Intern. J. Fuzzy Mathematical Archive
Vol. 18, No. 2, 2020, 61-64
ISSN: 2320 –3242 (P), 2320 –3250 (online)
Published on 17 September 2020
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/ijfma.v18n1a02216

International Journal of Fuzzy Mathematical Archive

Counting Distinct Fuzzy Subgroups of Symmetric Group S_5 by a New Equivalence Relation

***M.E.Ogiugo*[1], *M. EniOluwafe*[2*] and *S.A.Adebisi*[3]**

[1]Department of Mathematics, School of Science, Yaba College of Technology, Nigeria
E-mail: ekpenogiugo@gmail.com
[2]Department of Mathematics, University of Ibadan, Nigeria
E-mail: michael.enioluwafe@gmail.com
[3]Department of Mathematics, University of Lagos, Nigeria
E-mail: adesinasunday77@yahoo.com
*Corresponding author.

Received 15 August 2020; accepted 15 September 2020

Abstract. One of the significant aspects of fuzzy group theory is a classification of the fuzzy subgroups of finite groups under a suitable equivalence relation. In this paper, we determine the number of distinct fuzzy subgroups of finite symmetric group S_5 by the new equivalence relation introduced by Tărnăuceanu. In this case, the corresponding equivalence classes of fuzzy subgroups of a group G are closely connected to the automorphism group and the chains of subgroups of G.

Keywords: Equivalence Relation, Chains of Subgroups, Automorphism group

AMS Mathematics Subject Classification (2010): 20B30, 20B25, 20E15

1. Introduction

Classifying the fuzzy subgroups of a finite group has undergone rapid development, in recent years. The fuzzy subgroups of a group G can be classified up to some natural equivalence relations on the set consisting of all fuzzy subsets of G. Many papers have treated the classification of the fuzzy subgroups for particular cases of finite groups with respect to suitable equivalence relation. Murali and Makamba studied equivalence classes of fuzzy subgroups of a given group under a suitable equivalence relation. Ogiugo and EniOluwafe have also the number of fuzzy subgroups of S_5 under the natural equivalence relation. We have computed the number of fuzzy subgroups of S_4 by this new equivalence relation [1]. This new equivalence relation generalizes the natural equivalence relation defined on the lattice of fuzzy subgroups. The paper, we present some preliminary definitions and necessary results on fuzzy subgroups and recall the new equivalence relation. Other types of fuzzy algebraic structures are available in [7-9].

2. Preliminaries

A **fuzzy subset** of a set X is a function $\mu: X \rightarrow [0,1]$. Let G be a group with a multiplicative binary operation and identity e, and let $\mu: G \rightarrow [0,1]$ be a fuzzy subset of G. Then μ is said to be a **fuzzy subgroup** of G if

$$(1)\ \mu(xy) \geq min\{\mu(x),\mu(y)\},$$
$$(2)\ \mu(x^{-1}) \geq \mu(x) \text{ for all } x,y \in G$$

The following elementary facts about fuzzy subgroups follow easily from the axioms: $\mu(x) = \mu(x^{-1})$ and $\mu(x) \leq \mu(e)$, for all $x \in G$. Also, μ satisfies conditions (1) and (2) of Definition if and only if $\min\{\mu(x),\mu(y)\} \leq \mu(xy^{-1})$, for all x, y ∈ G.

The set $\{\mu(x)|x \in G\}$ is called **the image of μ** and is denoted by $\mu(G)$, For each $\alpha \in \mu(G)$, the set μ_α: $= \{x \in G|\mu(x) \geq \alpha\}$ is called **a level subset of μ** .It follows that μ is a fuzzy subgroup of G if and only if its level subsets are subgroups of G . These subsets are useful in the characterization of fuzzy subgroups.(see [6])

Suppose G is a finite group; then the number of subgroups of G is finite whereas the number of level subgroups of a fuzzy subgroup A appears to be infinite. But, since every level subgroup is indeed a subgroup of G, not all these level subgroups are distinct. In this paper, we count the classified distinct fuzzy subgroups of S_5 .

Without any equivalence relation on fuzzy subgroups of group G, the number of fuzzy subgroups is infinite, even for the trivial group {e}. So we define an equivalence relation on the set of all fuzzy subgroups of a given group. We say that μ is equivalent to v, written as $\mu \sim v$, if we have

$$\mu(x) > \mu(y) \Leftrightarrow v(x) > v(y), \text{ for all } x,y \in G$$

and

$$\mu(x) = 0 \Leftrightarrow v(x) = 0, \text{ for all } x \in G.$$

Note that the condition $\mu(x) = 0$ holds if and only if $v(x) = 0$ simply says that the supports of μ and v are equal and two fuzzy subgroups μ, v of G are said to be distinct if $\mu \neq v$.

3. Methodology

Let G be a finite group. Then it is well-defined the following action of $Aut(G)$ on FL(G)

$$\rho : FL(G) \times Aut(G) \to FL(G)$$
$$\rho(\mu, f) = \mu \circ f, \text{ for all } (\mu,f) \in FL(G) \times Aut(G)$$

Let us denote by $\approx_\rho$ the equivalence relation on FL(G) induced by ρ, namely $\mu \approx_\rho v$ if and only if there exists $f \in Aut(G)$ such that $v = \mu \circ f$

In this paper, it is called a new equivalence relation,[5]. There are other different versions of fuzzy equivalence relations in Literatures.

The problem of classifying the fuzzy subgroup of finite group G by using a new equivalence relation $\approx$ on the lattice of all fuzzy subgroups of G, its definition has a consistent group theoretical foundation, by involving the knowledge of the automorphism group associated to G. The approach is motivated by the realization that in a theoretical study of fuzzy groups, fuzzy subgroups are distinguished by their level subgroups and not by their images in [0,1] . Various enumeration techniques that are used in the counting of distinct fuzzy subgroups of a finite group. These counting techniques are derived from the interpretation of the definition of fuzzy equivalence relations used. The equivalence classes are called the orbits of the action, the orbit of a chain $C \in \bar{C}$ is $\{f(C) \mid f \in Aut(G)\}$, while the set of all chains in $\bar{C}$ that are fixed by an automorphism f of G is

$$Fix_C(f) = \{C \in \bar{C} \mid f(C) = C\}$$

Now, the number N is obtained by applying the Burnside's lemma:

$$N = \frac{1}{|Aut(G)|} \sum_{f \in Aut(G)} | Fix_{\overline{C}}(f) | \qquad (\#)$$

The Burnside's lemma plays an important role in the explicit formula to compute the number of distinct fuzzy subgroups(N) of a finite group G with respect to a certain equivalence relation on the lattice of fuzzy subgroups, induced by an action of the automorphism group $Aut(G)$ associated to G (see[5]).

4. Main results

Let $C \in Fix_C(f_\sigma)$, where $C : H_1 \subset H_2 \subset \cdots \subset H_m = S_n$. Then $f_\sigma(C) = C$, that is $f_\sigma(H_i) = H_i$, for all $i = 1 \leq i \leq m$.

Then every automorphism of S_n is of the form f_σ with $\sigma \in S_n$. In fact, for n 6= 2,6 the symmetric group is complete group. It is well-known that S_5 has 120 elements. The arrangement of elements according to their cycle structure reveals the conjugacy classes in S_5.

Proposition 4.1. Two elements of S_n are conjugate if and only if they have the same cycle structure. It is well-known that cycle types determine the conjugacy classes in S_n.

Proposition 4.2. Two elements of S_n are conjugate if and only if they have the same cycle structure.

It is well-known from classical group theory, an automorphism of a group G permutes the conjugacy classes in G, and the inner automorphisms preserve each conjugacy class.

Theorem 4.1. [1] Let α,β of S_n be conjugate, then the set of the number of chains of subgroups of S_n for $n \neq 2,6$ fixed by the automorphism f is equal.

The set of chains of subgroups of S_5 fixed by the automorphism f can be represented by the cycle structure of S_5. The inner automorphisms of S_5 preserve each conjugacy class in S_5.

Theorem 4.2. [3] The number of chains of subgroups of S_5 that ends in S_5 is 3784.

We shall compute the number of all distinct fuzzy subgroups of S_5 using the counting technique (#) from the definition of the new equivalence relation induced by the action of automorphism groups.

$$f\tau : S_5 \rightarrow S_5, \quad f_\tau(\sigma) = \tau^{-1}\sigma\,\tau$$

Then every automorphism of S_5 is of the form f_σ with $\sigma \in S_5$

$Aut(S_5) = \{f_1, f_2, f_3, f_4, f_5, ,f_6, f_7,\cdots,f_{118}, f_{119}, f_{120}\}$
$f_1 = f_e = 3784$

$f_2 = f_3 = f_4 = f_5 = \cdots f_{11} = 10 \times 740 = 7400$
$f_{12} = f_{13} = f_{14} = f_{15} = \cdots = f_{26} = 15 \times 492 = 7380$
$f_{27} = f_{28} = f_{29} = f_{30} = \cdots = f_{46} = 20 \times 110 = 2200$
$f_{47} = f_{48} = f_{49} = f_{50} = \cdots = f_{76} = 30 \times 154 = 4620$
$f_{77} = f_{78} = f_{79} = f_{80} = \cdots = f_{96} = 20 \times 46 = 920$
$f_{97} = f_{98} = f_{99} = f_{100} = \cdots = f_{120} = 24 \times 24 = 576$

The number of the set of all chains in S_5 that are fixed by an automorphism f of S_5 is 26880

$$h(S_5) = \frac{1}{|Aut(S_5)|} \sum_{f \in Aut(S_5)} | Fix_{\overline{C}}(f) |$$

$$h(S_5) = \frac{26880}{120} = 224$$

Theorem 4.3. The number of all distinct fuzzy subgroups with respect to ≈ of the symmetric group S_5 is 224

5. Conclusion

In this paper, we have determined the number of distinct fuzzy subgroups for symmetric group S_5 by the new equivalence relation which generalizes the natural equivalence relation used in [2].

Acknolegement. The authors are greatful to the revierews for their valuable suggestions for improvement of the paper.

REFERENCES

1. M.Ogiugo and M.EniOluwafe, A new equivalence relation for the classification of fuzzy subgroups of symmetric group S_4, *Transactions of the Nigerian Association of Mathematical Physics,* 6 (2018) 168– 172.
2. M.Ogiugo and M.EniOluwafe, On the number of fuzzy subgroups of symmetric group S_5 , (2017) 10pp, arXiv:1706.08419v1
3. M.Ogiugo A.Sehgal and M.EniOluwafe. The number of chains of subgroups for certain finite symmetric groups, submitted.
4. V.Murali and B.B.Makamba, On an equivalence of fuzzy subgroups-I, *Fuzzy Sets and Systems,* 123 (2016) 259-264.
5. M.Tarnauceanu, A new Equivalence relation to classify the fuzzy subgroups of finite groups, *Fuzzy Sets and Systems,* 289 (2016) 113-121.
6. A.C.Volf, Counting fuzzy subgroups and chains of subgroups. *Fuzzy Systems & Artificial Intelligence*, 10 (2004) 191–200.
7. T.Senapati, C.Jana, M.Bhowmik and M.Pal, L-fuzzy G-subalgebras of G-algebras, *Journal of the Egyptian Mathematical Society*, 23 (2) (2015) 219-223.
8. T.Senapati, M.Bhowmik and M.Pal, Fuzzy dot subalgebras and fuzzy dot ideals of B-algebras, *Journal of Uncertain Systems,* 8 (1) (2014) 22-30.
9. S.Dogra and M.Pal, Picture fuzzy subring of a crisp ring, *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.* (2020). https://doi.org/10.1007/s40010-020-00668-y