Intern. J. Fuzzy Mathematical Archive
Vol. 18, No. 1, 2020, 29-40
ISSN: 2320 –3242 (P), 2320 –3250 (online)
Published on 22 April 2020
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/ijfma.v18n1a04144

International Journal of
Fuzzy Mathematical Archive

Fixed Point Theorem in Fuzzy Metric Space Satisfying a Class of Implicit Relations

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Received 8 January 2020; accepted 23 March 2020

Abstract. In this Paper we establish a common fixed point theorem for weakly compatible maps on complete ε −chainable fuzzy metric space satisfying a class of implicit relations. The established results generalize, extend, unify and fuzzify several existing fixed point results in metric space and fuzzy metric space.

Keywords: Fuzzy Metric Space, ε −Chainable Fuzzy Metric Space, Semi- Compatible Maps, Weakly Compatibility, Implicit Relations, Common Fixed Point.

AMS Mathematics Subject Classification (2010): 03E72

1. Introduction

The theory of fuzzy sets was first introduced by Zadeh [18] in 1965. Since then, due to the wide applicability of this notion in various fields, many authors have expansively developed the theory of fuzzy sets and its applications. In this context Deng [4], Erceg, [5], Fang and Gao [6], Kaleva and Seikkala [10], Kramosil and Michalek [11] have introduced the concept of fuzzy metric spaces in different ways. In 1994 George and Veeramani [7] modified this concept of fuzzy metric space and obtain a Hausdroff topology for this kind of fuzzy metric spaces. It appears that the study of Kramosil and Michalek [11] of fuzzy metric spaces paves the way for developing the smooth machinery in the field of fixed point theory for the study of contractive maps. Sessa [13] initiated the tradition of improving commutativity conditions in fixed point theorems by introducing the notion of weakly commuting maps in metric spaces. Jungck [9] soon enlarged this concept to compatible maps. The notion of compatible mappings in fuzzy metric spaces was introduced by Cho [2]. Vasuki [17] introduced the concept of R – weakly commuting map and proved a fixed point theorem for fuzzy metric space using this concept. In 2000, Singh and Chauhan [14] introduced the concept of compatibility in fuzzy metric spaces. Singh and Jain [15] studied the notions of semi compatibility and weak compatibility of maps in fuzzy metric spaces. Popa [12] established some results on fixed point theorems for weakly compatible non continuous mappings using implicit relations. Imdad and Khan [8] extended the work of Popa [12]. Cho et al. [3] introduced the concept of ε −chainable fuzzy metric space and obtained common fixed point theorems for four weakly compatible mappings of ε −chainable fuzzy metric spaces. Singh and Bhadauriya [16] proved a fixed point theorem in ε −chainable fuzzy metric spaces using implicit relations.

In this Paper we establish a common fixed point theorem for weakly compatible maps on complete ε −chainable fuzzy metric space satisfying a class of implicit relations. The established results generalize, extend, unify and fuzzify several existing fixed point results in metric space and fuzzy metric space.

2. Preliminaries

Definition 1. A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous $t-norm$ if $*$ satisfies following conditions:

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Examples of continuous $t-norm$ are:

$$a * b = ab$$
$$a * b = \min(a, b)$$

Definition 2. A 3 – tuple $(X, \mathcal{M}, *)$ is called a $\mathcal{M}-$ fuzzy metric space if X is an arbitrary (non - empty) set, $*$ is a continuous $t-norm$, and $\mathcal{M}$ is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(i) $\mathcal{M}(x, y, t) > 0$,
(ii) $\mathcal{M}(x, y, t) = 1$ *if and only if* $x = y$,
(iii) $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$,
(iv) $\mathcal{M}(x, y, t) * \mathcal{M}(y, z, s) \leq \mathcal{M}(x, z, t + s)$,
(v) $\mathcal{M}(x, y, .) : (0, \infty) \to [0,1]$ is continuous.

Example 1. Let (X, d) be a metric space. Define $a * b = \min(a, b)$, and

$$\mathcal{M}(x, y, t) = \frac{t}{t + d(x, y)}$$

induced by the metric d is often called the standard fuzzy metric.

Definition 3. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}-$ fuzzy metric space. For $t > 0$, the open ball $B_{\mathcal{M}}(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by $B_{\mathcal{M}}(x, r, t) = \{y \in X : \mathcal{M}(x, y, t) > 1 - r\}$.

A subset A of X is called an open set if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $B_{\mathcal{M}}(x, r, t) \subseteq A$.

Definition 4. A sequence $\{x_n\}$ in a fuzzy metric space $(X, \mathcal{M}, *)$ is said to be a Cauchy sequence if for each $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{M}\ (x_n, x_m, t) > 1 - \varepsilon \text{ for all } n, m \geq n_0.$$

A sequence $\{x_n\}$ in a fuzzy metric space $(X, \mathcal{M}, *)$ is said to be convergent to $x \in X$ if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\lim_{n \to \infty} \mathcal{M}\ (x_n, x, t) > 1 - \varepsilon$ for all $t > 0\ \&\ n \geq n_0$. George and Veeramani [7] proved that a sequence $\{x_n\}$ in a fuzzy metric space $(X, \mathcal{M}, *)$ converges to a point $x \in X$ if and only if $\mathcal{M}\ (x_n, x, t) = 1$, for all $t > 0$.

A fuzzy metric space $(X, \mathcal{M}, *)$ is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 5. Two self mappings$A$ and B of a fuzzy metric space $(X, \mathcal{M}, *)$ are said to be compatible if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to \infty} \mathcal{M}\ (ABx_n, BAx_n, t) = 1, \text{ for all } t > 0,$$

whenever $\lim_{n \to \infty} A\, x_n = \lim_{n \to \infty} B\, x_n = x$ for some $x \in X$.

Definition 6. Two self mappings$A$ and B of a fuzzy metric space $(X, \mathcal{M}, *)$ are said to be weakly compatible if $ABx = BAx$ whenever $Ax = Bx$ for some $x \in X$. If the self mappings A and B of a fuzzy metric space $(X, \mathcal{M}, *)$ are compatible, then they are weakly compatible, but the converse is not necessarily true.

Example 2. Let $X = [0, 4]$ and $a * b = \min\{a, b\}$. Let $\mathcal{M}$ be the standard fuzzy metric induced by d, where $d\,(x, y) = |x - y|$ for $x, y \in X$. Define two self mappings A and B of the fuzzy metric space $(X, \mathcal{M}, *)$ by:

$$Ax = \begin{cases} 4 - x, & 0 \leq x \leq 2 \\ 4, & 2 \leq x \leq 4 \end{cases}$$

$$Bx = \begin{cases} x, & 0 \leq x \leq 2 \\ 4, & 2 \leq x \leq 4 \end{cases}$$

Let $\{x_n\} = \{1 - (1/n)\}$. Then it can be easily proved that the self mappings A and B are weakly compatible but they are not compatible.

Definition 7. A finite sequence $x = x_0, x_1, \cdots, x_n = y$ in a fuzzy metric space $(X, \mathcal{M}, *)$ is called $\varepsilon -$ chain from x to y if there exists $\varepsilon > 0$ such that $\mathcal{M}\ (x_i, x_{i-1}, t) > 1 - \varepsilon$ for all $t > 0$ and $i = 1, 2, \cdots, n$.

A fuzzy metric space $(X, \mathcal{M}, *)$ is called $\varepsilon -$ chainable if there exists an $\varepsilon -$ chain from x to y, for any $x, y \in X$.

Lemma 1. $\mathcal{M}\ (x, y, .)$ is non-decreasing for all $x, y \in X$.
Proof: Suppose $\mathcal{M}\ (x, y, t) > M\ (x, y, s)$ for some $0 < t < s$.
Then $\mathcal{M}\ (x, y, t) * \mathcal{M}\ (y, y, s - t) \leq \mathcal{M}\ (x, y, s) < M\ (x, y, t)$.
Since $\mathcal{M}\ (y, y, s - t) = 1$, therefore, $\mathcal{M}\ (x, y, t) \leq \mathcal{M}\ (x, y, s) < M\ (x, y, t)$, which is a contradiction. Thus, $\mathcal{M}\ (x, y, .)$ is non-decreasing for all $x, y \in X$.

Lemma 2. If for all $x, y \in X$, $t > 0$ and $0 < k < 1$,

$\mathcal{M}(x,y,kt) \geq \mathcal{M}(x,y,t)$, then $x = y$.
Proof: Suppose that there exists $0 < k < 1$ such that
$\mathcal{M}(x,y,kt) \geq \mathcal{M}(x,y,t)$ for all $x, y \in X$ and $t > 0$.
Then, $\mathcal{M}(x,y,t) \geq \mathcal{M}(x,y,t/k)$,
and $\mathcal{M}(x,y,t) \geq \mathcal{M}(x,y,t/k^n)$
for positive integer n. Taking limit as $n \to \infty$, $\mathcal{M}(x,y,t) \geq 1$ and hence $x = y$.

Definition 8. [1] ***A Class of Implicit Relations***
Let ψ be the set of all real and continuous functions $F: \mathbb{R}_+^6 \to \mathbb{R}$, non decreasing in the first argument satisfying the following conditions:

(a) For $u, v \geq 0, F(u,v,1,v,1,u) \geq 0$ implies that $u \geq v$.
(b) $F(u,1,1,1,1,u) \geq 0$ or $F(u,1,u,u,u,1) \geq 0$ or $F(u,u,u,1,u,1) \geq 0$ implies that $u \geq 1$.

Example 3. Let $F: \mathbb{R}_+^6 \to \mathbb{R}$ be defined by

$$F(t_1,t_2,t_3,t_4,t_5,t_6) = 20t_1 - 18t_2 + t_3 - 14t_4 - t_5 + 12t_6$$

Then we see that

$$F(u,v,1,v,1,u) \geq 0 \Rightarrow 32(u - v) \geq 0 \Rightarrow u \geq v$$
$$F(u,1,1,1,1,u) \geq 0 \Rightarrow 32(u-1) \geq 0 \Rightarrow u \geq 1$$
$$F(u,1,u,u,u,1) \geq 0 \Rightarrow 6(u-1) \geq 0 \Rightarrow u \geq 1$$
$$F(u,u,u,1,u,1) \geq 0 \Rightarrow 2(u-1) \geq 0 \Rightarrow u \geq 1$$

Therefore, $F \in \psi$.

Ali et al. [1] proved the following fixed point theorem for weakly compatible maps on complete ε −chainable fuzzy metric spaces satisfying an implicit relation

Theorem 1. Let $(X, \mathcal{M}, *)$ be a complete $\varepsilon-$chainable fuzzy metric space and let A, B, S *and* T be the self mappings of X, satisfying the following conditions:

(1) $AX \subset TX$and B$X \subset SX$;
(2) The pair (A,T) and (B,S) are weakly compatible;
(3) $T(X)$ or $S(X)$ is complete;
(4) There exists $k \in (0,1)$ such that

$$F\begin{pmatrix} \mathcal{M}(Ax,By,kt), \mathcal{M}(Sx,Ty,t), \mathcal{M}(Ax,Ty,t), \mathcal{M}(Sx,Ax,t), \\ \dfrac{a\,\mathcal{M}(Ax,By,t) + b\,\mathcal{M}(Ax,Ty,t)}{a\mathcal{M}(By,Ty,t) + b}, \mathcal{M}(By,Ty,t) \end{pmatrix} \geq 0$$

for every $x, y \in X$ and $t > 0$, where $a, b \geq 0$ with a & b cannot be simultaneously 0.
Then, A, B, S and T have a unique common fixed point in X.
We are now extending Ali et al. [1] work as the following results.

3. The main results

Definition 9. ***A class of implicit relations***
Let ψ be the set of all real and continuous functions $F: \mathbb{R}_+^7 \to \mathbb{R}$, non decreasing in the first argument satisfying the following conditions:

(a) For $u, v \geq 0, F(u,1,v,1,v,u,u) \geq 0$ implies that $u \geq v$.
(b) $F(u,1,1,1,1,u,u) \geq 0$ or $F(u,u,1,u,u,1,1) \geq 0$ or $F(u,u,u,u,1,1,1) \geq 0$ implies that $u \geq 1$.

Example 4. Let $F: \mathbb{R}_+^7 \to \mathbb{R}$ be defined by

$$F(t_1,t_2,t_3,t_4,t_5,t_6,t_7) = 22t_1 + t_2 - 18t_3 - t_4 - 14\,t_5 - 2t_6 + 12t_7$$

Then we see that

$$F(u,1,v,1,v,u,u) \geq 0 \implies 32(u - v) \geq 0 \implies u \geq v$$
$$F(u,1,1,1,1,u,u) \geq 0 \implies 32(u-1) \geq 0 \implies u \geq 1$$
$$F(u,u,1,u,u,1,1) \geq 0 \implies 8(u-1) \geq 0 \implies u \geq 1$$
$$F(u,u,u,u,1,1,1) \geq 0 \implies 4(u-1) \geq 0 \implies u \geq 1$$

Therefore, $F \in \psi$.

Theorem 2. Let $(X,\mathcal{M},*)$ be a complete $\varepsilon -$ chainable fuzzy metric space and let $A,B,S\ and\ T$ be the self mappings of X, satisfying the following conditions:

(1) $AX \subset TX$ and B$X \subset SX$;
(2) The pair (A,T) and (B,S) are weakly compatible;
(3) $T\,(X)$ or $S\,(X)$ is complete;
(4) There exists $k \in (0,1)$ such that

$$F\left(\begin{array}{c}\mathcal{M}\,(Ax,By,kt),\mathcal{M}\,(Ax,Ty,t),\\ \mathcal{M}\,(Sx,Ty,t),\dfrac{a\,\mathcal{M}\,(Ax,By,t)+\ b\,\mathcal{M}\,(Ax,Ty,t)}{a\,\mathcal{M}\,(By,Ty,t)+b},\\ \mathcal{M}\,(Sx,Ax,t),\dfrac{c\,\mathcal{M}\,(Ax,By,t)+\ d\,\mathcal{M}\,(By,Ty,t)}{c\,\mathcal{M}\,(Ax,Ty,t)+d},\\ \mathcal{M}\,(By,Ty,t)\end{array}\right) \geq 0$$

for all $x,y \in X$ and $t > 0$, where $k \in (0,1)$ and $a,b,c,d \geq 0$ with $a\,\&\,b$and $c\,\&\,d$ cannot be simultaneously 0. Then A,B,S and T have a unique common fixed point in X.

Proof: Let x_0 be any arbitrary point. As $AX \subset TX$, B$X \subset SX$ so, there exists x_1 , $x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Inductively we construct the sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n}$$
$$y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$$

For $n = 0,1,2,\cdots$.Now using condition (4) with $x = x_{2n}, y = x_{2n+1}$, we get

$$F\left(\begin{array}{c}\mathcal{M}\,(Ax_{2n},Bx_{2n+1},kt),\mathcal{M}\,(Ax_{2n},Tx_{2n+1},t),\mathcal{M}\,(Sx_{2n},Tx_{2n+1},t),\\ \mathcal{M}\,(Sx_{2n},Tx_{2n+1},t),\dfrac{a\,\mathcal{M}\,(Ax_{2n},Bx_{2n+1},t)+\ b\,\mathcal{M}\,(Ax_{2n},Tx_{2n+1},t)}{a\,\mathcal{M}\,(Bx_{2n+1},Tx_{2n+1},t)+b},\\ \mathcal{M}\,(Sx_{2n},Ax_{2n},t),\dfrac{c\,\mathcal{M}\,(Ax_{2n},Bx_{2n+1},t)+\ d\,\mathcal{M}\,(Bx_{2n+1},Tx_{2n+1},t)}{c\,\mathcal{M}\,(Ax_{2n},Tx_{2n+1},t)+d},\\ \mathcal{M}\,(Bx_{2n+1},Tx_{2n+1},t)\end{array}\right) \geq 0$$

that is $F\begin{pmatrix} \mathcal{M}\ (y_{2n}, y_{2n+1}, kt), \mathcal{M}\ (y_{2n}, y_{2n}, t), \\ \mathcal{M}\ (y_{2n-1}, y_{2n}, t), \frac{a\,\mathcal{M}\ (y_{2n}, y_{2n+1}, t) + b\,\mathcal{M}\ (y_{2n}, y_{2n}, t)}{a\,\mathcal{M}\ (y_{2n+1}, y_{2n}, t) + b}, \\ \mathcal{M}\ (y_{2n-1}, y_{2n}, t), \frac{c\,\mathcal{M}\ (y_{2n}, y_{2n+1}, t) + d\,\mathcal{M}\ (y_{2n+1}, y_{2n}, t)}{c\,\mathcal{M}\ (y_{2n}, y_{2n}, t) + d}, \\ \mathcal{M}\ (y_{2n+1}, y_{2n}, t) \end{pmatrix} \geq 0$

that is $F\begin{pmatrix} \mathcal{M}\ (y_{2n}, y_{2n+1}, kt), \mathcal{M}\ (y_{2n}, y_{2n}, t), \\ \mathcal{M}\ (y_{2n-1}, y_{2n}, t), \frac{a\,\mathcal{M}\ (y_{2n}, y_{2n+1}, t) + b}{a\,\mathcal{M}\ (y_{2n+1}, y_{2n}, t) + b}, \\ \mathcal{M}\ (y_{2n-1}, y_{2n}, t), \frac{c\,\mathcal{M}\ (y_{2n}, y_{2n+1}, t) + d\,\mathcal{M}\ (y_{2n+1}, y_{2n}, t)}{c + d}, \\ \mathcal{M}\ (y_{2n+1}, y_{2n}, t) \end{pmatrix} \geq 0$

that is $F\begin{pmatrix} \mathcal{M}\ (y_{2n}, y_{2n+1}, kt), 1, \\ \mathcal{M}\ (y_{2n-1}, y_{2n}, t), 1, \\ \mathcal{M}\ (y_{2n-1}, y_{2n}, t), \mathcal{M}\ (y_{2n}, y_{2n+1}, t), \\ \mathcal{M}\ (y_{2n}, y_{2n+1}, t) \end{pmatrix} \geq 0$

Thus we have $\mathcal{M}\ (y_{2n}, y_{2n+1}, kt)\ \geq\ \mathcal{M}\ (y_{2n}, y_{2n-1}, t) * \mathcal{M}\ (y_{2n+1}, y_{2n}, t)$
that is $\mathcal{M}\ (y_{2n}, y_{2n+1}, kt)\ \geq\ \mathcal{M}\ (y_{2n}, y_{2n-1}, t)$
Similarly, we have $\mathcal{M}\ (y_{2n+1}, y_{2n+2}, kt)\ \geq\ \mathcal{M}\ (y_{2n+1}, y_{2n}, t)$
Therefore, for all even and odd n, we have $\mathcal{M}\ (y_n, y_{n+1}, kt)\ \geq\ \mathcal{M}\ (y_n, y_{n-1}, t)$
Thus, for any n and t, we have

$$\mathcal{M}\ (y_n, y_{n+1}, kt)\ \geq\ \mathcal{M}\ (y_n, y_{n-1}, t)$$

$$\mathcal{M}\ (y_{n+1}, y_n, t)\ \geq\ \mathcal{M}\ \left(y_n, y_{n-1}, \frac{t}{k}\right)\ \geq\ \mathcal{M}\ \left(y_{n-1}, y_{n-2}, \frac{t}{k^2}\right)\ \geq\ \cdots$$

$\geq\ \mathcal{M}\ \left(y_1, y_0, \frac{t}{k^n}\right)\ \to 1\ as\ n\ \to\ \infty$. So, the result holds for $m = 1$. As our induction hypothesis suppose that the result holds for $m = p$.

Now,
$\mathcal{M}\ \left(y_n, y_{n-p+1}, t\right)\ \geq\ \mathcal{M}\ \left(y_n, y_{n-p}, \frac{t}{2}\right) * \mathcal{M}\ \left(y_{n+1}, y_{n-p+1}, \frac{t}{2}\right)\ \to 1 * 1 = 1$. Thus, the result holds for $m = p + 1$. Hence $\{y_n\}$ is a Cauchy sequence in X, which is complete. Therefore, $\{y_n\}$ converges to z, that is $y_n\ \to z$ for some $z\ \in X$. Then it follows that the sequences $\{Ax_{2n}\}$,$\{Sx_{2n}\}$, $\{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ also converge to z. Now, we prove that $\{x_n\}$ is a Cauchy sequence in X. Since X is $\varepsilon -$ chainable, there exists an $\varepsilon -$ chain from x_n to x_{n+1}, that is, there exists a finite sequence $x_n =\ y_1, y_2, \cdots, y_n =\ x_{n+1}$ such that
$\mathcal{M}\ (y_m, y_{m-1}, t) > (1 -\ \varepsilon)$ for all $t > 0$ and $i = 1, 2, \cdots, m$. Thus, we have

$$\mathcal{M}\ (x_n, x_{n+1}, t)\ \geq\ \mathcal{M}\ \left(y_1, y_2, \frac{t}{l}\right) * \mathcal{M}\ \left(y_2, y_3, \frac{t}{l}\right) * \cdots * \mathcal{M}\ \left(y_{m-1}, y_m, \frac{t}{l}\right)$$

$$\geq\ (1 -\ \varepsilon) *\ (1 -\ \varepsilon) *\ \cdots *\ (1 -\ \varepsilon) \geq\ (1 -\ \varepsilon)$$

For $m > n$,

$$\mathcal{M}(x_n, x_m, t) \geq \mathcal{M}\left(x_n, x_{n+1}, \frac{t}{m-n}\right) * \mathcal{M}\left(x_{n+1}, x_{n+2}, \frac{t}{m-n}\right) * \cdots * \mathcal{M}\left(x_{m-1}, x_m, \frac{t}{m-n}\right)$$

$$\geq (1-\varepsilon) * (1-\varepsilon) * \cdots * (1-\varepsilon) \geq (1-\varepsilon)$$

Hence $\{x_n\}$ is a Cauchy sequence in X, which is complete. Therefore, $\{x_n\}$ converges to z, that is $x_n \to z$ for some $z \in X$. Then it follows that its sub sequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ also converge to z.

Case I: When $T(X)$ is complete.

If we take $z \in T(X)$, then there exists $u \in X$, such that $z = Tu$.

Step I: Put $x = x_{2n}$ and $y = u$ in condition (4), we obtain,

$$F\left(\begin{array}{c} \mathcal{M}(Ax_{2n}, Bu, kt), \mathcal{M}(Ax_{2n}, Tu, t), \\ \mathcal{M}(Sx_{2n}, Tu, t), \dfrac{a\,\mathcal{M}(Ax_{2n}, Bu, t) + b\,\mathcal{M}(Ax_{2n}, Tu, t)}{a\,\mathcal{M}(Bu, Tu, t) + b}, \\ \mathcal{M}(Sx_{2n}, Ax_{2n}, t), \dfrac{c\,\mathcal{M}(Ax_{2n}, Bu, t) + d\,\mathcal{M}(Bu, Tu, t)}{c\,\mathcal{M}(Ax_{2n}, Tu, t) + d}, \\ \mathcal{M}(Bu, Tu, t) \end{array}\right) \geq 0$$

Taking limit $n \to \infty$ in the above, we get

$$F\left(\begin{array}{c} \mathcal{M}(z, Bu, kt), \mathcal{M}(z, Tu, t), \\ \mathcal{M}(z, Tu, t), \dfrac{a\,\mathcal{M}(z, Bu, t) + b\,\mathcal{M}(z, Tu, t)}{a\,\mathcal{M}(Bu, Tu, t) + b}, \\ \mathcal{M}(z, z, t), \dfrac{c\,\mathcal{M}(z, Bu, t) + d\,\mathcal{M}(Bu, Tu, t)}{c\,\mathcal{M}(z, Tu, t) + d}, \\ \mathcal{M}(Bu, Tu, t) \end{array}\right) \geq 0$$

$$\Rightarrow F\left(\begin{array}{c} \mathcal{M}(z, Bu, kt), \mathcal{M}(z, z, t), \\ \mathcal{M}(z, z, t), \dfrac{a\,\mathcal{M}(z, Bu, t) + b\,\mathcal{M}(z, z, t)}{a\,\mathcal{M}(Bu, z, t) + b}, \\ \mathcal{M}(z, z, t), \dfrac{c\,\mathcal{M}(z, Bu, t) + d\,\mathcal{M}(Bu, z, t)}{c\,\mathcal{M}(z, z, t) + d}, \\ \mathcal{M}(Bu, z, t) \end{array}\right) \geq 0$$

$$\Rightarrow F\left(\begin{array}{c} \mathcal{M}(z, Bu, kt), 1, \\ 1, \dfrac{a\,\mathcal{M}(z, Bu, t) + b}{a\,\mathcal{M}(Bu, z, t) + b}, \\ 1, \dfrac{c\,\mathcal{M}(z, Bu, t) + d\,\mathcal{M}(Bu, z, t)}{c + d}, \\ \mathcal{M}(Bu, z, t) \end{array}\right) \geq 0$$

$$\Rightarrow F\begin{pmatrix}\mathcal{M}\ (z,Bu,kt),1,\\ 1,1,\\ 1,\mathcal{M}\ (Bu,z,t),\\ \mathcal{M}\ (Bu,z,t)\end{pmatrix}\geq 0$$

Since F is non-decreasing in the first argument, therefore,

$$F\left(\mathcal{M}\ (z,Bu,kt),1,1,1,1,\mathcal{M}\ (Bu,z,t),\mathcal{M}\ (Bu,z,t)\right)\geq 0$$

So that $\mathcal{M}\ (z,Bu,t)\ \geq 1$. Hence $z=Bu$. Since, $B\subset S$, therefore, $z=Bu\in S$ and so $z=Bu=Su$. Therefore,$z=Bu=Su=Tu$. Now, (B,S) is weakly compatible, so $BSu=SBu$ and so $Bz=Sz$.

Step II: Put $x=x_{2n}$ and $y=z$ in condition (4), we obtain,

$$F\begin{pmatrix}\mathcal{M}\ (Ax_{2n},Bz,kt),\mathcal{M}\ (Ax_{2n},Tz,t),\\ \mathcal{M}\ (Sx_{2n},Tz,t),\dfrac{a\,\mathcal{M}\ (Ax_{2n},Bz,t)+\ b\,\mathcal{M}\ (Ax_{2n},Tz,t)}{a\,\mathcal{M}\ (Bz,Tz,t)+b},\\ \mathcal{M}\ (Sx_{2n},Ax_{2n},t),\dfrac{c\,\mathcal{M}\ (Ax_{2n},Bz,t)+\ d\,\mathcal{M}\ (Bz,Tz,t)}{c\,\mathcal{M}\ (Ax_{2n},Tz,t)+d},\\ \mathcal{M}\ (Bz,Tz,t)\end{pmatrix}\geq 0$$

Taking limit $n\ \rightarrow\ \infty$ in the above, we get

$$F\begin{pmatrix}\mathcal{M}\ (z,Bz,kt),\mathcal{M}\ (z,Tz,t),\\ \mathcal{M}\ (z,Tz,t),\dfrac{a\,\mathcal{M}\ (z,Bz,t)+\ b\,\mathcal{M}\ (z,Tz,t)}{a\,\mathcal{M}\ (Bz,Tz,t)+b},\\ \mathcal{M}\ (z,z,t),\dfrac{c\,\mathcal{M}\ (z,Bz,t)+\ d\,\mathcal{M}\ (Bz,Tz,t)}{c\,\mathcal{M}\ (z,Tz,t)+d},\\ \mathcal{M}\ (Bz,Tz,t)\end{pmatrix}\geq 0$$

Since F is non-decreasing in the first argument, $z=Tz$ and $z\ \in T\ (X)$, therefore,

$$F\begin{pmatrix}\mathcal{M}\ (z,Bz,kt),\mathcal{M}\ (z,z,t),\\ \mathcal{M}\ (z,z,t),\dfrac{a\,\mathcal{M}\ (z,Bz,t)+\ b\,\mathcal{M}\ (z,z,t)}{a\,\mathcal{M}\ (Bz,z,t)+b},\\ \mathcal{M}\ (z,z,t),\dfrac{c\,\mathcal{M}\ (z,Bz,t)+\ d\,\mathcal{M}\ (Bz,z,t)}{c\,\mathcal{M}\ (z,z,t)+d},\\ \mathcal{M}\ (Bz,z,t)\end{pmatrix}\geq 0$$

$$\Rightarrow\ F\begin{pmatrix}\mathcal{M}\ (z,Bz,kt),1,\\ 1,\dfrac{a\,\mathcal{M}\ (z,Bz,t)+\ b}{a\,\mathcal{M}\ (Bz,z,t)+b},\\ 1,\dfrac{c\,\mathcal{M}\ (z,Bz,t)+\ d\,\mathcal{M}\ (Bz,z,t)}{c\ +d},\\ \mathcal{M}\ (Bz,z,t)\end{pmatrix}\geq 0$$

$$\Rightarrow\ F\begin{pmatrix}\mathcal{M}\ (z,Bz,kt),1,\\ 1,1,\\ 1,\mathcal{M}\ (Bz,z,t),\\ \mathcal{M}\ (Bz,z,t)\end{pmatrix}\geq 0$$

$\Rightarrow F\big(\mathcal{M}\ (z, Bz, kt), 1,1,1,1, \mathcal{M}(Bz, z, t), \mathcal{M}\ (Bz, z, t)\big) \geq 0$

So that $\mathcal{M}\ (z, Bz, t)\ \geq 1$. Hence $z = Bz$ and so $z = Bz = Tz$.

Step III: As $B\ (X) \subset S\ (X)$, there exists $v \in X$ such that $z = Bz = Sv$.

Put $x = v$ and $y = z$ in condition (4), we obtain,

$$F\begin{pmatrix} \mathcal{M}\ (Av, Bz, kt), \mathcal{M}\ (Av, Tz, t), \\ \mathcal{M}\ (Sv, Tz, t), \dfrac{a\ \mathcal{M}\ (Av, Bz, t) +\ b\ \mathcal{M}\ (Av, Tz, t)}{a\ \mathcal{M}\ (Bz, Tz, t) + b}, \\ \mathcal{M}\ (Sv, Av, t), \dfrac{c\ \mathcal{M}\ (Av, Bz, t) +\ d\ \mathcal{M}\ (Bz, Tz, t)}{c\ \mathcal{M}\ (Av, Tz, t) + d}, \\ \mathcal{M}\ (Bz, Tz, t) \end{pmatrix} \geq 0$$

that is $$F\begin{pmatrix} \mathcal{M}\ (Av, z, kt), \mathcal{M}\ (Av, z, t), \\ \mathcal{M}\ (z, z, t), \frac{a\ \mathcal{M}\ (Av,z,t) + b\ \mathcal{M}\ (Av,z,t)}{a\ \mathcal{M}\ (z,z,t) + b}, \\ \mathcal{M}\ (z, Av, t), \frac{c\ \mathcal{M}\ (Av,z,t) + d\ \mathcal{M}\ (z,z,t)}{c\ \mathcal{M}\ (Av,z,t) + d}, \\ \mathcal{M}\ (z, z, t) \end{pmatrix} \geq 0$$

$$\Rightarrow F\begin{pmatrix} \mathcal{M}\ (Av, z, kt), \mathcal{M}\ (Av, z, t), \\ 1, \dfrac{a\ \mathcal{M}\ (Av, z, t) +\ b\ \mathcal{M}\ (Av, z, t)}{a\ + b}, \\ \mathcal{M}\ (z, Av, t), \dfrac{c\ \mathcal{M}\ (Av, z, t) +\ d}{c\ \mathcal{M}\ (Av, z, t) + d}, \\ 1 \end{pmatrix} \geq 0$$

$$\Rightarrow F\begin{pmatrix} \mathcal{M}\ (Av, z, kt), \mathcal{M}\ (Av, z, t), \\ 1, \mathcal{M}\ (Av, z, t), \\ \mathcal{M}\ (Av, z, t), 1, \\ 1 \end{pmatrix} \geq 0$$

$\Rightarrow\ F\ \{\mathcal{M}\ (Av, z, kt), \mathcal{M}\ (Av, z, t), 1, \mathcal{M}\ (Av, z, t), \mathcal{M}\ (Av, z, t), 1, 1\} \geq 0$

Since F is non-decreasing in the first argument, we have

$\Rightarrow\ F\ \{\mathcal{M}\ (Av, z, t), \mathcal{M}\ (Av, z, t), 1, \mathcal{M}\ (Av, z, t), \mathcal{M}\ (Av, z, t), 1, 1\} \geq 0$

that is $\mathcal{M}\ (Av, z, t)\ \geq 1$. So, $z = Av$. Now, since $A\ \subset T$, therefore $z = Av \in T$ and so $z = Av = Tv$. As (A, T) is weakly compatible, therefore, $ATv = TAv$ so that $Az = Tz$. Combining all the results, we have $Az = Tz = Bz = Sz = z$.

Step IV: Put $x = Sz$ and $y = z$ in condition (4), we obtain,

$$F\begin{pmatrix} \mathcal{M}\ (ASz, Bz, kt), \mathcal{M}\ (ASz, Tz, t), \\ \mathcal{M}\ (SSz, Tz, t), \dfrac{a\ \mathcal{M}\ (ASz, Bz, t) +\ b\ \mathcal{M}\ (ASz, Tz, t)}{a\ \mathcal{M}\ (Bz, Tz, t) + b}, \\ \mathcal{M}\ (SSz, ASz, t), \dfrac{c\ \mathcal{M}\ (ASz, Bz, t) +\ d\ \mathcal{M}\ (Bz, Tz, t)}{c\ \mathcal{M}\ (ASz, Tz, t) + d}, \\ \mathcal{M}\ (Bz, Tz, t) \end{pmatrix} \geq 0$$

that is $F\begin{pmatrix} \mathcal{M}(Az,Bz,kt), \mathcal{M}(Az,Tz,t), \\ \mathcal{M}(Sz,Tz,t), \frac{a\,\mathcal{M}(Az,Bz,t)+b\,\mathcal{M}(Az,Tz,t)}{a\,\mathcal{M}(Bz,Tz,t)+b}, \\ \mathcal{M}(Sz,Az,t), \frac{c\,\mathcal{M}(Az,Bz,t)+d\,\mathcal{M}(Bz,Tz,t)}{c\,\mathcal{M}(Az,Tz,t)+d}, \\ \mathcal{M}(Bz,Tz,t) \end{pmatrix} \geq 0$

$$F\begin{pmatrix} \mathcal{M}(Az,z,kt), \mathcal{M}(Az,z,t), \\ \mathcal{M}(Sz,z,t), \frac{a\,\mathcal{M}(Az,z,t)+\ b\,\mathcal{M}(Az,z,t)}{a\,\mathcal{M}(z,z,t)+b}, \\ \mathcal{M}(Sz,Az,t), \frac{c\,\mathcal{M}(Az,z,t)+\ d\,\mathcal{M}(z,z,t)}{c\,\mathcal{M}(Az,z,t)+d}, \\ \mathcal{M}(z,z,t) \end{pmatrix} \geq 0$$

$$\Rightarrow F\begin{pmatrix} \mathcal{M}(Az,z,kt), \mathcal{M}(Az,z,t), \\ \mathcal{M}(z,z,t), \frac{a\,\mathcal{M}(Az,z,t)+\ b\,\mathcal{M}(Az,z,t)}{a\,\mathcal{M}(z,z,t)+b}, \\ \mathcal{M}(z,Az,t), \frac{c\,\mathcal{M}(Az,z,t)+\ d\,\mathcal{M}(z,z,t)}{c\,\mathcal{M}(Az,z,t)+d}, \\ \mathcal{M}(z,z,t) \end{pmatrix} \geq 0$$

$$\Rightarrow F\begin{pmatrix} \mathcal{M}(Az,z,kt), \mathcal{M}(Az,z,t), \\ 1, \frac{a\,\mathcal{M}(Az,z,t)+\ b\,\mathcal{M}(Az,z,t)}{a\ +b}, \\ \mathcal{M}(z,Az,t), \frac{c\,\mathcal{M}(Az,z,t)+\ d}{c\,\mathcal{M}(Az,z,t)+d}, \\ 1 \end{pmatrix} \geq 0$$

$$\Rightarrow F\begin{pmatrix} \mathcal{M}(Az,z,kt), \mathcal{M}(Az,z,t), \\ 1, \mathcal{M}(Az,z,t), \\ \mathcal{M}(z,Az,t), 1, \\ 1 \end{pmatrix} \geq 0$$

$\Rightarrow\ F\{\mathcal{M}(Az,z,kt), \mathcal{M}(Az,z,t), 1, \mathcal{M}(Az,z,t), \mathcal{M}(z,Az,t), 1, 1\} \geq 0$

Since F is non-decreasing in the first argument, we have

$\Rightarrow\ F\{\mathcal{M}(Az,z,t), \mathcal{M}(Az,z,t), 1, \mathcal{M}(Az,z,t), \mathcal{M}(z,Az,t), 1, 1\} \geq 0$

that is $\mathcal{M}(Az,z,t) \geq 1$. Therefore, $Az = z$. Similarly, we can show that $Bz = z$, $Tz = z$ and $Sz = z$. Hence $z = Az = Tz = Bz = Sz$.

Case II: When $S(X)$ is complete.

If we take $z \in S(X)$, then there exists $w \in X$, such that $z = Tw$. Proceeding exactly as in case I, we can show that $Az = z$, $Bz = z$, $Tz = z$ and $Sz = z$. Hence, $z = Az = Tz = Bz = Sz$. Thus z is the common fixed point of A, B, S and T.

Uniqueness: Let w and z be two common fixed points of the mappings A, B, S and T. Put $x = z$ and $y = w$ in condition (4), we obtain,

$$F\begin{pmatrix} \mathcal{M}(Az, Bw, kt), \mathcal{M}(Az, Tw, t), \\ \mathcal{M}(Sz, Tw, t), \dfrac{a\,\mathcal{M}(Az, Bw, t) + b\,\mathcal{M}(Az, Tw, t)}{a\,\mathcal{M}(Bw, Tw, t) + b}, \\ \mathcal{M}(Sz, Az, t), \dfrac{c\,\mathcal{M}(Az, Bw, t) + d\,\mathcal{M}(Bw, Tw, t)}{c\,\mathcal{M}(Az, Tw, t) + d}, \\ \mathcal{M}(Bw, Tw, t) \end{pmatrix} \geq 0$$

that is $F\begin{pmatrix} \mathcal{M}(z, w, kt), \mathcal{M}(z, w, t), \\ \mathcal{M}(z, w, t), \frac{a\,\mathcal{M}(z,w,t) + b\,\mathcal{M}(z,w,t)}{a\,\mathcal{M}(w,w,t) + b}, \\ \mathcal{M}(z, z, t), \frac{c\,\mathcal{M}(z,w,t) + d\,\mathcal{M}(w,w,t)}{c\,\mathcal{M}(z,w,t) + d}, \\ \mathcal{M}(w, w, t) \end{pmatrix} \geq 0$

$$\Rightarrow F\begin{pmatrix} \mathcal{M}(z, w, kt), \mathcal{M}(z, w, t), \\ \mathcal{M}(z, w, t), \dfrac{a\,\mathcal{M}(z, w, t) + b\,\mathcal{M}(z, w, t)}{a + b}, \\ 1, \dfrac{c\,\mathcal{M}(z, w, t) + d}{c\,\mathcal{M}(z, w, t) + d}, \\ 1 \end{pmatrix} \geq 0$$

$$\Rightarrow F\begin{pmatrix} \mathcal{M}(z, w, kt), \mathcal{M}(z, w, t), \\ \mathcal{M}(z, w, t), \mathcal{M}(z, w, t), \\ 1, 1, \\ 1 \end{pmatrix} \geq 0$$

Since F is non-decreasing in the first argument, we have

$$\Rightarrow F(\mathcal{M}(z, w, kt), \mathcal{M}(z, w, kt), \mathcal{M}(z, w, kt), \mathcal{M}(z, w, kt), 1,1,1) \geq 0$$

that is $\mathcal{M}(z, w, t) \geq 1$.

Thus $z = w$. Hence z is the unique common fixed point of A, B, S and T.

4. Conclusion

In this chapter we have extended the work of Ali et al. [1] and established a common fixed point theorem for four weakly compatible maps on complete ε-chainable fuzzy metric space satisfying a class of implicit relations. The established results can be extended for more number of maps satisfying a more complex class of implicit relations.

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