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The Modular Group of the form : $M_{2^n} \times C_2$

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Abstract. In this paper, the classification of finite p-groups is extended to the group of the modular structure $M_{2^n} \times C_2$, and the number of distinct subgroups were computed, making the classification of the given structure possible for the given prime $p = 2$

Keywords: Finite p-groups, nilpotent group, fuzzy subgroups, Dihedral group, inclusion-exclusion principle, maximal subgroups

AMS Mathematics Subject Classification (2010): 20D15, 20E28, 20F18, 20N25, 20K27

1. Introduction

The classifications in finite p-grouos is fast gaining considerable grounds. Of recent, various contributions have been made of which many viable, resourceful, benefita and enriching l publications have been made. In [1], the case of dihedral was considered while [2] dealt with the generalisd quaternion. In [3], the work was extended to the quasidihedral (otherwise refered to as the semidihedral groups), and In [4], efforts are being made to close these aspect up by generalising the conceps up to the entire prime p i.e including both the even as well as the odd cases. Many works on BCI/BCK algebras and related algebras are available in literature, some of them are discussed in [7-15].

The following properties for the fuzzy subgroups of G were known.

1. The level sets of a fuzzy subset of a finite set form a chain .
2. λ is a fuzzy subgroup of G iff its level sets are subgroups of G'
3. The relation $\sim$ is an equivalence relation on fuzzy subgroups of G,where

for fuzzy subgroups μ, ν of G , $\mu \sim \nu$ iff $\forall x, y \in G, (\mu(x) > \mu(y) iff \nu(x) > \nu(y))$.

2. Preliminaries

Suppose that $(G, \cdot, e)$ is a group with identity e. Let $S(G)$ denote the collection of all fuzzy subsets of G. An element $\lambda \in S(G)$ is said to be a fuzzy subgroup of G if the following two conditions are sat.

1. $\lambda(ab) \geq \in \{\lambda(a), \lambda(b)\}, \quad \forall\ a, b \in G;$
2. $\lambda(a^{-1} \geq \lambda(a)$ for any $a \in G$.

And, since $(a^{-1})^{-1} = a$, we have that $\lambda(a^{-1}) = \lambda(a)$, for any $a \in G$.
Also, by this notation and definition, $\lambda(e) = \sup\lambda(G)$. [Marius [5]].

Now, concerning the subgroups, the set $FL(G)$ possessing all fuzzy subgroups of

G forms a lattice under the usual ordering of fuzzy set inclusion. This is called the fuzzy subgroup lattice of G.

In what follows, the method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite p-group G is described. (See [2] and [3]) Suppose that $M_1, M_2, \dots, M_t$ are the maximal subgroups of G. Let $h(G)$ denote the number of chains of subgroups of G which ends in G. The method of computing $h(G)$ is based on the application of the Inclusion-Exclusion Principle. If A is the set of chains in G of type $C_1 \subset C_2 \subset \cdots \subset C_r = G$, and A' represents the set of chains of A' which are contained in $M_r,\ r = 1, \dots, t$.

Then we have:

$$|A| = 1 + |A'| = |\cup_{r=1}^{t} A_r|$$
$$= 1 + \sum_{r=1}^{t} |A_r| - \sum_{1\le r_1 \le r_2 \le t} |A_{r_1} \cap A_{r_2}| + \cdots + (-1)^{t-1} |\cap_{r=1}^{t} A_r|$$

Observe that, for every $1 \le w \le t$ and $1 \le r_1 < r_2 < \cdots < r_w \le t$, the set $\cap_{i=1}^{w} A_{r_i}$ consists of all chains of A' which are included in $\cap_{i=1}^{w} M_{r_i}$. We have that

$$\left|\cap_{i=1}^{w} A_{r_i}\right| = 2h\left(\cap_{i=1}^{w} M_{r_i}\right)^{-1}$$

$$\therefore \quad |A| = 1 + \sum_{r=1}^{t} (2h(M_r) - 1) - \sum_{1\le r_1 < r_2 \le t} (2h(M_{r_1} \cap M_{r_2}) - 1)$$

$$+ \cdots + (-1)^{t-1}(2h(\cap_{r=1}^{t} M_r) - 1)$$

$$= 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1\le r_1<r_2\le t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h(\cap_{r=1}^{t} M_r)\right) + C$$

And

$$C = 1 + \sum_{r=1}^{t} (-1) - \sum_{1 \le r_1 < r_2 \le t} (-1) + \cdots + (-1)^{t-1}(-1)$$
$$= (1-1)^t = 0$$

we have that:

$$h(G) = 2\big(\sum_{r=1}^{t} h(M_r) - \sum_{1\le r_1<r_2\le t} h(M_{r_1} \cap M_{r_2})$$

$$+ \cdots + (-1)^{t-1} h(\cap_{r=1}^{t} M_r)\big) (3)$$

In [6], (3) was used to obtain the explicit formulas of $h(D_{2n})$ for some positive integers n. (Also see [1]) and [2]

Theorem 2. [5] The number of distinct fuzzy subgroups of a finite p-group of order p^n which have a cyclic maximal subgroup is:

1. $h(\mathbb{Z}_{p^n}) = 2^n$
2. $h(D_{2^n}) = 2^{2n-1}$
3. $h(\varphi_{2^n}) = 2^{2n-2}$
4. $h(S_{2^n}) = 3.\,2^{2n-3}$
5. $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}[2 + (n-1)p]$

3. The nilpotent 2-group of the form $M_{2^n} \times C_2$

Recall that

$$M_{2^n} = \langle x, y | x^{2^{n-1}} = y^2 = 1,\quad y^{-1}xy = x^{1+2^{n-2}} \rangle$$

The Modular Group of the form : $M_{2^n} \times C_2$

Since $n = 3$ is not defined for this particular structure, we begin by taking the case, $n = 4$. We have:

$$M_{2^4} = \langle x, y | x^8 = y^2 = 1,\ y^{-1}xy = x^5 \rangle$$
$$= \begin{Bmatrix} 1, x, x^2, x^3, x^4, x^5, x^6, x^7, y, xy, \\ x^2y, x^3y, x^4y, x^5y, x^6y, x^7y \end{Bmatrix}$$

From here, $M_{2^4} \times C_2 = \begin{Bmatrix} 1, x, x^2, x^3, x^4, x^5, x^6, x^7, y, xy, \\ x^2y, x^3y, x^4y, x^5y, x^6y, x^7y \end{Bmatrix} \times \{1, a\}$

$$= \begin{Bmatrix} (1,1), (1,a), (x,1), (x,a), (x^2,1), (x^2,a), (x^3,1), (x^3,a), (x^4,1), (x^4,a), \\ (x^5,1), (x^5,a), (x^6,1), (x^6,a), (x,^7,1), (x^7,a), (y,1), (y,a), (xy,1), (xy,a), \\ (x^2y,1), (x^2y,a), (x^3y,1), (x^3y,a), (x^4y,1), (x^4y,a), (x^5y,1), (x^5y,a), \\ (x^6y,1), (x^6y,a), (x^7y,1), (x^7y,a) \end{Bmatrix}$$

We have the maximal subgroups for $M_{2^4} \times C_2$ as follows:

$$M_1 = \begin{Bmatrix} (1,1), (1,a), (x,1), (x,a), (x^2,1), (x^2,a), (x^3,1), (x^3,a), (x^4,1), (x^4,a), \\ (x^5,1), (x^5,a), (x^6,1), (x^6,a), (x,^7,1), (x^7,a), \end{Bmatrix}$$

$$M_2 = \begin{Bmatrix} (1,1), (x,1), (x^2,1), (x^3,1), (x^4,1), (x^5,1), (x^6,1), (x^7,1), \\ (y,1), (xy,1), (x^2y,1), (x^3y,1), (x^4y,1), (x^5y,1), (x^6y,1), (x^7y,1) \end{Bmatrix}$$

$$M_3 = \begin{Bmatrix} (1,1), (x,1), (x^2,1), (x^3,1), (x^4,1), (x^5,1), (x^6,1), (x^7,1), \\ (y,a), (xy,a), (x^2y,a), (x^3y,a), (x^4y,a), (x^5y,a), (x^6y,a), (x^7y,a) \end{Bmatrix}$$

$$M_4 = \begin{Bmatrix} (1,1), (x,a), (x^2,1), (x^3,a), (x^4,1), (x^5,a), (x^6,1), (x^7,a), \\ (y,a), (xy,1), (x^2y,a), (x^3y,1), (x^4y,a), (x^5y,1), (x^6y,a), (x^7y,1) \end{Bmatrix}$$

$$M_5 = \begin{Bmatrix} (1,1), (x,a), (x^2,1), (x^3,a), (x^4,1), (x^5,a), (x^6,1), (x^7,a), \\ (y,1), (xy,a), (x^2y,1), (x^3y,a), (x^4y,1), (x^5y,a), (x^6y,1), (x^7y,a) \end{Bmatrix}$$

$M_6 =$

$$\left\{\begin{matrix}(1,1),(1,a),(x^2,1),(x^2,a),(x^4,1),(x^4,a),(x^6,1),(x^6,a),\\ (y,1),(y,a),(x^2y,1),(x^2y,a),(x^4y,1),(x^4y,a),(x^6y,1),(x^6y,a)\end{matrix}\right\}$$

$$M_7 = \left\{\begin{matrix}(1,1),(1,a),(x^2,1),(x^2,a),(x^4,1),(x^4,a),(x^6,1),(x^6,a),\\ (xy,1),(xy,a),(x^3y,1),(x^3y,a),(x^5y,1),(x^5y,a),(x^7y,1),(x^7y,a)\end{matrix}\right\}$$

From here, $M_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^3}$,

$$M_2 \cong M_3 \cong M_4 \cong M_5 \cong M_7 \cong M_{2^4}$$

And, $M_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^2}$.

By making the application of equation (c), we have,

$$\begin{aligned}\frac{1}{2}h(M_{2^4} \times C_2) &= [2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^3}) + 4h(M_{2^4}) + h(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^2})] \\ &\quad -3[h(\mathbb{Z}_{2^3}) + h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) + 2h(\mathbb{Z}_{2^3}) + 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2})] \\ &\quad +[28h(\mathbb{Z}_{2^2}) + 2h(\mathbb{Z}_{2^3}) + h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) + 2h((\mathbb{Z}_{2^3}) \\ &\quad +2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2})] - 35h(\mathbb{Z}_{2^2}) + 21h(\mathbb{Z}_{2^2}) - 7h(\mathbb{Z}_{2^2}) + h(\mathbb{Z}_{2^2}) \\ &= 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^3}) + 4h(M_{2^4}) + h(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^2}) \\ &\quad -8h(\mathbb{Z}_{2^3}) - 6h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) + 8h(\mathbb{Z}_{2^2}) \\ &= 4(64) + 2(64) + 304 - 8(8) - 6(24) + 8(4) \\ &= 320 + 304 + 32 - 144 \\ &= 512.\end{aligned}$$

$\therefore \;\; h(M_{2^4} \times C_2) = 2 \times 512 = 1024.$

The number of fuzzy subgroups of the nilpotent group : $M_{2^5} \times C_2$

$$\begin{aligned}\frac{1}{2}h(M_{2^5} \times C_2) &= [2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^4}) + 4h(M_{2^5}) + h(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^3})] \\ &\quad -3[h(\mathbb{Z}_{2^4}) + h(\mathbb{Z}_2 \times \mathbb{Z}_{2^3}) + 2h(\mathbb{Z}_{2^4}) + 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^3})] \\ &\quad +[28h(\mathbb{Z}_{2^3}) + 2h(\mathbb{Z}_{2^4}) + h(\mathbb{Z}_2 \times \mathbb{Z}_{2^3}) + 2h(\mathbb{Z}_{2^4}) + 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^3})] \\ &\quad -35h(\mathbb{Z}_{2^3}) + 21h(\mathbb{Z}_{2^3}) - 7h(\mathbb{Z}_{2^3}) + h(\mathbb{Z}_{2^3}) \\ &= 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^4}) + 4h(M_{2^5}) + h(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^3}) \\ &\quad -8h(\mathbb{Z}_{2^4}) - 6h(\mathbb{Z}_2 \times \mathbb{Z}_{2^3}) + 8h(\mathbb{Z}_{2^3}) \\ &= h(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^3}) + 2(160) + 4(160) - 8(16) - 6(64) \times 8(8) \\ &= h(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^3}) + 512 \\ &= 1024 + 512 = 1536 \\ &\therefore \;\; h(M_{2^5} \times C_2) = 2 \times 5136 \\ &= 3072.\end{aligned}$$

4. The number of fuzzy subgroups of the nilpotent group: $M_{2^n} \times C_2$

In general, for any positive integer $n > 4$

$$h(M_{2^n} \times C_2) = 2\begin{bmatrix}2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}) + 4h(M_{2^n}) + h(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}})) \\ -8h(\mathbb{Z}_{2^{n-1}}) - 6h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}) + 8h(\mathbb{Z}_{2^{n-2}})\end{bmatrix}$$

$$= 2h(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}) + 2^{n+1}(3n+1)$$

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$$= 2^n(3n^2 - n - 6) + 2^{n+1}(3n + 1)$$

$$h(M_{2^n} \times C_2) = 2^n(3n^2 + 5n - 4), \quad n > 4$$
$$= 2^n\left[3\left(n + \frac{5}{6}\right)^2 - \frac{73}{12}\right], \quad p = 2$$

5. Conclusion

Finally, the classification for the nilpotent 2-groups of the specified modular structure given by : $M_{2^n} \times C_2$ is thus hereby clearly made with the number of distinct fuzzy subgroups explicitly computed for the prime $p = 2$ and every non zero integer $n \geq 3$

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