

Graceful Labeling of Some Theta Related Graphs

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Abstract . A labeled graph G which can be gracefully numbered is said to be graceful. Labeling the nodes of G with distinct nonnegative integers and then labeling the e edges of G with the absolute differences between node values, if the graph edge numbers run from 1 to e , the graph G is gracefully numbered. In this paper, we have discussed the gracefulness of some the graphs formed from theta graphs.

Keywords: Labeling; Graceful graph; Theta graph

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1. Introduction

Labeled graphs form useful models for a wide range of applications such as coding theory, X-ray crystallography, radar, astronomy, circuit design, communication network addressing and database management.

An *graceful labeling* f of a graph G with q edges is an injective function from the set of vertices of G to the set $\{0,1,2,\dots,q\}$ such that when each edge xy is assigned the label $|f(x)-f(y)|$, the resulting edge labels are distinct and nonzero. The concept above was put forward by Rosa in 1967.

The Ringel-Kotzig conjecture that all trees are graceful, has been the focus of many mathematicians. Among the trees known to be graceful are caterpillars, trees with at most 4-end vertices, trees with diameter at most 5, trees with at most 35 vertices, symmetrical trees, regular olive trees, lobsters, firecrackers, banana trees and bamboo trees.

Graphs consisting of any number of pair wise disjoint paths with common end vertices are called generalized theta graphs. Various labelings have been found for these graphs.

In this paper, some new classes of graphs have been constructed by combining some *theta* graphs with the star graphs $St(n)$, ($n \geq 1$). Only finite simple undirected graphs are considered here. Our notations and terminology are as in [1]. We refer to [2] for some basic concepts.

2. Some results on gracefulness of few theta related graphs

Definition 2.1. The theta graph $\theta(\alpha, \beta, \gamma)$ consists of three edge disjoint paths of lengths $\alpha, \beta,$ and γ having the same end points. Let the theta graph $\theta(2, 2, 3)$ have the paths $P_1: v_2, v_1, v_5; P_2: v_2, v_6, v_5$ and $P_3: v_2, v_3, v_4, v_5$.

Definition 2.2. Let A be any graph. Let B and C be any tree graphs. $A_{ij} \circ (B, C)$ denotes the new graph formed by attaching a center vertex of B to a vertex v_i of A and a center vertex of C to a vertex v_j of A , where i and j are distinct.

Theorem 2.3. $\theta_{\alpha\beta}(2,2,3) \circ (St(m), St(n))$ is graceful, for $m \geq 1, n \geq 1$.

Proof. Consider the theta graph $\theta(2,2,3)$ with 6 vertices v_1, v_2, \dots, v_6 . Let $St(m)$ be a star graph with $(m + 1)$ vertices w, w_1, w_2, \dots, w_m for $m \geq 1$, where w is the center vertex and w_1, w_2, \dots, w_m are pendant vertices.

Label the star graph $St(n)$ on $(n+1)$ vertices, by naming the vertices u, u_1, u_2, \dots, u_n ($n \geq 1$). Here u is the center vertex and the other vertices are pendant vertices. To form the graph $G = \theta_{\alpha\beta}(2,2,3) \circ (St(m), St(n))$ attach the vertex w of $St(m)$ with a vertex v_α ($1 \leq \alpha \leq 6$) of $\theta(2,2,3)$ and rename it as v_α and attach the vertex u of $St(n)$ with a vertex v_β ($1 \leq \beta \leq 6$) of $\theta(2,2,3)$ and rename it as v_β , for distinct α and β . G has $m + n + 6$ vertices and $m + n + 7$ edges.

The vertex set $V(G) = \{v_i, w_j, u_k / i = 1, 2, \dots, 6, j = 1, 2, \dots, m, k = 1, 2, \dots, n\}$. The edge set $E(G) = \{v_i v_{i+1} / i=1, 2, \dots, 5\} \cup \{v_i v_{i+4} / i=1, 2\} \cup \{v_i w_j / i=1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5 \text{ or } 6 \ \& \ j=1, 2, \dots, m\} \cup \{v_j u_k / j=1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5 \text{ or } 6 \ \& \ k=1, 2, \dots, n\}$.

Let f be the labeling on the set of vertices of G and g be the induced labeling on the set of edges of G . The vertex label set of G can be written as $A \cup B \cup C$ where $A = \{f(v_i) / i=1, 2, \dots, 6\}, B = \{f(w_j) / j=1, 2, \dots, m\}, C = \{f(u_k) / k=1, 2, \dots, n\}$.

The edge label set of G can be written as $D \cup E \cup F \cup G$ where $D = \{g(v_i v_{i+1}) / i=1, 2, \dots, 5\}, E = \{g(v_i v_{i+4}) / i=1, 2\}, F = \{g(v_i w_j) / i=1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5 \text{ or } 6 \ \& \ j=1, 2, \dots, m\}, G = \{g(v_j u_k) / j=1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5 \text{ or } 6 \ \& \ k = 1, 2, \dots, n\}$. Consider all the cases upto isomorphism.

Case 1. $\alpha=1, \beta=4$.

Let the labeling f on the vertices of G be defined by $f(v_i) = i+1$ for $i = 1$; $f(v_i) = m + n + i + 5$ for $i = 2$; $f(v_i) = m + n + i + 2$ for $i = 3$; $f(v_i) = i - 1$ for $i = 4$; $f(v_i) = m + n + i + 1$ for $i = 5$; $f(v_i) = i - 6$ for $i = 6$; $f(w_j) = j + 4$ for $1 \leq j \leq m$; $f(u_k) = k + 3$ for $k = 1$; $f(u_k) = m + k + 4$ for $2 \leq k \leq n$.

The induced labeling g on the edges of G is defined by $g(v_i v_{i+1}) = m + n + 5$ for $i = 1$; $g(v_i v_{i+1}) = 2$ for $i = 2$; $g(v_i v_{i+1}) = m + n + 2$ for $i = 3$; $g(v_i v_{i+1}) = m + n + 3$ for $i = 4$; $g(v_i v_{i+1}) = m + n + 6$ for $i = 5$; $g(v_i v_{i+4}) = m + n + 4$ for $i = 1$; $g(v_i v_{i+4}) = m + n + 7$ for $i = 2$; $g(v_i w_j) = j + 2$ for $i = 1, 1 \leq j \leq m$; $g(v_i u_k) = 1$ for $i = 4, k = 1$; $g(v_i u_k) = m + k + 1$ for $i = 4, 2 \leq k \leq n$.

The vertex labels of G can be arranged in the following order.

$A = \{0, 2, 3, m + n + 5, m + n + 6, m + n + 7\}, B = \{5, 6, 7, \dots, m + 4\}, C = \{4, m + 6, m + 7, \dots, m + n + 4\}$. The set of vertex labels of G is $A \cup B \cup C = \{0, 2, 3, \dots, m + 4, m + 6, \dots, m + n + 7\}$.

The edge labels of G can be arranged in the following order.

$D = \{ 2, m + n + 2, m + n + 3, m + n + 5, m + n + 6 \}$, $E = \{m + n + 4, m + n + 7\}$, $F = \{ 3, 4, \dots, m + 2 \}$, $G = \{ 1, m + 3, m + 4, \dots, m + n + 1 \}$. The set of edge labels of G is $D \cup E \cup F \cup G = \{1, 2, 3, \dots, m + n + 7\}$.

Therefore the set of vertex labels and edge labels are distinct. So f is a graceful labeling. Hence $G = \theta_{14}(2,2,3) \circ (St(m),St(n))$, for $m \geq 1, n \geq 1$, is a graceful graph.

Case 2. $\alpha = 1, \beta = 5$.

Let the labeling f on the vertices of G be defined by $f(v_i) = i$ for $i = 1$; $f(v_i) = m + n + i + 5$ for $i = 2$; $f(v_i) = i - 1$ for $i = 3$; $f(v_i) = m + n + i + 2$ for $i = 4$; $f(v_i) = i - 2$ for $i = 5$; $f(v_i) = i - 6$ for $i = 6$; $f(w_j) = j + 4$ for $1 \leq j \leq m$; $f(u_k) = k + 3$ for $k = 1$; $f(u_k) = m + k + 5$ for $2 \leq k \leq n$.

The induced labeling g on the edges of G is defined by $g(v_i v_{i+1}) = m + n + 6$ for $i = 1$; $g(v_i v_{i+1}) = m + n + 5$ for $i = 2$; $g(v_i v_{i+1}) = m + n + 4$ for $i = 3$; $g(v_i v_{i+1}) = m + n + 3$ for $i = 4$; $g(v_i v_{i+1}) = 3$ for $i = 5$; $g(v_i v_{i+4}) = 2$ for $i = 1$; $g(v_i v_{i+4}) = m + n + 7$ for $i = 2$; $g(v_i w_j) = j + 3$ for $i = 1, 1 \leq j \leq m$; $g(v_i u_k) = 1$ for $i = 5, k = 1$; $g(v_i u_k) = m + k + 2$ for $i = 5, 2 \leq k \leq n$.

The vertex labels of G can be arranged in the following order.

$A = \{ 0, 1, 2, 3, m + n + 6, m + n + 7 \}$, $B = \{ 5, 6, 7, \dots, m + 4 \}$, $C = \{ 4, m + 7, m + 8, \dots, m + n + 5 \}$. The set of vertex labels of G is $A \cup B \cup C = \{0, 1, 2, 3, \dots, m + 4, m + 7, \dots, m + n + 7\}$.

The edge labels of G can be arranged in the following order.

$D = \{ 3, m + n + 3, \dots, m + n + 6 \}$, $E = \{ 2, m + n + 7 \}$, $F = \{ 4, 5, \dots, m + 3 \}$, $G = \{ 1, m + 4, m + 5, \dots, m + n + 2 \}$. The set of edge labels of G is $D \cup E \cup F \cup G = \{1, 2, 3, \dots, m + n + 7\}$.

Therefore the set of vertex labels and edge labels are distinct. So f is a graceful labeling. Hence $G = \theta_{15}(2,2,3) \circ (St(m),St(n))$, for $m \geq 1, n \geq 1$, is a graceful graph.

Case 3. $\alpha = 1, \beta = 6$.

Let the labeling f on the vertices of G be defined by $f(v_i) = i + 1$ for $i = 1$; $f(v_i) = m + n + i + 5$ for $i = 2$; $f(v_i) = i - 3$ for $i = 3, 6$; $f(v_i) = m + n + i + 2$ for $i = 4$; $f(v_i) = m + n + i$ for $i = 5$; $f(w_j) = j + 4$ for $1 \leq j \leq m$; $f(u_k) = k$ for $k = 1$; $f(u_k) = m + k + 4$ for $2 \leq k \leq n$.

The induced labeling g on the edges of G is defined by $g(v_i v_{i+1}) = m + n + 5$ for $i = 1$; $g(v_i v_{i+1}) = m + n + 7$ for $i = 2$; $g(v_i v_{i+1}) = m + n + 6$ for $i = 3$; $g(v_i v_{i+1}) = 1$ for $i = 4$; $g(v_i v_{i+1}) = m + n + 2$ for $i = 5$; $g(v_i v_{i+4}) = m + n + 3$ for $i = 1$; $g(v_i v_{i+4}) = m + n + 4$ for $i = 2$; $g(v_i w_j) = j + 2$ for $i = 1, 1 \leq j \leq m$; $g(v_i u_k) = 2$ for $i = 6, k = 1$; $g(v_i u_k) = m + k + 1$ for $i = 6, 2 \leq k \leq n$.

The vertex labels of G can be arranged in the following order.

$A = \{ 0, 2, 3, m + n + 5, m + n + 6, m + n + 7 \}$, $B = \{ 5, 6, 7, \dots, m + 4 \}$, $C = \{ 1, m + 6, m + 7, \dots, m + n + 4 \}$. The set of vertex labels of G is $A \cup B \cup C = \{0, 1, 2, 3, 5, \dots, m + 4, m + 6, \dots, m + n + 7\}$.

The edge labels of G can be arranged in the following order.

$D = \{ 1, m + n + 2, m + n + 5, m + n + 6, m + n + 7 \}$, $E = \{ m + n + 3, m + n + 4 \}$, $F = \{ 3, 4, \dots, m + 2 \}$, $G = \{ 2, m + 3, m + 4, \dots, m + n + 1 \}$. The set of edge labels of G is $D \cup E \cup F \cup G = \{1, 2, 3, \dots, m + n + 7\}$.

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Therefore the set of vertex labels and edge labels are distinct. So f is a graceful labeling. Hence $G = \theta_{16}(2,2,3) \circ (\text{St}(m), \text{St}(n))$, for $m \geq 1, n \geq 1$, is a graceful graph.

Case 4. $\alpha = 2, \beta = 4$.

Let the labeling f on the vertices of G be defined by $f(v_i) = i + 2$ for $i = 1$; $f(v_i) = i$ for $i = 2$; $f(v_i) = m + i + 2$ for $i = 3$; $f(v_i) = i - 3$ for $i = 4$; $f(v_i) = m + n + i + 2$ for $i = 5$; $f(v_i) = i - 6$ for $i = 6$; $f(w_j) = j + 4$ for $1 \leq j \leq m$; $f(u_k) = m + k + 5$ for $1 \leq k \leq n-1$; $f(u_k) = m + k + 6$ for $k = n$.

The induced labeling g on the edges of G is defined by $g(v_i v_{i+1}) = 1$ for $i = 1$; $g(v_i v_{i+1}) = m + 3$ for $i = 2$; $g(v_i v_{i+1}) = m + 4$ for $i = 3$; $g(v_i v_{i+1}) = m + n + 6$ for $i = 4$; $g(v_i v_{i+1}) = m + n + 7$ for $i = 5$; $g(v_i v_{i+4}) = m + n + 4$ for $i = 1$; $g(v_i v_{i+4}) = 2$ for $i = 2$; $g(v_i w_j) = j + 2$ for $i = 2, 1 \leq j \leq m$; $g(v_i u_k) = m + k + 4$ for $i = 4, 1 \leq k \leq n-1$; $g(v_i u_k) = m + k + 5$ for $i = 4, k = n$.

The vertex labels of G can be arranged in the following order.

$A = \{0, 1, 2, 3, m + 5, m + n + 7\}$, $B = \{5, 6, 7, \dots, m + 4\}$, $C = \{m + 6, m + 7, \dots, m + n + 4, m + n + 6\}$. The set of vertex labels of G is $A \cup B \cup C = \{0, 1, 2, 3, 5, \dots, m + n + 4, m + n + 6, \dots, m + n + 7\}$.

The edge labels of G can be arranged in the following order.

$D = \{1, m + 3, m + 4, m + n + 6, m + n + 7\}$, $E = \{2, m + n + 4\}$, $F = \{3, 4, \dots, m + 2\}$, $G = \{m + 5, m + 6, \dots, m + n + 3, m + n + 5\}$. The set of edge labels of G is $D \cup E \cup F \cup G = \{1, 2, 3, \dots, m + n + 7\}$.

Therefore the set of vertex labels and edge labels are distinct. So f is a graceful labeling. Hence $G = \theta_{24}(2,2,3) \circ (\text{St}(m), \text{St}(n))$, for $m \geq 1, n \geq 1$, is a graceful graph.

Case 5. $\alpha = 2, \beta = 5$.

Let the labeling f on the vertices of G be defined by $f(v_i) = i + 2$ for $i = 1$; $f(v_i) = i - 1$ for $i = 2, 3$; $f(v_i) = m + i + 2$ for $i = 4$; $f(v_i) = i - 5$ for $i = 5$; $f(v_i) = m + n + i + 1$ for $i = 6$; $f(w_j) = j + 4$ for $1 \leq j \leq m$; $f(u_k) = m + k + 4$ for $k = 1$; $f(u_k) = m + k + 5$ for $2 \leq k \leq n$.

The induced labeling g on the edges of G is defined by $g(v_i v_{i+1}) = 2$ for $i = 1$; $g(v_i v_{i+1}) = 1$ for $i = 2$; $g(v_i v_{i+1}) = m + 4$ for $i = 3$; $g(v_i v_{i+1}) = m + 6$ for $i = 4$; $g(v_i v_{i+1}) = m + n + 7$ for $i = 5$; $g(v_i v_{i+4}) = 3$ for $i = 1$; $g(v_i v_{i+4}) = m + n + 6$ for $i = 2$; $g(v_i w_j) = j + 3$ for $i = 2, 1 \leq j \leq m$; $g(v_i u_k) = m + k + 4$ for $i = 5, k = 1$; $g(v_i u_k) = m + k + 5$ for $i = 5, 2 \leq k \leq n$.

The vertex labels of G can be arranged in the following order.

$A = \{0, 1, 2, 3, m + 6, m + n + 7\}$, $B = \{5, 6, 7, \dots, m + 4\}$, $C = \{m + 5, m + 7, \dots, m + n + 5\}$. The set of vertex labels of G is $A \cup B \cup C = \{0, 1, 2, 3, 5, \dots, m + n + 5, m + n + 7\}$.

The edge labels of G can be arranged in the following order.

$D = \{1, 2, m + 4, m + 6, m + n + 7\}$, $E = \{3, m + n + 6\}$, $F = \{4, 5, \dots, m + 3\}$, $G = \{m + 5, m + 7, \dots, m + n + 5\}$. The set of edge labels of G is $D \cup E \cup F \cup G = \{1, 2, 3, \dots, m + n + 7\}$.

Therefore the set of vertex labels and edge labels are distinct. So f is a graceful labeling. Hence $G = \theta_{25}(2,2,3) \circ (\text{St}(m), \text{St}(n))$, for $m \geq 1, n \geq 1$, is a graceful graph.

Case 6. $\alpha = 3, \beta = 4$.

Let the labeling f on the vertices of G be defined by $f(v_i) = i + 2$ for $i = 1$; $f(v_i) = m + n + i + 4$ for $i = 2$; $f(v_i) = i + 1$ for $i = 3$; $f(v_i) = i - 2$ for $i = 4$; $f(v_i) = m + n + 2 + i$ for $i = 5$; $f(v_i) = i - 6$ for $i = 6$; $f(w_j) = n + j + 5$ for $1 \leq j \leq m$; $f(u_k) = k$ for $k = 1$; $f(u_k) = k + 3$ for $2 \leq k \leq n$.

The induced labeling g on the edges of G is defined by $g(v_i v_{i+1}) = m + n + 3$ for $i = 1$; $g(v_i v_{i+1}) = m + n + 2$ for $i = 2$; $g(v_i v_{i+1}) = 2$ for $i = 3$; $g(v_i v_{i+1}) = m + n + 5$ for $i = 4$; $g(v_i v_{i+1}) = m + n + 7$ for $i = 5$; $g(v_i v_{i+4}) = m + n + 4$ for $i = 1$; $g(v_i v_{i+4}) = m + n + 6$ for $i = 2$; $g(v_i w_j) = n + j + 1$ for $i = 3$, $1 \leq j \leq m$; $g(v_i u_k) = k$ for $i = 4$, $k = 1$; $g(v_i u_k) = k + 1$ for $i = 4$, $2 \leq k \leq n$.

The vertex labels of G can be arranged in the following order.

$A = \{0, 2, 3, 4, m + n + 6, m + n + 7\}$, $B = \{n + 6, n + 7, \dots, m + n + 5\}$, $C = \{1, 5, \dots, n + 3\}$. The set of vertex labels of G is $A \cup B \cup C = \{0, 1, 2, 3, 4, 5, \dots, n + 3, n + 6, n + 7, \dots, m + n + 7\}$.

The edge labels of G can be arranged in the following order.

$D = \{2, m + n + 2, m + n + 3, m + n + 5, m + n + 7\}$, $E = \{m + n + 4, m + n + 6\}$, $F = \{n + 2, n + 3, \dots, m + n + 1\}$, $G = \{1, 3, 4, 5, \dots, n + 1\}$. The set of edge labels of G is $D \cup E \cup F \cup G = \{1, 2, 3, \dots, m + n + 7\}$.

Therefore the set of vertex labels and edge labels are distinct. So f is a graceful labeling. Hence $G = \theta_{34}(2, 2, 3) \circ (St(m), St(n))$, for $m \geq 1, n \geq 1$, is a graceful graph.

Case 7. $\alpha = 4, \beta = 5$.

Let the labeling f on the vertices of G be defined by $f(v_i) = n + i + 4$ for $i = 1$; $f(v_i) = i - 2$ for $i = 2$; $f(v_i) = m + n + i + 4$ for $i = 3$; $f(v_i) = i - 3$ for $i = 4, 5$; $f(v_i) = n + i$ for $i = 6$; $f(w_j) = j + 2$ for $j = 1$; $f(w_j) = n + j + 6$ for $2 \leq j \leq m$; $f(u_k) = k + 4$ for $1 \leq k \leq n$.

The induced labeling g on the edges of G is defined by $g(v_i v_{i+1}) = n + 5$ for $i = 1$; $g(v_i v_{i+1}) = m + n + 7$ for $i = 2$; $g(v_i v_{i+1}) = m + n + 6$ for $i = 3$; $g(v_i v_{i+1}) = 1$ for $i = 4$; $g(v_i v_{i+1}) = n + 4$ for $i = 5$; $g(v_i v_{i+4}) = n + 3$ for $i = 1$; $g(v_i v_{i+4}) = n + 6$ for $i = 2$; $g(v_i w_j) = 2$ for $i = 4, j = 1$; $g(v_i w_j) = n + j + 5$ for $i = 4, 2 \leq j \leq m$; $g(v_i u_k) = k + 2$ for $i = 5, 1 \leq k \leq n$.

The vertex labels of G can be arranged in the following order.

$A = \{0, 1, 2, n + 5, n + 6, m + n + 7\}$, $B = \{5, 6, \dots, n + 4\}$, $C = \{3, n + 8, n + 9, \dots, m + n + 6\}$. The set of vertex labels of G is $A \cup B \cup C = \{0, 1, 2, 3, 5, \dots, n + 6, n + 8, \dots, m + n + 7\}$.

The edge labels of G can be arranged in the following order.

$D = \{1, n + 4, n + 5, m + n + 6, m + n + 7\}$, $E = \{n + 3, n + 6\}$, $F = \{3, 4, \dots, n + 2\}$, $G = \{2, n + 7, \dots, m + n + 5\}$. The set of edge labels of G is $D \cup E \cup F \cup G = \{1, 2, 3, \dots, m + n + 7\}$.

Therefore the set of vertex labels and edge labels are distinct. So f is a graceful labeling. Hence $G = \theta_{45}(2, 2, 3) \circ (St(m), St(n))$, for $m \geq 1, n \geq 1$, is a graceful graph.

Case 8. $\alpha = 4, \beta = 6$.

Let the labeling f on the vertices of G be defined by $f(v_i) = i - 1$ for $i = 1$; $f(v_i) = m + n + i + 4$ for $i = 2$; $f(v_i) = i - 2$ for $i = 3, 6$; $f(v_i) = i - 1$ for $i = 4$; $f(v_i) = m + n + i + 2$ for $i = 5$; $f(w_j) = j + 1$ for $j = 1$; $f(w_j) = j + 4$ for $2 \leq j \leq m$; $f(u_k) = m + k + 5$ for $1 \leq k \leq n$.

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The induced labeling g on the edges of G is defined by $g(v_i v_{i+1}) = m + n + 6$ for $i = 1$; $g(v_i v_{i+1}) = m + n + 5$ for $i = 2$; $g(v_i v_{i+1}) = 2$ for $i = 3$; $g(v_i v_{i+1}) = m + n + 4$ for $i = 4$; $g(v_i v_{i+1}) = m + n + 3$ for $i = 5$; $g(v_i v_{i+4}) = m + n + 7$ for $i = 1$; $g(v_i v_{i+4}) = m + n + 2$ for $i = 2$; $g(v_i w_j) = 1$ for $i = 4, j = 1$; $g(v_i w_j) = j + 1$ for $i = 4, 2 \leq j \leq m$; $g(v_i u_k) = m + k + 1$ for $i = 6, 1 \leq k \leq n$.

The vertex labels of G can be arranged in the following order.

$A = \{0, 1, 3, 4, m + n + 6, m + n + 7\}$, $B = \{2, 6, \dots, m + 4\}$, $C = \{m + 6, \dots, m + n + 5\}$. The set of vertex labels of G is $A \cup B \cup C = \{0, 1, 2, 3, 4, 6, \dots, m + 4, m + 6, \dots, m + n + 7\}$.

The edge labels of G can be arranged in the following order.

$D = \{2, m + n + 3, \dots, m + n + 6\}$, $E = \{m + n + 2, m + n + 7\}$, $F = \{1, 3, 4, \dots, m + 1\}$, $G = \{m + 2, \dots, m + n + 1\}$. The set of edge labels of G is $D \cup E \cup F \cup G = \{1, 2, 3, \dots, m + n + 7\}$.

Therefore the set of vertex labels and edge labels are distinct. So f is a graceful labeling. Hence $G = \theta_{46}(2, 2, 3) \circ (St(m), St(n))$, for $m \geq 1, n \geq 1$, is a graceful graph.

Case 9. $\alpha = 5, \beta = 6$.

Let the labeling f on the vertices of G be defined by $f(v_i) = i + 3$ for $i = 1$; $f(v_i) = m + n + i + 5$ for $i = 2$; $f(v_i) = i - 3$ for $i = 3, 6$; $f(v_i) = m + n + i + 2$ for $i = 4$; $f(v_i) = i - 4$ for $i = 5$; $f(w_j) = j + 4$ for $1 \leq j \leq m$; $f(u_k) = k + 1$ for $k = 1$; $f(u_k) = m + k + 5$ for $2 \leq k \leq n$.

The induced labeling g on the edges of G is defined by $g(v_i v_{i+1}) = m + n + 3$ for $i = 1$; $g(v_i v_{i+1}) = m + n + 7$ for $i = 2$; $g(v_i v_{i+1}) = m + n + 6$ for $i = 3$; $g(v_i v_{i+1}) = m + n + 5$ for $i = 4$; $g(v_i v_{i+1}) = 2$ for $i = 5$; $g(v_i v_{i+4}) = 3$ for $i = 1$; $g(v_i v_{i+4}) = m + n + 4$ for $i = 2$; $g(v_i w_j) = j + 3$ for $i = 5, 1 \leq j \leq m$; $g(v_i u_k) = 1$ for $i = 6, k = 1$; $g(v_i u_k) = m + k + 2$ for $i = 6, 2 \leq k \leq n$.

The vertex labels of G can be arranged in the following order.

$A = \{0, 1, 3, 4, m + n + 6, m + n + 7\}$, $B = \{5, 6, \dots, m + 4\}$, $C = \{2, m + 7, \dots, m + n + 5\}$. The set of vertex labels of G is $A \cup B \cup C = \{0, 1, 2, 3, 4, 6, \dots, m + 4, m + 7, \dots, m + n + 7\}$.

The edge labels of G can be arranged in the following order.

$D = \{2, m + n + 3, m + n + 5, \dots, m + n + 7\}$, $E = \{3, m + n + 4\}$, $F = \{4, 5, 6, \dots, m + 3\}$, $G = \{1, m + 4, \dots, m + n + 2\}$. The set of edge labels of G is $D \cup E \cup F \cup G = \{1, 2, 3, \dots, m + n + 7\}$.

Therefore the set of vertex labels and edge labels are distinct. So f is a graceful labeling. Hence $G = \theta_{56}(2, 2, 3) \circ (St(m), St(n))$, for $m \geq 1, n \geq 1$, is a graceful graph.

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