

Some Features of Fuzzy α -Compactness

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Abstract. In this paper, we study several features of fuzzy α -compactness due to Gantner et al. [4] in fuzzy topological spaces and obtain its several other properties of this concept.

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1. Introduction

The concept of fuzzy sets and fuzzy set operations was first introduced by Zadeh in his classical papers [9] in the year 1965 describing fuzziness mathematically. Many researchers have worked on Fuzzy topological spaces [2,3,7] as a generalization of results on topological spaces[6]. Compactness occupies a very important place in fuzzy topological spaces. The purpose of this paper is to study the concept due to Gantner et al. in more detail and to obtain several other features.

2. Preliminaries

We briefly touch upon the terminological concepts and some definitions, which are needed in the sequel. The following are essential in our study and can be found in the paper referred.

Definition 2.1. [9] Let X be a non-empty set and I is the closed unit interval $[0, 1]$. A fuzzy set in X is a function $u : X \rightarrow I$ which assigns to every element $x \in X$. $u(x)$ denotes a degree or the grade of membership of x . The set of all fuzzy sets in X is denoted by I^X . A member of I^X may also be called fuzzy subset of X .

Definition 2.2. [8] A fuzzy set is empty iff its grade of membership is identically zero. It is denoted by 0 or ϕ .

Definition 2.3. [8] A fuzzy set is whole iff its grade of membership is identically one in X . It is denoted by 1 or X .

Definition 2.4. [3] Let u and v be two fuzzy sets in X . Then we define

- (i) $u = v$ iff $u(x) = v(x)$ for all $x \in X$
- (ii) $u \subseteq v$ iff $u(x) \leq v(x)$ for all $x \in X$
- (iii) $\lambda = u \cup v$ iff $\lambda(x) = (u \cup v)(x) = \max [u(x), v(x)]$ for all $x \in X$
- (iv) $\mu = u \cap v$ iff $\mu(x) = (u \cap v)(x) = \min [u(x), v(x)]$ for all $x \in X$
- (v) $\gamma = u^c$ iff $\gamma(x) = 1 - u(x)$ for all $x \in X$.

Definition 2.5. [3] In general, if $\{ u_i : i \in J \}$ is family of fuzzy sets in X , then union

$\cup u_i$ and intersection $\cap u_i$ are defined by

$$\cup u_i(x) = \sup \{ u_i(x) : i \in J \text{ and } x \in X \}$$

$$\cap u_i(x) = \inf \{ u_i(x) : i \in J \text{ and } x \in X \}, \text{ where } J \text{ is an index set.}$$

Definition 2.6. [3] Let $f : X \rightarrow Y$ be a mapping and u be a fuzzy set in X . Then the image of u , written $f(u)$, is a fuzzy set in Y whose membership function is given by

$$f(u)(y) = \begin{cases} \sup \{ u(x) : x \in f^{-1}(y) \} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases}.$$

Definition 2.7. [3] Let $f : X \rightarrow Y$ be a mapping and v be a fuzzy set in Y . Then the inverse of v , written $f^{-1}(v)$, is a fuzzy set in X whose membership function is given by $(f^{-1}(v))(x) = v(f(x))$.

Definition 2.8. [3] Let X be a non-empty set and $t \subseteq I^X$, i.e. t is a collection of fuzzy set in X . Then t is called a fuzzy topology on X if

- (i) $0, 1 \in t$
- (ii) $u_i \in t$ for each $i \in J$, then $\bigcup_i u_i \in t$
- (iii) $u, v \in t$, then $u \cap v \in t$

The pair (X, t) is called a fuzzy topological space and in short, fts. Every member of t is called a t -open fuzzy set. A fuzzy set is t -closed iff its complements is t -open. In the sequel, when no confusion is likely to arise, we shall call a t -open (t -closed) fuzzy set simply an open (closed) fuzzy set.

Definition 2.9. [3] Let (X, t) and (Y, s) be two fuzzy topological spaces. A mapping $f : (X, t) \rightarrow (Y, s)$ is called an fuzzy continuous iff the inverse of each s -open fuzzy set is t -open.

Definition 2.10. [8] Let (X, t) be an fts and $A \subseteq X$. Then the collection $t_A = \{ u|_A : u \in t \} = \{ u \cap A : u \in t \}$ is fuzzy topology on A , called the subspace fuzzy topology on A and the pair (A, t_A) is referred to as a fuzzy subspace of (X, t) .

Some Features of Fuzzy α -Compactness

Definition 2.11. [4] An fts (X, t) is said to be fuzzy Hausdorff iff for all $x, y \in X, x \neq y$, there exist $u, v \in t$ such that $u(x) = 1, v(y) = 1$ and $u \cap v = 0$.

Distributive laws 2.12. [9] Distributive laws remain valid for fuzzy sets in X , i.e. if u, v and w are fuzzy sets in X , then

$$(i) u \cup (v \cap w) = (u \cup v) \cap (u \cup w)$$

$$(ii) u \cap (v \cup w) = (u \cap v) \cup (u \cap w).$$

Definition 2.13. [1] Let (X, T) be a topological space. A function $f : X \rightarrow \mathbf{R}$ (with usual topology) is called lower semi-continuous (l.s.c.) if for each $a \in \mathbf{R}$, the set $f^{-1}(a, \infty) \in T$. For a topology T on a set X , let $\omega(T)$ be the set of all l.s.c. functions from (X, T) to I (with usual topology); thus

$$\omega(T) = \{u \in I^X : u^{-1}(a, 1] \in T, a \in I_1\}.$$

It can be shown that $\omega(T)$ is a fuzzy topology on X .

Let P be a property of topological spaces and FP be its fuzzy topology analogue. Then FP is called a ‘good extension’ of P ‘iff the statement (X, T) has P iff $(X, \omega(T))$ has FP ’ holds good for every topological space (X, T) . Thus characteristic functions are l.s.c.

Definition 2.14. [4] An fts (X, t) is said to be α -compact (respectively α^* -compact) if each α -shading (respectively α^* -shading) of X by open fuzzy sets has a finite α -subshading (respectively α^* -subshading) where $\alpha \in I$.

Definition 2.15. [5] Let (X, t) be an fts and $0 \leq \alpha < 1$, then the family $t_\alpha = \{ \alpha(u) : u \in t \}$ of all subsets of X of the form $\alpha(u) = \{ x \in X : u(x) > \alpha \}$ is called α -level sets, forms a topology on X and is called the α -level topology on X and the pair (X, t_α) is called α -level topological space.

3. Characterizations of fuzzy α -compactness

Now we obtain some tangible features of fuzzy α -compact spaces.

Theorem 3.1. Let $0 \leq \alpha < 1$ and F be a closed subset of an fts (X, t) . If (X, t) is α -compact, then F is compact as a subspace of (X, t_α) .

Proof : Suppose (X, t) is α -compact. Let $M = \{ U_i : i \in J \}$ be an open cover of F where $U_i \in t_\alpha$. Then, since for each U_i , there exists a $g_i \in t$ such that $U_i = \alpha(g_i)$, we have $M = \{ \alpha(g_i) : i \in J \}$. Then the family $V = \{ g_i : i \in J \}$ is an α -shading of (X, t) . Let $H = \{ A \in t_\alpha : A \cap F \in M \}$. Then $W = H \cup \{ X - F \}$ is a family and is an open cover of (X, t_α) . To show this, let $x \in X$. Since M is an open cover of F , then there is an $U_{i_0} \in M$ such that $x \in U_{i_0}$. But $U_{i_0} = \alpha(g_{i_0})$ for some $g_{i_0} \in t$. Therefore

$x \in \alpha (g_{i_0})$ which implies that $g_{i_0} (x) > \alpha$. By α -compactness of (X, t) , V has a finite α -subshading, say $\{ g_{i_k} \} (k \in J_n)$. Again, let $A' \in t_\alpha$ such that $A' | F = U_{i_0}$. Then $A' \in H$ and $\{ U_{i_0} \}$ is a family of open cover of (X, t_α) . By hypothesis, (X, t_α) is compact, W has a finite subcover $\{ U_{i_k} \} (k \in J_n)$ i.e. $\{ \alpha (g_{i_k}) \} (k \in J_n)$ and hence $\{ \alpha (g_{i_k}) \} (k \in J_n)$ forms a finite subcover of M . Hence F is compact as a subspace of (X, t_α) .

Theorem 3.2. Let (X, t) be an fts and $A \subseteq X$. Then the following are equivalent :

(i) A is α -compact with respect to t .

(ii) A is α -compact with respect to the subspace fuzzy topology t_A on A .

Proof : (i) \Rightarrow (ii) : Let $M = \{ u_i : i \in J \}$ be an α -shading of A by t_A -open fuzzy sets. Let $H = \{ g \in t : g|A \in M \}$. Then $H \cup \{1 - A\}$ is a family and is an α -shading of X . To show this, let $x \in X$. If $x \in A$, there is an $u_{i_0} \in M$ such that $u_{i_0} (x) > \alpha$. Let $g' \in t$ such that $g' | A = u_{i_0}$. Then $g' \in H$ and $g' (x) > \alpha$. If $x \in 1 - A$, then $(1 - A) (x) = 1 > \alpha$. By α -compactness of X , the family $H \cup \{1 - A\}$ has a finite α -subshading, say $\{ g_1, g_2, \dots, g_n, 1 - A \}$. Then the family $\{ g_1|A, g_2|A, \dots, g_n|A \}$ is a finite α -subshading of M . It is clear that (A, t_A) is α -compact.

(ii) \Rightarrow (i) : Let $H = \{ u_i : i \in J \}$ be an α -shading of A by t -open fuzzy sets. Let $M = \{ g \in t_A : g|A \in H \}$. Then $M \cup \{1 - A\}$ is a family and is an α -shading of X . To show this, let $x \in X$. If $x \in A$, there is an $u_{i_0} \in H$ such that $u_{i_0} (x) > \alpha$. Let $g' \in t_A$. Then $g' | A = u_{i_0}$. Then $g' \in M$ and $g' (x) > \alpha$. If $x \in 1 - A$, then $(1 - A) (x) = 1 > \alpha$. By α -compactness of X , the family $M \cup \{1 - A\}$ has a finite α -subshading, say $\{ g_1, g_2, \dots, g_n, 1 - A \}$. Then the family $\{ g_1|A, g_2|A, \dots, g_n|A \}$ is a finite α -subshading of H . It is clear that (A, t) is α -compact.

Theorem 3.3. [6] If T is a cofinite topology on X , then (X, T) is compact.

Theorem 3.4. Let (X, t) be an fts and if t_α becomes a cofinite topology on X , then (X, t) is α -compact, where $0 \leq \alpha < 1$.

Proof : Let $M = \{ u_i : i \in J \}$ be an open α -shading of (X, t) . Then $t_\alpha = \{ \alpha (u_i) : u_i \in t \}$, where $\alpha (u_i) = \{ x \in X : u_i (x) > \alpha \}$ and by the theorem t_α is a cofinite topology on X . We see that $H = \{ \alpha (u_i) : i \in J \}$ is an open cover of (X, t_α) . For let, $x \in X$, then there exists a $u_{i_0} \in M$ such that $u_{i_0} (x) > \alpha$. Therefore, $x \in \alpha (u_{i_0})$

and $u_{i_0} \in H$. As (X, t_α) is cofinite, hence compact which implies that H has a finite subcover, say $\{\alpha(u_{i_k})\} (k \in J_n)$, where $u_{i_k} \in t$ and $\alpha(u_{i_k}) \in t_\alpha$. Then the family $\{u_{i_k}\} (k \in J_n)$ forms a finite α -subshading of M and hence (X, t) is α -compact.

Theorem 3.5. Let (X, t) and (Y, s) be two fuzzy topological spaces with (X, t) is α -compact. Let $f: (X, t) \rightarrow (Y, s)$ be a continuous surjection. Then (Y, s) is α -compact.

Proof : Let $\{u_i : u_i \in s\}$ be an open α -shading of (Y, s) for every $i \in J$. Since f is continuous, then $f^{-1}(u_i) \in t$. We see that, for every $x \in X$, $f^{-1}(u_i)(x) > \alpha$ and so $\{f^{-1}(u_i)\}$ is an open α -shading of (X, t) , $i \in J$. Since (X, t) is α -compact, then $\{f^{-1}(u_i)\}$ has a finite α -subshading, say $\{f^{-1}(u_k)\} (k \in J_n)$. Now, if $y \in Y$, then $y = f(x)$ for some $x \in X$. Then there exists $u_{i_k} \in \{u_i\}$ such that $f^{-1}(u_{i_k})(x) > \alpha$ which implies that $u_{i_k}(f(x)) > \alpha$ or $u_{i_k}(y) > \alpha$. Thus $\{u_i\}$ has a finite α -subshading $\{u_{i_k}\} (k \in J_n)$. Hence (Y, s) is α -compact.

Theorem 3.6. Let (X, t) and (Y, s) be two fuzzy topological spaces with (X, t) is α^* -compact. Let $f: (X, t) \rightarrow (Y, s)$ be a continuous surjection. Then (Y, s) is α^* -compact.

Proof: The proof is similar as above.

Theorem 3.7. Let (X, t) be an fts and $A \subseteq X$. If (X, t) is α -compact and A is closed, then (A, t_A) is α -compact subspace of (X, t) , where $0 \leq \alpha < 1$.

Proof : Let $M = \{u_i : i \in J\}$ be an open α -shading of (A, t_A) . Then by definition of subspace fuzzy topology, there exists $v_i \in t$ such that $u_i = A \cap v_i$. Let $H = \{v_i \in t : v_i \cap A \in M\}$. Then $H \cup \{X - A\}$ is a family and is an open α -shading of (X, t) . To prove this, let $x \in X$. If $x \in A$, then there exists $u_{i_0} \in M$ such that $u_{i_0}(x) > \alpha$. Let $v'_{i_0} \in t$ such that $v'_{i_0} \cap A = u_{i_0}$. Thus $v'_{i_0} \in H$ and $v'_{i_0}(x) > \alpha$. If $x \in X - A$, then $(X - A)(x) > \alpha$. Since (X, t) is α -compact, so $H \cup \{X - A\}$ has a finite α -subshading, say $\{v_{i_k}, X - A\} (k \in J_n)$. Also A and $(X - A)$ are disjoint, so we can exclude $(X - A)$ from this α -shading. Hence $\{v_{i_k} \cap A\} (k \in J_n)$ is a finite α -subshading of M . For if $x \in A$ and $\{v_{i_k}, X - A\} (k \in J_n)$ is an open α -shading of (X, t) , then there exists $v_{i_0} \in M$ such that $v_{i_0}(x) > \alpha$. Therefore $(v_{i_0} \cap A)(x) > \alpha$ and $v_{i_0} \cap A \in M$. Hence (A, t_A) is α -compact.

Theorem 3.8. Let (X, t) be an fts and $A \subseteq X$. If (X, t) is α^* -compact and A is closed, then (A, t_A) is α^* -compact subspace of (X, t) , where $0 \leq \alpha < 1$.

Proof: The proof is similar as above.

Theorem 3.9. Let (X, t) be a fuzzy Hausdorff space. Let A be an α -compact ($0 \leq \alpha < 1$) subset of (X, t) and suppose $x \in A^c$. Then there exist $u, v \in t$ such that $u(x) = 1, A \subseteq v^{-1}(0, 1]$ and $u \cap v = 0$.

Proof: Let $y \in A$. Since $x \notin A$ ($x \in A^c$), then clearly $x \neq y$. As (X, t) is fuzzy Hausdorff, then there exist $u_y, v_y \in t$ such that $u_y(x) = 1, v_y(y) = 1$ and $u_y \cap v_y = 0$. Let us take $\alpha \in I_1$ such that $v_y(y) > \alpha > 0$. Thus we see that $\{v_y : y \in A\}$ is an α -shading of A . Since A is α -compact in (X, t) , so it has a finite α -subshading, say $\{v_{y_k} : y_k \in A\}$ ($k \in J_n$). Now, let $v = v_{y_1} \cup v_{y_2} \cup \dots \cup v_{y_n}$ and $u = u_{y_1} \cap u_{y_2} \cap \dots \cap u_{y_n}$. Thus we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. $v, u \in t$. Moreover, $A \subseteq v^{-1}(0, 1]$ and $u(x) = 1$, as $u_{y_k}(x) = 1$ for each k .

Finally, we claim that $u \cap v = 0$. As $u_{y_k} \cap v_{y_k} = 0$ implies that $u \cap v_{y_k} = 0$, by distributive law, we see that $u \cap v = u \cap (v_{y_1} \cup v_{y_2} \cup \dots \cup v_{y_n}) = 0$.

Theorem 3.10. Let (X, t) be a fuzzy Hausdorff space and A, B be disjoint α -compact ($0 \leq \alpha < 1$) subsets of (X, t) . Then there exist $u, v \in t$ such that $A \subseteq u^{-1}(0, 1], B \subseteq v^{-1}(0, 1]$ and $u \cap v = 0$.

Proof : Let $y \in A$. Then $y \notin B$, as A and B are disjoint. Since B is α -compact, then by theorem (3.9), there exist $u_y, v_y \in t$ such that $u_y(y) = 1, B \subseteq v_y^{-1}(0, 1]$ and $u_y \cap v_y = 0$. Let us take $\alpha \in I_1$ such that $v_y(y) > \alpha > 0$. As $u_y(y) = 1$, then we see that $\{u_y : y \in A\}$ is an α -shading of A . Since A is α -compact in (X, t) , so it has a finite α -subshading, say $\{u_{y_k} : y_k \in A\}$ ($k \in J_n$). Furthermore, since B is α -compact, so B has a finite α -subshading, say $\{v_{y_k} : y_k \in B\}$ ($k \in J_n$) as $B \subseteq v_{y_k}^{-1}(0, 1]$ for each k . Now, let $u = u_{y_1} \cup u_{y_2} \cup \dots \cup u_{y_n}$ and $v = v_{y_1} \cap v_{y_2} \cap \dots \cap v_{y_n}$. Thus we see that $A \subseteq u^{-1}(0, 1]$ and $B \subseteq v^{-1}(0, 1]$. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively, i.e. $u, v \in t$.

Some Features of Fuzzy α -Compactness

Lastly, we have to show that $u \cap v = 0$. First, we observe that $u_{y_k} \cap v_{y_k} = 0$ for each k implies that $u_{y_k} \cap v = 0$, by distributive law, we see that $u \cap v = (u_{y_1} \cup u_{y_2} \cup \dots \cup u_{y_n}) \cap v = 0$.

Theorem 3.11. A topological space (X, T) is compact iff $(X, \omega(T))$ is α -compact.

Proof: Suppose (X, T) is compact. Let $M = \{u_i : i \in J\}$ be an open α -shading of $(X, \omega(T))$. Then $u_i^{-1}(a, 1] \in T$ and $\{u_i^{-1}(a, 1] : u_i^{-1}(a, 1] \in T\}$ is an open cover of (X, T) . As (X, T) is compact, so it has a finite subcover, i.e. there exist $u_{i_k}^{-1}(a, 1] \in T$ ($k \in J_n$) such that $X = u_{i_1}^{-1}(a, 1] \cup u_{i_2}^{-1}(a, 1] \cup \dots \cup u_{i_n}^{-1}(a, 1]$. Now, we observe that there exist $u_{i_k} \in \{u_i\}$ ($k \in J_n$) such that $u_{i_k}(x) > \alpha$ for all $x \in X$ and it is shows that $\{u_{i_k}\}$ ($k \in J_n$) is a finite α -subshading of M . Therefore $(X, \omega(T))$ is α -compact.

Conversely, suppose that $(X, \omega(T))$ is α -compact. Let $\{V_j : j \in J\}$ be open cover of (X, T) , i.e. $X = \bigcup_{j \in J} \{V_j : V_j \in T\}$. Since 1_{V_j} is l.s.c, then $1_{V_j} \in \omega(T)$ and $\{1_{V_j} : 1_{V_j} \in \omega(T)\}$ is an open α -shading of $(X, \omega(T))$. As $(X, \omega(T))$ is α -compact, so it has a finite α -shading, say $\{1_{V_{j_k}} : 1_{V_{j_k}} \in \omega(T)\}$ ($k \in J_n$) such that $1_{V_{j_k}}(x) > \alpha$ for all $x \in X$. Therefore, we can write $X = V_{j_1} \cup V_{j_2} \cup \dots \cup V_{j_n}$ and it is clear that $\{V_{j_k}\}$ ($k \in J_n$) is a finite subcover of (X, T) . Hence (X, T) is compact.

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