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Column-Average-Max-Norm of Fuzzy Matrix

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Abstract: Fuzzy matrices helps to solve different types of model in a fuzzy environment. In this paper, we have defined a norm on fuzzy matrices, the namely column-average-max norm. We also investigated different properties and lemma of this norm.

Keywords: Norm of a fuzzy matrix, column-average-max norm, properties of the column-average-max norm.

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1. Introduction

The study of linear algebra has become more and more famous over the last decades, besides its beauty and connection with many other pure and applied areas. That's while People are attracted to this subject. In the theoretical development of this subject and several applications, one often needs to evaluate the length of vectors. For this determination, A norm functions are considered on a vector space.

A norm be defined on a real vector space V is a function $\|.\|: V \to R$ satisfying

1. $\|v\| > 0$ for any nonzero $v \in V$.
2. $\|mv\| = |m|\,\|v\|$ for any $m \in R$ and $v \in$ V.
3. $\|v+w\| \le \|v\| + \|w\|$ for any $v, w \in V$.

The norm is to evaluate the size of the vector u where equation(1) gives the size to be positive, equation (2) demands if the vector is scaled, then the size to be scaled, and equation (3) is similar to the triangle inequality and its origin in the notion of distance in R^3. The equation (2) is said to be a homogeneous condition and ensures that the norm of the zero vector in V is 0; this condition is often involved in defining a norm.

A familiar example of norms on R^n are the l_p norms, where $1 \le p \le \infty$, and it is defined by

$$l_p(u) = \{\sum_{j=1}^{n} |u_j|^p\}^{\frac{1}{p}} \quad \text{if } 1 \le p < \infty \text{ and}$$

$$l_p(u) = \max_{1 \le j \le n} |u_j| \quad \text{if } p = \infty$$

for any $u = (u_1, u_2, ..., u_n)^t \in R^n$. Note that if one defines an l_p function on R^n as defined above with $0 < p < 1$, then the triangle inequality is not satisfied. Therefore, it is not a norm.

Given that a norm is defined on a real vector space V, one can compare the vector norms, study the convergence of a sequence of vectors, discuss limits and continuity of transformations, and consider approximation problems. These problems arise naturally in analysis, numerical analysis other than differential equations, Markov chains etc. For example, finding the nearest element in a subset or a subspace of V to a given vector.

The norm of a matrix is given by how large its elements are. It is a way of determining the "size" of a matrix that is necessarily related to how many rows or columns are present in this matrix. The norm of a square matrix P is denoted by $\|P\|$, is a non-negative real number. There are some different ways of defining a matrix norm, but they all discuss the following properties:

1. $\|P\| \geq 0$ for any square matrix P.
2. $\|P\| = 0$ iff the matrix $P = 0$.
3. $\|MP\| = |M| \|P\|$ for any scaler M .
4. $\|P + Q\| \leq \|P\| + \|Q\|$ for any square matrix P, Q.
5. $\|PQ\| \leq \|P\| \|Q\|$.

Fuzzy matrix norm

Matrix norm and fuzzy matrix norm are almost same. Norm of a fuzzy matrix is also a function $\|.\| : M_n(F) \to [0,1]$ which satisfies the following properties

1. $\|P\| \geq 0$ for any fuzzy matrix P .
2. $\|P\| = 0$ iff the fuzzy matrix P =0.
3. $\|MP\| = |M| \|P\|$ for any scaler $M \in [0,1]$.
4. $\|P + Q\| \leq \|P\| + \|Q\|$ for any two fuzzy matrices P and Q.
5. $\|PQ\| \leq \|P\| \|Q\|$ for any two fuzzy matrices P and Q.

In this project paper, we have defined different types of norm on fuzzy matrices.

1.1. Why study different norms?

We know that different types of the norm on a vector space can give rise to different geometrical and analytical structures. In an infinite dimensional vector space, the convergence of a sequence can vary depending on the choice of norm. This spectacle gives many interesting questions and research in analysis and functional analysis.

In a finite-dimensional vector space V, two norm $\|.\|_1$ and $\|.\|_2$ are said to be equivalent if there exist one positive constant such that

$$a\|v\|_1 \leq \|v\|_2 \leq \|v\|_1 \text{ for all} v \in V.$$

First, it may be easier to prove convergence for a given sequence with respect to one norm rather than another. In an application such as numerical analysis one would like to use a norm that can determine convergence efficiently. Therefore, this idea is good to have knowledge of different norms.

Second, a specific norm may sometimes be needed to deal with a particular problem. For instance, if one travels in Manhattan and wants to measure the distance from a location marked as the origin $(0,0)$ to a destination marke as (x, y) on the map, one may use the l_2 norm of (x, y), which measures the straight line distance between two points, or one may need to use the l_1 norm of v, which measures the distance for a taxi cab to drive from $(0,0)$ to (x, y). The l_1 norm is sometimes called the taxi cab norm for this reason.

In approximation theory, solutions of a problem can vary with different problems. For an example, if W is a subspace of R^n and v does not belongs to W, then for $1 < p < \infty$ there is a unique $u_0 \in W$ such that

$$\|v - u_0\| \le \|v - u\| \qquad \text{for all } u \in W,$$

but the uniqueness condition may fail if $p = 1$ or ∞. To see a concrete example, let $v = (1,0)$ and $W = \{(0, y) : y \in R\}$.

Then for all $y \in [-1,1]$ we have $1 = \|v - (0, y)\| \le \|v - w\|$ for all $w \in W$. For some problems, having a unique approximation is good, but for others, it may be better to have many so that one of them can be chosen to satisfy additional conditions.

1.2. Fuzzy matrix

We know that matrices play an important role in various areas, not only mathematics, but also physics, statistics, engineering, social sciences and many other subjects. Some works on classical matrices are available in individual journals even in books also. But, crisp data is not always involved, such in our real-life problems in social science, medical science, environment etc. Therefore, the various types of uncertainties present in our daily life problems. So we cannot successfully use traditional classical matrices. At the moment, probabilities, fuzzy sets, intuitionistic fuzzy sets, vague sets, and rough sets are used as mathematical tools for dealing with uncertainties. Fuzzy matrices derive a lot of applications, one kind of which is as adjacency matrices of fuzzy relations and fuzzy relational equations have all important applications in pattern classification and in handling fuzziness under knowledge-based systems. First-time fuzzy matrices were introduced by Thomason [43], who studied the convergence of powers of fuzzy matrices. Some properties of the min-max composition of fuzzy matrices are presented by Ragab et al. [34, 35]. Hashimoto [18, 19] presented the canonical form of a transitive fuzzy matrix. Iterates of fuzzy circulant matrices investigated by Hemashina et al.[20]. Tan [42] considered powers and nilpotent conditions of matrices over a distributive lattice. After that, Pal, Bhowmik, Adak, Shyamal, Mondal have done a lot of work on fuzzy, intuitionistic fuzzy, interval-valued fuzzy matrices [1-12, 26-33,36-40]. The values of fuzzy matrix elements lie in the closed interval [0, 1]. We see that every fuzzy matrix are matrices, but all matrix, in general, is not a fuzzy matrix. We see that [0,1] is a fuzzy interval, i.e. the unit interval is a subset of reals. Since the unit interval [0, 1] is contained in the set of reals, a matrix, in general, is not a fuzzy matrix. Now a big query is can we add two fuzzy matrices A and B and get the sum of them to be a fuzzy matrix? The result in general, is not possible for the sum of two fuzzy matrices may turn out to be a matrix which is not a fuzzy matrix. If we added two fuzzy matrix A and B then we get A+B whose elements may not not lie in [0,1]. Hence A+B is not a fuzzy matrix. It is only a simple matrix. So only in case, we have defined the average or min operation of fuzzy. Now add this above fuzzy matrix under the max or min operation, and we get the resultant matrix is again a fuzzy matrix. In general, to add two fuzzy matrices, we use the max operation. We see the product of two fuzzy matrices under simple matrix multiplication is not a fuzzy matrix. Similarly, we need to define a compatible operation analogous to the product of two fuzzy matrices; otherwise, the product of two fuzzy matrices is not a fuzzy matrix. However, even for this new operation, if the product AB is to be defined, we must be needed the order of two fuzzy matrices to be the same, i.e., the number of columns of A is equal to the number of rows of B. We can have the two types of operation are max-min and min-max. Maity [23, 24] introduced max-norm, square-max norm, row and column-max-average nom of fuzzy matrices and some properties of these two norms. In this paper, we have defined the column-average-max norm with some properties.

2. Preliminaries

Definition 1. *An $n \times n$ fuzzy matrix M is said to be reflexive iff $m_{ii} = 1$ for all i=1,2,...,n. It is said to be α-reflexive iff $m_{ii} \geq \alpha$ for all i=1,2,...,n where $\alpha \in [0,1]$. It is said to be weakly reflexive iff $m_{ii} \geq m_{ij}$ for all i, j=1,2,...,n. An $n \times n$ fuzzy matrix M is called irreflexive iff $m_{ii} = 0$ for all i=1,2,...,n.*

Definition 2. *An $n \times n$ fuzzy matrix S is said to be symmetric iff $s_{kl} = s_{lk}$ for all k,l=1,2,...,n. It is said to be antisymmetric iff $S \wedge S' \leq I_n$ where I_n is unit usual matrix.*

Note that the condition $S \wedge S' \leq I_n$, means that $s_{kl} \wedge s_{lk} = 0$ for all $k \neq l$ and $s_{kk} \leq 1$ for all k. So if $S_{kl} = 1$ then $s_{lk} = 0$, which the crisp case.

Definition 3. *An $n \times n$ fuzzy matrix N, $N^n = 0$ (the zero matrix) iff the matrix is said to be nilpotent. If $N^{p-1} \neq 0$ and $N^p = 0$; $1 \leq p \leq n$ then N is called nilpotent of degree p. An $n \times n$ fuzzy matrix E is said to be idempotent iff $E^2 = E$. It is called transitive iff $E^2 \leq E$ And it is said to be compact iff $E^2 \geq E$.*

Definition 4. *A fuzzy matrix of order $m \times n$ is defined as $A = (a_{ij})_{m \times n}$, is called triangular fuzzy matrix where $a_{ij} = \langle m_{ij}, \alpha_{ij}, \beta_{ij} \rangle$ is the ij^{th} element of A, m_{ij} is the mean value of a_{ij} and α_{ij}, β_{ij} are left and right spread of a_{ij} respectively.*

This section discusses different types of matrix and fuzzy matrix norms.

2.1. Matrix norm

In this section different types of matrix norm and fuzzy matrix norm are discussed.

Definition 5. *(the maximum absolute column sum). Simply at first,we sum the absolute values of each column and then take the maximum values. (A useful reminder is that "1" is a tall, thin character and a column is a tall, thin quantity.)*

$$\|B\|_1 = \max_{1\le j\le n}\left(\sum_{i=1}^{n} |b_{ij}|\right).$$

Definition 6. *The infinity norm of a square matrix is the maximum of the absolute row sum. Simply at first,we sum the absolute values down each row and then take the maximum value. The infinity norm of a matrix B is defined by*

$$\|B\|_\infty = \max_{1\le i\le n}\left(\sum_{j=1}^{n} |b_{ij}|\right).$$

Definition 7. *The Euclidean norm of a square matrix is the square root of the sum of the squares of all elements. This is similar to ordinary "Pythagorean" length, where a vector's size is found by taking the square root of the sum of all the squares of the elements.The Euclidean norm of a matrix B is defined by*

$$\|B\|_E = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(b_{ij})^2}.$$

Any definition you can define that satisfies the five conditions mentioned at the beginning of this section is a definition of a norm. There are many possibilities, but the three above are among the most commonly used.

2.2. Fuzzy matrix norm

Definition 8. ***Max norm*** *(Maity [23]): Max norm of a fuzzy matrix $A \in M_n(F)$ is denoted by $\|A\|_M$ which gives the maximum element of the fuzzy matrix, and it is defined by*

$$\|A\|_M = \bigvee_{i,j=1}^{n} a_{ij}$$

Definition 9. *(Maity [23]): Square-max norm of a fuzzy matrix A is denoted by $\|A\|_{SM}$ and defined by $\|A\|_{SM} = (\bigvee_{i,j=1}^{n} a_{ij})^2 = (\|A\|_M)^2$.*

In this norm,we will first find the maximum element of the fuzzy matrix and then square it.

Definition 10. ***Row-max-average Norm*** *(Maity [24])**:** Row-max-average norm of a fuzzy matrix A is denoted by $\|A\|_{RMA}$ and defined by $\|A\|_{RMA} = \frac{1}{n}\sum_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij})$*

Here, at first, we find the maximum element in each row. Then we determine the average of the maximum element.

Definition 11. ***Column-max-average norm*** *(Maity [24]): The Column-max-average norm of a fuzzy matrix A is denoted by $\|A\|_{CMA}$ and defined by $\|A\|_{CMA} = \frac{1}{n}\sum_{j=1}^{n}(\bigvee_{i=1}^{n} a_{ij})$.*

Here we find the maximum element in each column and then the average of the maximum elements.

Definition 12. ***Pseudo norm on fuzzy matrix*** *(Maity [24]): A norm of a fuzzy matrix is called pseudo norm of a fuzzy matrix if it fulfills the following conditions*

1. $\|A\| \geq 0$ for any fuzzy matrix A.
2. if $A = 0$ then $\|A\| = 0$.
3. $\|kA\| = |k|\,\|A\|$ for any scaler $k \in [0,1]$.
4. $\|A+B\| \leq \|A\| + \|B\|$ for any two fuzzy matrices A and B.
5. $\|AB\| \leq \|A\|\|B\|$ for any two fuzzy matrices A and B.

Definition 13. ***Max-min Norm*** *(Maity [24]): Max-Min norm of a fuzzy matrix A is denoted by $\|A\|_{MM}$ and defined by $\|A\|_{MM} = \bigwedge_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij})$*

Here, we find the maximum element in each row and then the minimum of the maximum elements.

2.3. Addition and multiplication of fuzzy matrices

Maity [24]defined two new types of operators of fuzzy matrices denoted by the symbol $\oplus$ and $\otimes$. The operator $\oplus$is used for addition and the operator $\otimes$ is used for multiplication of fuzzy matrices. The following way defines these two operators.

If $C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$ and $D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix}$

Then $C \oplus D = \begin{bmatrix} \frac{c_{11}+d_{11}}{2} & \frac{c_{12}+d_{12}}{2} & \cdots & \frac{c_{1n}+d_{1n}}{2} \\ \frac{c_{21}+d_{21}}{2} & \frac{c_{22}+d_{22}}{2a} & \cdots & \frac{c_{2n}+d_{2n}}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{c_{n1}+d_{n1}}{2} & \frac{c_{n2}+d_{n2}}{2} & \cdots & \frac{c_{nn}+d_{nn}}{2} \end{bmatrix}$

and C $\otimes$ D $= \begin{bmatrix} \wedge\{c_{11}, d_{11}\} & \wedge\{c_{12}, d_{12}\} & & \wedge\{c_{1n}, d_{1n}\} \\ \wedge\{c_{21}, d_{21}\} & \wedge\{c_{22}, d_{22}\} & & \wedge\{c_{2n}, d_{2n}\} \\ \vdots & \vdots & \ddots & \vdots \\ \wedge\{c_{n1}, d_{n1}\} & \wedge\{c_{n2}, d_{n2}\} & \cdots & \wedge\{c_{nn}, d_{nn}\} \end{bmatrix}$

In this type of multiplication, fuzzy matrices will be of same order.

Example:

If $C = \begin{bmatrix} 0.2 & 0.6 & 0.3 \\ 0.4 & 0.8 & 0.5 \\ 0.8 & 0.4 & 0.7 \end{bmatrix}$ and $D = \begin{bmatrix} 0.6 & 0.3 & 0.7 \\ 0.2 & 0.1 & 0.3 \\ 0.1 & 0.4 & 0.3 \end{bmatrix}$

Then $C \oplus D = \begin{bmatrix} 0.4 & 0.45 & 0.5 \\ 0.3 & 0.45 & 0.4 \\ 0.45 & 0.4 & 0.5 \end{bmatrix}$ and $\mathrm{C} \otimes \mathrm{D} = \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.2 & 0.1 & 0.3 \\ 0.1 & 0.4 & 0.3 \end{bmatrix}$

3. Column-average-max-norm (CAM)

Here we have defined a new type of norm called column-average-max-norm. We have used new type of operators of fuzzy matrices for this norm. Here at first we determine the average value in each column. Then we find the maximum of these average values. The column-average-max-norm of a fuzzy matrix P is denoted by$\|P\|_{CAM}$ and defined by

$$\|P\|_{CAM} = \bigvee_{j=1}^{n} \left(\frac{1}{n} \sum_{i=1}^{n} p_{ij} \right)$$

Lemma 1. *All the conditions of norm are satisfied by* $\|P\|_{CAM} = \vee_{j=1}^{n} \left(\frac{1}{n} \sum_{i=1}^{n} p_{ij} \right)$

Proof:

Let us consider, $P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$ and $Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix}$

$\|P\|_{CAM} = \vee_{j=1}^{n} \left(\frac{1}{n} \sum_{i=1}^{n} p_{ij} \right)$ and $\|Q\|_{CAM} = \vee_{j=1}^{n} \left(\frac{1}{n} \sum_{i=1}^{n} q_{ij} \right)$

i. As all $p_{ij} \geq 0$

So, according to the definition of column-average-max-norm obviously $\|P\|_{CAM} \geq 0$

Now, $\|P\|_{CAM} = 0$

$\Leftrightarrow \vee_{j=1}^{n} \left(\frac{1}{n} \sum_{i=1}^{n} p_{ij} \right) = 0$

$\Leftrightarrow \sum_{i=1}^{n} p_{ij} = 0$ for all $j = 1,2,..n$

$\Leftrightarrow p_{1j} = p = \cdots p_{nj} = 0$ for all $j = 1,2,..n$

$\Leftrightarrow p_{ij} = 0$ for all $j = 1,2,..n$

$\Leftrightarrow P = 0$

So, $\|P\|_{CAM} = 0$ iff $P = 0$

ii. Here, we have defined a new type of scalar multiplication as follows

$$\beta p_{ij} = \begin{Bmatrix} |\alpha| if |\alpha| \leq \|P\|_{CAM} \\ \|P\|_{CAM} if |\alpha| > \|P\|_{CAM} \end{Bmatrix}$$

So, if$|\beta| \leq \|P\|_{CAM}$ then $\|\beta P\|_{CAM} = |\beta| \|P\|_{CAM}$

and if $|\beta| > \|P\|_{CAM}$ then $\|\beta P\|_{CAM} = |\beta| \|P\|_{CAM}$

Therefore, $\|\beta P\|_{CAM} = |\beta| \|P\|_{CAM}$for all$\beta \in [0,1]$

$$P \oplus Q = \begin{bmatrix} \frac{p_{11}+q_{11}}{2} & \frac{p_{12}+q_{12}}{2} & \cdots & \frac{p_{1n}+q_{1n}}{2} \\ \frac{p_{21}+q_{21}}{2} & \frac{p_{22}+q_{22}}{2a} & \cdots & \frac{p_{2n}+q_{2n}}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{p_{n1}+q_{n1}}{2} & \frac{p_{n2}+q_{n2}}{2} & \cdots & \frac{p_{nn}+q_{nn}}{2} \end{bmatrix}$$

$$\| \text{ P} \oplus \text{Q}\|_{CAM} = \vee_{j=1}^{n}\left[\frac{\frac{p_{1j}+q_{1j}}{2}+\frac{p_{2j}+q_{2j}}{2}+\cdots+\frac{p_{nj}+q_{nj}}{2}}{n}\right]$$
$$= \vee_{j=1}^{n}\left[\frac{(p_{1j}+q_{1j})+(p_{2j}+q_{2j})+\cdots+(p_{nj}+q_{nj})}{2n}\right]$$
$$\leq \frac{\vee_{j=1}^{n}[p_{1j}+p_{2j}+\cdots+p_{nj}]+\vee_{j=1}^{n}[q_{1j}+q_{2j}+\cdots+q_{nj}]}{2n}$$
$$= \frac{\vee_{j=1}^{n}(\sum_{i=1}^{n}p_{ij})+\vee_{j=1}^{n}(\sum_{i=1}^{n}q_{ij})}{2n}$$
$$= \frac{\vee_{j=1}^{n}\left(\frac{1}{n}\sum_{i=1}^{n}p_{ij}\right)+\vee_{j=1}^{n}\left(\frac{1}{n}\sum_{i=1}^{n}q_{ij}\right)}{2}$$
$$= \frac{\|P\|_{CAM}+\|Q\|_{CAM}}{2} = \|P\|_{CAM} \oplus \|Q\|_{CAM}$$

So, $\| \text{ P} \oplus \text{Q}\|_{CAM} \leq \|P\|_{CAM} \oplus \|Q\|_{CAM}$

iii. $$P \otimes Q = \begin{bmatrix} \wedge\{p_{11},q_{11}\} & \wedge\{p_{12},q_{12}\} & & \wedge\{p_{1n},q_{1n}\} \\ \wedge\{p_{21},q_{21}\} & \wedge\{p_{22},q_{22}\} & & \wedge\{p_{2n},q_{2n}\} \\ \vdots & \vdots & \ddots & \vdots \\ \wedge\{p_{n1},q_{n1}\} & \wedge\{p_{n2},q_{n2}\} & \cdots & \wedge\{p_{nn},q_{nn}\} \end{bmatrix}$$

Now $\wedge\{p_{ij},q_{ij}\} \leq p_{ij}$ and q_{ij} for all i,j

$\sum_{i=1}^{n}\{\wedge(p_{ij},q_{ij})\} \leq \sum_{i=1}^{n}p_{ij}$ and q_{ij} for all i,j

$\vee_{j=1}^{n}\left[\frac{1}{n}\sum_{i=1}^{n}\{\wedge(p_{ij},q_{ij})\}\right] \leq \vee_{j=1}^{n}\left[\frac{1}{n}\sum_{i=1}^{n}p_{ij}\right]$ and $\vee_{j=1}^{n}\left[\frac{1}{n}\sum_{i=1}^{n}q_{ij}\right]$

$\Rightarrow \|\text{P} \otimes \text{Q}\|_{CAMN} \leq \|P\|_{CAMN} \otimes \|Q\|_{CAMN}$

Hence, all the conditions of norm are satisfied by column-average-max norm.

3.1. Properties of column-average-max norm

Properties 1. *If C and D are two fuzzy matrices then $\|(C \oplus D)^T\|_{CAM} \leq \|C^T\|_{CAM} \oplus \|D^T\|_{CAM}$ hold.*

Proof:

Let us consider $C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$ and $B = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix}$

Then $(\text{C} \oplus \text{D}) = \begin{bmatrix} \frac{c_{11}+d_{11}}{2} & \frac{c_{12}+d_{12}}{2} & \cdots & \frac{c_{1n}+d_{1n}}{2} \\ \frac{c_{21}+d_{21}}{2} & \frac{c_{22}+d_{22}}{2a} & \cdots & \frac{c_{2n}+d_{2n}}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{c_{n1}+d_{n1}}{2} & \frac{c_{n2}+d_{n2}}{2} & \cdots & \frac{c_{nn}+d_{nn}}{2} \end{bmatrix}$ and

$$(\text{C} \oplus \text{D})^T = \begin{bmatrix} \frac{c_{11}+d_{11}}{2} & \frac{c_{21}+d_{21}}{2} & \cdots & \frac{c_{n1}+d_{n1}}{2} \\ \frac{c_{12}+d_{12}}{2} & \frac{c_{22}+d_{22}}{2} & \cdots & \frac{c_{n2}+d_{n2}}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{c_{1n}+d_{1n}}{2} & \frac{c_{2n}+d_{2n}}{2} & \cdots & \frac{c_{nn}+d_{nn}}{2} \end{bmatrix}$$

$$\|(\text{C} \oplus \text{D})^T\|_{CAM} = \vee_{j=1}^{n}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{c_{ji}+d_{ji}}{2}\right)\right] \leq \vee_{j=1}^{n}\left[\frac{1}{2n}\sum_{i=1}^{n}c_{ji}\right] + \vee_{j=1}^{n}\left[\frac{1}{2n}\sum_{i=1}^{n}d_{ji}\right]$$

$\leq \frac{\bigvee_{j=1}^{n}\left[\frac{1}{n}\sum_{i=1}^{n} c_{ji}\right]+\bigvee_{j=1}^{n}\left[\frac{1}{n}\sum_{i=1}^{n} d_{ji}\right]}{2} = \frac{\|C^T\|_{CAM}+\|D^T\|_{CAM}}{2} = \|C^T\|_{CAM} \oplus \|D^T\|_{CAM}$

Example 1.

Let $C = \begin{bmatrix} 0.5 & 0.3 & 0.4 \\ 0.4 & 0.6 & 0.4 \\ 0.4 & 0.7 & 0.8 \end{bmatrix}$ and $D = \begin{bmatrix} 0.8 & 0.3 & 0.2 \\ 0.9 & 0.6 & 0.1 \\ 0.1 & 0.7 & 0.5 \end{bmatrix}$

$(\mathrm{C} \oplus \mathrm{D}) = \begin{bmatrix} 0.65 & 0.3 & 0.3 \\ 0.65 & 0.6 & 0.25 \\ 0.25 & 0.7 & 0.65 \end{bmatrix}$; $(\mathrm{C} \oplus \mathrm{D})^T = \begin{bmatrix} 0.65 & 0.65 & 0.25 \\ 0.3 & 0.6 & 0.7 \\ 0.3 & 0.25 & 0.65 \end{bmatrix}$

Now, $\|(\mathrm{C} \oplus \mathrm{D})^T\|_{CAM} = max\{0.4167,0.5,0.534\} = 0.534$

$$\|C^T\|_{CAM} = max\{0.4,0.47,0.64\} = 0.64$$
$$\|D^T\|_{CAM} = max\{0.44,0.54,0.44\} = 0.54$$
$$\|C^T\|_{CAM} \oplus \|D^T\|_{CAM} = 0.59$$

So, $\|(\mathrm{C} \oplus \mathrm{D})^T\|_{CAM} < \|C^T\|_{CAM} \oplus \|D^T\|_{CAM}$

Properties 2. *If C and D are two fuzzy matrices and $C \leq D$ then $\|C\|_{CAM} \leq \|D\|_{CAM}$*

Proof:

As $C \leq D$ so, $c_{ij} \leq d_{ij}$

$\Rightarrow \sum_{i=1}^{n} c_{ij} \leq \sum_{i=1}^{n} d_{ij}$ for all j

$$\Rightarrow \bigvee_{j=1}^{n}\left[\frac{1}{n}\sum_{i=1}^{n} c_{ij}\right] \leq \bigvee_{j=1}^{n}\left[\frac{1}{n}\sum_{i=1}^{n} d_{ij}\right]$$
$$\Rightarrow \|C\|_{CAM} \leq \|D\|_{CAM}$$

Example 2.

Let $C = \begin{pmatrix} 0.3 & 0.1 \\ 0.5 & 0.2 \end{pmatrix}$ and $D = \begin{pmatrix} 0.6 & 0.5 \\ 0.8 & 0.7 \end{pmatrix}$

$\|C\|_{CAM} = 0.4$ and $\|D\|_{CAM} = 0.7$

So, $\|C\|_{CAM} < \|D\|_{CAM}$

Properties 3. *If C, D and E are two fuzzy matrices and $C \leq D$ then $\|C \otimes E\|_{CAM} \leq \|D \otimes E\|_{CAM}$ hold.*

Proof:

Let us consider, $C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$, $D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix}$ and

$E = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{bmatrix}$.

Then $\mathrm{C} \otimes \mathrm{E} = \begin{bmatrix} \wedge\{c_{11}, e_{11}\} & \wedge\{c_{12}, e_{12}\} & & \wedge\{c_{1n}, e_{1n}\} \\ \wedge\{c_{21}, e_{21}\} & \wedge\{c_{22}, e_{22}\} & & \wedge\{c_{2n}, e_{2n}\} \\ \vdots & \vdots & \ddots & \vdots \\ \wedge\{c_{n1}, e_{n1}\} & \wedge\{c_{n2}, e_{n2}\} & \cdots & \wedge\{c_{nn}, e_{nn}\} \end{bmatrix}$

and $D \otimes E = \begin{bmatrix} \wedge\{d_{11},e_{11}\} & \wedge\{d_{12},e_{12}\} & & \wedge\{d_{1n},e_{1n}\} \\ \wedge\{d_{21},e_{21}\} & \wedge\{d_{22},e_{22}\} & & \wedge\{d_{2n},e_{2n}\} \\ \vdots & \vdots & \ddots & \vdots \\ \wedge\{d_{n1},e_{n1}\} & \wedge\{d_{n2},e_{n2}\} & \cdots & \wedge\{d_{nn},e_{nn}\} \end{bmatrix}$

$\|C \otimes E\|_{CAM} = \vee_{j=1}^{n}\left[\frac{1}{n}\sum_{i=1}^{n}\{\wedge(c_{ij},e_{ij})\}\right]$ and

$\|D \otimes E\|_{CAM} = \vee_{j=1}^{n}\left[\frac{1}{n}\sum_{i=1}^{n}\{\wedge(d_{ij},e_{ij})\}\right]$

Now $C \leq D \Rightarrow c_{ij} \leq d_{ij}$ for all i,j

$\Rightarrow \wedge(c_{ij},e_{ij}) \leq \wedge(d_{ij},e_{ij})$ for all i,j

$\Rightarrow \sum_{i=1}^{n}\{\wedge(c_{ij},e_{ij})\} \leq \sum_{i=1}^{n}\{\wedge(d_{ij},e_{ij})\}$ for all j

$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\{\wedge(c_{ij},e)\} \leq \frac{1}{n}\sum_{i=1}^{n}\{\wedge(d_{ij},e_{ij})\}$ for all j

$\Rightarrow \vee_{j=1}^{n}\left[\frac{1}{n}\sum_{i=1}^{n}\{\wedge(c_{ij},e_{ij})\}\right] \leq \vee_{j=1}^{n}\left[\frac{1}{n}\sum_{i=1}^{n}\{\wedge(d_{ij},e_{ij})\}\right]$

$\Rightarrow \|C \otimes E\|_{CAM} \leq \|D \otimes E\|_{CAM}$

Example 3.

Let $C = \begin{bmatrix} 0.65 & 0.45 & 0.1 \\ 0.25 & 0.6 & 0.35 \\ 0.2 & 0.7 & 0.5 \end{bmatrix}$, $D = \begin{bmatrix} 0.8 & 0.55 & 0.3 \\ 0.3 & 0.7 & 0.45 \\ 0.2 & 0.8 & 0.9 \end{bmatrix}$ and

$$E = \begin{bmatrix} 0.6 & 0.4 & 0.3 \\ 0.9 & 0.7 & 0.1 \\ 0.2 & 0.5 & 0.8 \end{bmatrix}$$

Now, $C \otimes E = \begin{bmatrix} 0.6 & 0.4 & 0.1 \\ 0.25 & 0.6 & 0.1 \\ 0.2 & 0.5 & 0.5 \end{bmatrix}$ and $D \otimes E = \begin{bmatrix} 0.65 & 0.45 & 0.1 \\ 0.25 & 0.6 & 0.35 \\ 0.2 & 0.7 & 0.5 \end{bmatrix}$

$\|C \otimes E\|_{CAMN} = 0.5$ and $\|D \otimes E\|_{CAMN} = 0.54$

So, $\|C \otimes E\|_{CAMN} < \|D \otimes E\|_{CAMN}$

Properties 4. *If C and D are two fuzzy matrices, then $\|C \oplus D\|_{CAM} = \|D \oplus C\|_{CAM}$ hold.*

Proof:

Let us consider $C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$ and $D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix}$

Then $(C \oplus D) = \begin{bmatrix} \frac{c_{11}+d_{11}}{2} & \frac{c_{12}+d_{12}}{2} & \cdots & \frac{c_{1n}+d_{1n}}{2} \\ \frac{c_{21}+d_{21}}{2} & \frac{c_{22}+d_{22}}{2a} & \cdots & \frac{c_{2n}+d_{2n}}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{c_{n1}+d_{n1}}{2} & \frac{c_{n2}+d_{n2}}{2} & \cdots & \frac{c_{nn}+d_{nn}}{2} \end{bmatrix}$

Now, $\|C \oplus D\|_{CAM} = \vee_{j=1}^{n}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{c_{ij}+d_{ij}}{2}\right)\right]$

$= \vee_{j=1}^{n}\left[\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{d_{ij}+c_{ij}}{2}\right)\right]$

$= \|D \oplus C\|_{CAM}$

So, $\|C \oplus D\|_{CAM} = \|D \oplus C\|_{CAM}$

Example 4.

$C = \begin{bmatrix} 0.4 & 0.3 & 0.1 \\ 0.6 & 0.5 & 0.2 \\ 0.1 & 0.7 & 0.3 \end{bmatrix}$ and $D = \begin{bmatrix} 0.2 & 0.4 & 0.5 \\ 0.7 & 0.5 & 0.1 \\ 0.2 & 0.8 & 0.3 \end{bmatrix}$

Now'$(C \oplus D) = \begin{bmatrix} 0.3 & 0.35 & 0.3 \\ 0.65 & 0.5 & 0.15 \\ 0.15 & 0.75 & 0.45 \end{bmatrix}$ and $(D \oplus C) = \begin{bmatrix} 0.3 & 0.35 & 0.3 \\ 0.65 & 0.5 & 0.15 \\ 0.15 & 0.75 & 0.45 \end{bmatrix}$

$\| C \oplus D\|_{CAM} = 0.534$ and $\| D \oplus C\|_{CAM} = 0.534$

So, $\| C \oplus D\|_{CAM} = \| D \oplus C\|_{CAM}$

Properties 5. *For any two fuzzy matrix C and $D(\neq C)$, CD and DC may or may not be equal but $\|C \otimes D\|_{CAM} = \|D \otimes C\|_{CAM}$ always holds.*

Proof:

Let us consider $C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$ and $D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix}$

Then $C \otimes D = \begin{bmatrix} \wedge\{c_{11}, d_{11}\} & \wedge\{c_{12}, d_{12}\} & & \wedge\{c_{1n}, d_{1n}\} \\ \wedge\{c_{21}, d_{21}\} & \wedge\{c_{22}, d_{22}\} & & \wedge\{c_{2n}, d_{2n}\} \\ \vdots & \vdots & \ddots & \vdots \\ \wedge\{c_{n1}, d_{n1}\} & \wedge\{c_{n2}, d_{n2}\} & \cdots & \wedge\{c_{nn}, d_{nn}\} \end{bmatrix}$

Now,$\|C \otimes D\|_{CAM} = \vee_{j=1}^{n} \left[\frac{1}{n}\sum_{i=1}^{n}\{\wedge(c_{ij}, d_{ij})\}\right]$

$= \vee_{j=1}^{n} \left[\frac{1}{n}\sum_{i=1}^{n}\{\wedge(d_{ij}, c)\}\right]$

$= \|D \otimes C\|_{CAM}$

So, $\|C \otimes D\|_{CAM} = \|D \otimes C\|_{CAM}$

Example 5.

Let $C = \begin{bmatrix} 0.1 & 0.4 & 0.3 \\ 0.6 & 0.5 & 0.7 \\ 0.3 & 0.2 & 0.1 \end{bmatrix}$ and $D = \begin{bmatrix} 0.3 & 0.4 & 0.2 \\ 0.6 & 0.5 & 0.1 \\ 0.8 & 0.9 & 0.3 \end{bmatrix}$

Now $C \otimes D = \begin{bmatrix} 0.1 & 0.4 & 0.2 \\ 0.6 & 0.5 & 0.1 \\ 0.3 & 0.2 & 0.1 \end{bmatrix}$ and $D \otimes C = \begin{bmatrix} 0.1 & 0.4 & 0.2 \\ 0.6 & 0.5 & 0.1 \\ 0.3 & 0.2 & 0.1 \end{bmatrix}$

$\| C \otimes D\|_{CAM} = 0.364$ and $\| D \otimes C\|_{CAM} = 0.364$

So, $\| C \otimes D\|_{CAM} = \| D \otimes C\|_{CAM}$

Definition 14. *Define a mapping $d': M_n(F) \times M_n(F) \to [0,1]$ as*
$d'(C, D) = min\{\|C\|_{CAM}, \|D\|_{CAM}\}$ for all C, D in $M_n(F)$.

Proposition 1. *The above mapping d' satisfies the following condition for all C, D, E in $M_n(F)$.*

(i) $d'(C, D) \geq 0$ *and* $d'(C, D) = 0$ *iff* $C = 0$ *or* $D = 0$ *or both* $C = D = 0$

(ii) $d'(C, D) = d'(D, C)$

Proof:

(i) $d^{'}(C,D) = min\{\|C\|_{CAM}, \|D\|_{CAM}\} \geq 0$ as $\|C\|_{CAM} \geq 0$ and $\|D\|_{CAM} \geq 0$

Now, $d^{'}(C,D) = min\{\|C\|_{CAM}, \|D\|_{CAM}\} = 0$

$\Rightarrow \|C\|_{CAM} = 0$ or $\|D\|_{CAM} = 0$ or both $\|C\|_{CAM} = \|D\|_{CAM} = 0$

$\Rightarrow$ either $C = 0$ or $D = 0$ or both $C = D = 0$

ii. $d^{'}(C,D) = min\{\|C\|_{CAM}, \|D\|_{CAM}\} = min\{\|D\|_{CAM}, \|C\|_{CAM}\} = d^{'}(D,C)$

Proposition 2. *If* $C, D \in M_n(F)$ *and* $C \leq D$ *then* $d^{'}(C,E) \leq,$ $d^{'}(D,E)$ *for all* $E \in M_n(F)$.

Proof: Since $C \leq D$ so $\|C\|_{CAM} \leq \|D\|_{CAM}$

Now $d^{'}(C,E) = min\{\|C\|_{CAM}, \|E\|_{CAM}\}$

and $d^{'}(D,E) = min\{\|D\|_{CAM}, \|E\|_{CAM}\}$

Case1:

If$\|C\|_{CAM} \leq \|D\|_{CAM} \leq \|E\|_{CAM}$ then $d^{'}(C,E) = \|C\|_{CAM} \leq \|D\|_{CAM} = d^{'}(D,E)$ i.e $d^{'}(C,E) \leq d^{'}(D,E)$

Case2:

If $\|E\|_{CAM} \leq \|C\|_{CAM} \leq \|D\|_{CAM}$ then$d^{'}(C,E) = \|E\|_{CAM} = d^{'}(D,E)$

Case3:

If $\|C\|_{CAM} \leq \|E\|_{CAM} \leq \|D\|_{CAM}$ then $d^{'}(C,E) = \|C\|_{CAM}$ and $d^{'}(D,E) = \|E\|_{CAM}$

So,$d^{'}(C,E) \leq d^{'}(D,E)$

Therefore $d^{'}(C,E) \leq d^{'}(D,E)$ for all$E \in M_n(F)$

Definition 15. *For all* C *in* $M_n(F)$*we define*

$$C_{sup} = \{x \in M_n(F): \|X\|_{CAM} \geq \|C\|_{CAM}\}$$
$$C_{inf} = \{x \in M_n(F): \|X\|_{CAM} \leq \|C\|_{CAM}\}$$
$$C_{equ} = \{x \in M_n(F): \|X\|_{CAM} = \|C\|_{CAM}\}$$

Clearly, $M_n(F) = C_{sup} \cup C_{inf} \cup C_{equ}$

The set C_{sup} is called column-average-max-superior to C.

C_{inf} is called column-average-max-inferior to C.

C_{equ} is called column-average-max-equivent to C.

3.2. Algorithm

Input: A fuzzy matrix of order $n \times n$

Output: A real number in [0,1]

```
Max=0;
for  j=1 to n;
       ans=0;
      for i=1 to n
           ans=ans+a[i][j];
 end for
ans = ans/n;
if max < ans then max = ans;
end for
return max;
```

4. Conclusion

In this paper, we have defined the column-average-max norm and its properties. Using these norms, we can define conditional numbers to check whether a linear system of equations is ill-posed or well-posed. The norm of fuzzy matrices can effectively contribute to solving a fuzzy system of linear equations. In this paper, we define different types of norm on simple fuzzy matrices. Similarly, we can define norm on interval-valued fuzzy matrix, Circulant triangular fuzzy matrix, Fuzzy membership matrix etc. A fuzzy membership matrix is used in medical diagnosis and decision-making. So, if we define the norm on the fuzzy membership matrix, it will make a contribution to medical science.

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