2\mu}+2a_{i_2i_3\mu}+\ldots+2^{m-2}a_{i_{m-1}j\mu}}{2^{m-1}}, \frac{a_{si_1\nu}+a_{i_1i_2\nu}+2a_{i_2i_3\nu}+\ldots+2^{m-2}a_{i_{m-1}j\nu}}{2^{m-1}}\right\rangle.$$

By Theorem 3.1, we have

$$w(P_m(rj)) = w(P_m(sj)) + \frac{\langle a_{ri_1\mu}, a_{ri_1\nu}\rangle}{2^{m-1}} - \frac{\langle a_{si_1\mu}, a_{si_1\nu}\rangle}{2^{m-1}} \le [A^m]_{sj} + \frac{\langle 1,0\rangle}{2^{m-1}} \quad \text{and}$$

$$w(P_m(sj)) = w(P_m(rj)) + \frac{\langle a_{si_1\mu}, a_{si_1\nu}\rangle}{2^{m-1}} - \frac{\langle a_{ri_1\mu}, a_{ri_1\nu}\rangle}{2^{m-1}} \le [A^m]_{rj} + \frac{\langle 1,0\rangle}{2^{m-1}}.$$

The above two inequalities imply, $|[A^m]_{rj} - [A^m]_{sj}| \le \frac{\langle 1,0\rangle}{2^{m-1}}$.

Now as, $\lim_{m\to\infty} A^m = Å$, we can obtain $Å_{rj} = Å_{sj}$.

Theorem 3.3. *Let A be an $n\times n$ IFM and $\lim_{m\to\infty} A^m = Å$. Then the following statemements are equivalent.*

(i) All entries in the j-th column of $Å$ are $\langle 1,0\rangle$, that is, $[Å]_{sj} = \langle 1,0\rangle$ for all $s = 1,2,\ldots,n$.

(ii) There is a critical path in G from a critical vertex to the vertex j .

Proof: $(i) \Rightarrow (ii)$

Let $\beta = \langle \beta_{ij\mu}, \beta_{ij\nu}\rangle = \langle \max_{1\le i,j\le n} a_{ij\mu}, \min_{1\le i,j\le n} a_{ij\nu}\rangle$, then $\beta_{ij\mu} \in [0,1]$ and $\beta_{ij\nu} \in [0,1]$

Suppose that there is no critical path from a critical vertex to the vertex j . Let $P = (i_0, i_1, \ldots, i_{m-1}, i_m = j)$ be a path of length m from the vertex i_0 to the vertex j with $m \ge n$.

Let us we claim that,

$\langle a_{i_{m-n}i_{m-n+1}\mu}, a_{i_{m-n}i_{m-n+1}\nu}\rangle.\langle a_{i_{m-n+1}i_{m-n+2}\mu}, a_{i_{m-n+1}i_{m-n+2w}\nu}\rangle.\ldots.\langle a_{i_{m-1}i_m\mu}, a_{i_{m-1}i_m\nu}\rangle$ is dominated by $\langle 1,0\rangle$.

If $\langle a_{i_{m-n}i_{m-n+1}\mu}, a_{i_{m-n}i_{m-n+1}\nu}\rangle.\langle a_{i_{m-n+1}i_{m-n+2}\mu}, a_{i_{m-n+1}i_{m-n+2w}\nu}\rangle.\ldots.\langle a_{i_{m-1}i_m\mu}, a_{i_{m-1}i_m\nu}\rangle = \langle 1,0\rangle$ then it imply, $\langle a_{i_{m-n}i_{m-n+1}\mu}, a_{i_{m-n}i_{m-n+1}\nu}\rangle = \langle a_{i_{m-n+1}i_{m-n+2}\mu}, a_{i_{m-n+1}i_{m-n+2w}\nu}\rangle = \ldots = \langle a_{i_{m-1}i_m\mu}, a_{i_{m-1}i_m\nu}\rangle = \langle 1,0\rangle$.

Since $\{i_{m-n}, i_{m-n+1}, \ldots, i_m\} \subset \{1,2,\ldots,n\}$ with $(n+1)$ elements, there are $m-n \le r < s < m$ such that $i_r = i_s$. In this situation, the vertex i_r is a critical vertex. If we let $P' = (i_r, i_{r+1}, \ldots, i_m = j)$, then P' will be a path from a critical vertex i_r to vertex j, a contradiction. Hence we have,

$$\langle a_{i_{m-n}i_{m-n+1}\mu}, a_{i_{m-n}i_{m-n+1}\nu}\rangle.\langle a_{i_{m-n+1}i_{m-n+2}\mu}, a_{i_{m-n+1}i_{m-n+2w}\nu}\rangle.\ldots.\langle a_{i_{m-1}i_m\mu}, a_{i_{m-1}i_m\nu}\rangle < \langle 1,0\rangle$$

.

Moreover, there exists $a_{i_q i_{q+1}} \le \beta < \langle 1,0\rangle$ for some $m-n \le q \le m-1$. Thus,

$$w(P(i_0,i_1,\ldots,i_m = j) = \langle \frac{a_{i_0i_1\mu} + a_{i_1i_2\mu} + 2a_{i_2i_3\mu} + \ldots + 2^{q-1}a_{i_qi_{q+1}\mu} + \ldots + 2^{m-2}a_{i_{m-1}i_m\mu}}{2^{m-1}},$$

$$\frac{a_{i_0i_1\nu} + a_{i_1i_2\nu} + 2a_{i_2i_3\nu} + \ldots + 2^{q-1}a_{i_qi_{q+1}\nu} + \ldots + 2^{m-2}a_{i_{m-1}i_m\nu}}{2^{m-1}}\rangle$$

$$\le \{\frac{1+1+2+2^2+\ldots+2^{q-2}+\beta 2^{q-1}+2^q+\ldots+2^{m-2}}{2^{m-1}}\}\langle 1,0\rangle$$

$$= \{\frac{2^{m-1}+\beta 2^{q-1}-2^{q-1}}{2^{m-1}}\}\langle 1,0\rangle$$

$$= \langle 1,0\rangle - \frac{1-\beta}{2^{m-q}}\langle 1,0\rangle.$$

This leads, $[\text{Å}]_{i_oj} = \lim_{m\to\infty}[A^m]_{i_0j} \le (1-\frac{1-\beta}{2^{m-q}})\langle 1,0\rangle \le (1-\frac{1-\beta}{2^n})\langle 1,0\rangle \le \langle 1,0\rangle$, which is a contradiction of $[\text{Å}]_{i_oj} = \langle 1,0\rangle$.

So, our assumption is wrong, that is, there is a critical path in G from a critical vertex to the vertex j.

$(ii) \Rightarrow (i)$

Let $P^* = (i_0, i_1, \ldots, i_{s-1}, i_s = j)$ be a critical path from a critical vertex i_0 to vertex j and let $C = (r_0 = i_0, r_1, \ldots, r_h = i_0)$ be a critical circuit with length h. For m large enough, let $0 \le k \le h-1$ such that $m-s = hl+k$ for some positive integer l. Choose $1 \le t \le n$, let C_1 be a k path from vertex t to vertex i_0.

Then, $P = C_1 + C + C + \ldots + C + P^*$ is a path of length m from the vertex t to the vertex j and

$$w(P) \geq (\frac{2^k + 2^{k+1} + \ldots + 2^m}{2^{m-1}})\langle 1,0\rangle = (\frac{2^{m-1} - 2^k}{2^{m-1}})\langle 1,0\rangle.$$

This result provides, $[A^m]_{tj} \geq w(P) \geq (1 - \frac{1}{2^{m-1-k}})\langle 1,0\rangle$.

Therefore we conclude, $\lim_{m\to\infty}[A^m]_{tj} = \langle 1,0\rangle$.

Theorem 3.4. *Let* A *be an* $n \times n$ *IFM and* $\text{Å} = \lim_{m\to\infty} A^m$. *Then* $\text{Å} = J$ *if and only if there exists an entry* $\langle 1,0\rangle$ *in each column of* A.

Proof: The above theorem can be proved by the help of Theorem 3.3. Then, it is sufficient to show that for each j there is a critical path from a critical vertex to the vertex j. Since each column of A contains $\langle 1,0\rangle$, for this j there is i_0, $1 \leq i_0 \leq n$ such that $\langle a_{i_0 j\mu}, a_{i_0 j\nu}\rangle = \langle 1,0\rangle$. For this i_0 there is i_1, $1 \leq i_1 \leq n$ such that $\langle a_{i_1 i_0 \mu}, a_{i_1 i_0 \nu}\rangle = \langle 1,0\rangle$. Continuing by this way, we obtain $1 \leq i_{n-1}, \ldots, i_0, j \leq n$ such that $\langle a_{i_0 j\mu}, a_{i_0 j\nu}\rangle = \langle 1,0\rangle$ and $\langle a_{i_t i_{t-1}\mu}, a_{i_t i_{t-1}\nu}\rangle = \langle 1,0\rangle$ for all $t = 0,1,\ldots,n-1$. As $1 \leq i_{n-1}, \ldots, i_0, j \leq n$ there is $0 \leq r \leq n-1$ such that $i_r \in \{i_{r-1}, i_{r-2}, \ldots, i_0, j\}$. Therefore, the vertex i_r is a critical vertex and $P = (i_r, i_{r-1}, \ldots, i_0, j)$ is the required path.

Example 3.5. *Let us consider the IFM* $A = \begin{bmatrix} \langle 0,1\rangle & \langle 1,0\rangle & \langle 0.5,0.4\rangle \\ \langle 1,0\rangle & \langle 0,1\rangle & \langle 0.6,0.3\rangle \\ \langle 1,0\rangle & \langle 1,0\rangle & \langle 0,1\rangle \end{bmatrix}$.

Then the directed IFG corresponding to the IFM A is given in the Figure 2.

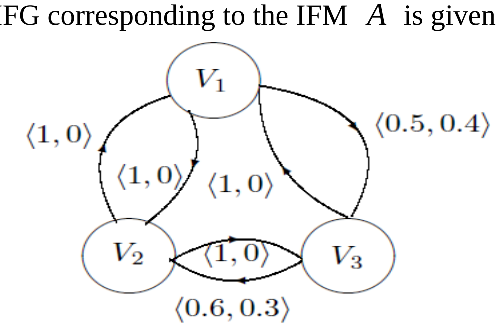

Figure 2: Directed IFG

One of its critical circuit is $V_1 \to V_2 \to V_1$ (see Figure 3).

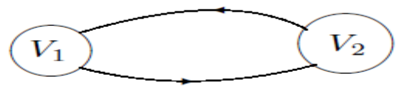

Figure 3: A critical circuit in Example 3.5

Then the set of all critical vertices in the directed intuitionistic fuzzy graph G is $\{V_1, V_2\}$ and there is no path from a critical vertex to the vertex V_3.

It is seen that the limit matrix is, $\mathring{A} = A^{10} = \begin{bmatrix} \langle 1,0\rangle & \langle 1,0\rangle & \langle 0.8,0.15\rangle \\ \langle 1,0\rangle & \langle 1,0\rangle & \langle 0.8,0.15\rangle \\ \langle 1,0\rangle & \langle 1,0\rangle & \langle 0.8,0.15\rangle \end{bmatrix}$.

Here all entries in column 1 and 2 are $\langle 1,0\rangle$ and all entries in column 3 are $\langle 0.8,0.15\rangle$, other than $\langle 1,0\rangle$.

Example 3.6. *Let us consider the IFM* $B = \begin{bmatrix} \langle 1,0\rangle & \langle 0,1\rangle & \langle 0.5,0.4\rangle \\ \langle 0,1\rangle & \langle 1,0\rangle & \langle 0.6,0.3\rangle \\ \langle 1,0\rangle & \langle 0,1\rangle & \langle 1,0\rangle \end{bmatrix}$.

The directed IFG corresponding to the IFM B is given in the Figure 4.

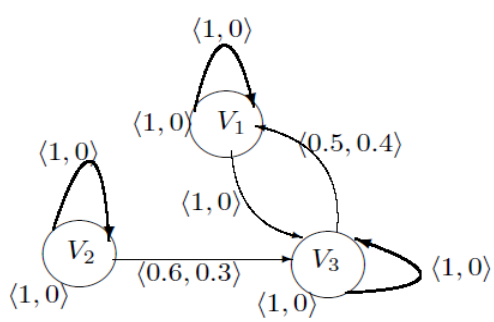

Figure 4: Directed IFG

Since the IFM B contain $\langle 1,0\rangle$ in each column and all the vertices of the directed IFG G, corresponding to B has a self-loop, so the set of all critical vertices is

$\{V_1, V_2, V_3\}$. So by Theorem 3.4, we have, $B = B^{12} = \begin{bmatrix} \langle 1,0\rangle & \langle 1,0\rangle & \langle 1,0\rangle \\ \langle 1,0\rangle & \langle 1,0\rangle & \langle 1,0\rangle \\ \langle 1,0\rangle & \langle 1,0\rangle & \langle 1,0\rangle \end{bmatrix} = J$.

Definition 3.7. *Let* R *be an intuitionistic fuzzy relation between the elements of a finite set* X *and* A *be an intuitionistic fuzzy set of* X. *Then the max-min composition of* A *and* R *gives* B, *an intuitionistic fuzzy set of* X. *When* B *equals to* A, *that is,* $A \circ R = A$ *then we say that* A *is an* **eigen intuitionistic fuzzy set** *associated with the* *given relation* R.

For simplicity we can write, for a given IFM R, $x \in F(X)$ (intuitionistic fuzzy set of the finite set X) is called an eigen intuitionistic fuzzy set of R if and only if, $x \circ R = x$. Here the operation ``$\circ$'' is the maxarithmetic mean-minarithmetic mean composition.

Theorem 3.8. *Let* A *be an IFM of order* n *and* $\mathring{A} = \lim_{k\to\infty} A^k$. *Then the row vector of* $\mathring{A}$ *is the unique eigen intuitionistic fuzzy set of* A.

Proof: Since $\mathring{A} = \lim_{k\to\infty} A^{k+1} = \lim_{k\to\infty}(A^k \circ A) = \mathring{A} \circ A$, we see that the identical row vectors of the limiting matrix $\mathring{A}$ are an eigen intuitionistic fuzzy set of A.

For any row vector $x = (\langle x_{1\mu}, x_{1\nu}\rangle, \langle x_{2\mu}, x_{2\nu}\rangle, \ldots, \langle x_{n\mu}, x_{n\nu}\rangle)$, consider $x^{(0)} = x$ and $x^{(k)} = x^{(k-1)} \circ A$, for all $k \geq 1$. By the same argument of the proof of Theorem 3.1, we see that for each $1 \leq j \leq n$,

$$[x^{(k)}]_j = \left\langle \max_{1\le i_1,i_2,\ldots,i_k\le n}\{x_{i_1\mu} \circ a_{i_1i_2\mu} \circ \ldots \circ a_{i_kj\mu}\}, \min_{1\le i_1,i_2,\ldots,i_k\le n}\{x_{i_1\nu} \circ a_{i_1i_2\nu} \circ \ldots \circ a_{i_kj\nu}\}\right\rangle.$$

Now,

$$\langle\{x_{i_1\mu} \circ a_{i_1i_2\mu} \circ \ldots \circ a_{i_kj\mu}\}, \{x_{i_1\nu} \circ a_{i_1i_2\nu} \circ \ldots \circ a_{i_kj\nu}\}\rangle = \langle \frac{x_{i_1\mu} + a_{i_1i_2\mu} + 2a_{i_2i_3\mu} + \ldots + 2^{k-1}a_{i_kj\mu}}{2^k}, \frac{x_{i_1\nu} + a_{i_1i_2\nu} + 2a_{i_2i_3\nu} + \ldots + 2^{k-1}a_{i_kj\nu}}{2^k}\rangle.$$

Suppose that y, z are two eigen intuitionistic fuzzy set of A. Then for $y \circ A = y$ and $z \circ A = z$, we have $y^{(k)} = y$ and $z^{(k)} = z$ for all $z = 1,2,\ldots$. Let $i_1, i_2, \ldots, i_k$ be such that

$$[y^{(k)}]_j = \langle\{y_{i_1\mu} \circ a_{i_1i_2\mu} \circ \ldots \circ a_{i_kj\mu}\}, \{y_{i_1\nu} \circ a_{i_1i_2\nu} \circ \ldots \circ a_{i_kj\nu}\}\rangle.$$

Then,

$$\begin{aligned} |y_j - z_j| &= |[y^{(k)}]_j - [z^{(k)}]_j| \\ &\leq [\langle\{y_{i_1\mu} \circ a_{i_1i_2\mu} \circ \ldots \circ a_{i_kj\mu}\}, \{y_{i_1\nu} \circ a_{i_1i_2\nu} \circ \ldots \circ a_{i_kj\nu}\}\rangle \\ &\quad - \langle\{z_{i_1\mu} \circ a_{i_1i_2\mu} \circ \ldots \circ a_{i_kj\mu}\}, \{z_{i_1\nu} \circ a_{i_1i_2\nu} \circ \ldots \circ a_{i_kj\nu}\}\rangle] \\ &\leq \frac{\delta}{2^k} \textit{ for all } k \geq 1, \textit{where } \delta = \langle \max_{1\le i,j\le n} a_{ij\mu}, \min_{1\le i,j\le n} a_{ij\nu}\rangle. \end{aligned}$$

By the same argument, we can write that, $|z_j - y_j| = |[z^{(k)}]_j - [y^{(k)}]_j| \leq \frac{\delta}{2^k}$.

From the above two inequalities, we have $y_j = z_j$ for each $1 \leq j \leq n$, that imply, $y = z$ and hence A contains at most one eigen intuitionistic fuzzy set.

4. Conclusions

Here we derive the procedure to get the power of an IFM under the maxarithmetic mean-minarithmetic mean operation using the graph theoretic concept. In this paper, we showed that the power of an IFM with the said operation are always convergent. Moreover, the limit IFM has the feature that all elements in each column are identical. Here we also define eigen intuitionistic fuzzy set and shown that the row vector of the limit IFM $\mathring{A}$ is the only eigen intuitionistic fuzzy set of A. In our further work we shall try to test the convergence of IFMs with respect to the different binary operations.

REFERENCES

1. K.T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20 (1986), 87-96.
2. M. Bhowmik and M. Pal, Some results on intuitionistic fuzzy matrices and intuitionistic circulant fuzzy matrices, *International Journal of Mathematical Sciences*, 7(1-2) (2008), 177-192.
3. J.J. Buckley, Note on convergence of powers of a fuzzy matrix, *Fuzzy Sets and Systems*, 121 (2001), 363-364.
4. Z.-T. Fan, A note on the power sequence of a fuzzy matrix, *Fuzzy Sets and Systems*, 102 (1999), 281-286.
5. Z.-T. Fan and D.-F. Liu, On the oscillating power sequence of a fuzzy matrix, *Fuzzy Sets and Systems*, 93 (1998), 75-85.
6. Z.-T. Fan and D.-F. Liu, Convergence of the power sequence of a nearly monotone increasing fuzzy matrix, *Fuzzy Sets and Systems*, 88 (1997), 363-372.
7. S.-M. Guu, Y.-Y. Lur and C.-T. Pang, On infinite products of fuzzy matrices, *SIAM J. Matrix Anal. Appl.*, 22(4) (2001), 1190-1203.
8. H. Hashimoto, Convergence of powers of a fuzzy transitive matrix, *Fuzzy Sets and Systems*, 9 (1983), 153-160.
9. R. Pradhan and M. Pal, Generalized inverse of block intuitionistic fuzzy matrices, *International Journal of Applications of Fuzzy Sets and Artificial Intelligence*, 3 (2013), 23-38.
10. R. Pradhan and M. Pal, Intuitionistic fuzzy linear transformations, *Annals of Pure and Applied Mathematics*, 1(1) (2012), 57-68.
11. R. Pradhan and M. Pal, Generalized inverse of Intuitionistic fuzzy matrices, (Communicated).
12. M.G. Thomason, Convergence of powers of a fuzzy matrix, *J. Math. Anal. Appl.*, 57 (1977), 476-480.
13. L.J. Xin, Convergence of power of a controllble fuzzy matrices, *Fuzzy Sets and Systems*, 45 v, 313-319.
14. Y.-Y. Lur, Y.-K. Wu and S.-M. Guu, Convergence of max-arithmetic mean powers of a fuzzy matrix, *Fuzzy Sets and Systems*, 158 (2007), 2516-2522.
15. L.A. Zadeh, Fuzzy sets, *Information and Control*, 8(1965), 338-353.