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Irreducible and Strongly Irreducible Bi-ideals of a Near-Algebra

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Abstract. The aim of this paper is to study the notion of irreducible and strongly irreducible bi-ideal of a near-algebra and fuzzy concepts of a near-algebra. Also, some characterizations are presented.

Keywords: Irreducible bi-ideal, strongly irreducible bi-ideal, fuzzy set, near algebra.

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1. Introduction

Generalization of ideals is necessary for further development in the study of algebraic systems. The notion of quasi-ideals in near-rings was pioneered by Yakabe [16]. Lajos and Szasz [6] introduced the concept of quasi-ideals in associate near-rings. The quasi ideals are generalization of left and right ideals, whereas bi-ideals are generalization of quasi ideals. We refer to Chelvam and Ganesan [15] for bi-ideals. Abbassi and Rizivi [1] studied the prime ideals in near rings. The ideals in fuzzy sub near rings was introduced by Zaid [2]. A near-algebra is a near ring which admits a field as a right operator domain. Pilz [9] wrote a monograph on near-rings. Brown [3], Srinivas [12], Irish [4], Swamy [7,8] have studied certain properties of near-algebras. The notion fuzzy set was introduced by Zadeh [17]. The concept of prime ideals and fuzzy prime ideals is initiated in [5,10,11,13,14]. In this paper, we introduce the concept of irreducible, strongly irreducible bi-ideals of a near algebra, and obtain certain characterizations. Also, we studied the fuzzy concepts of irreducible, strongly irreducible bi-ideals for a near algebra.

2. Preliminaries

Definition 2.1. A (right) near-algebra Y over a field X is a linear space Y over X on which multiplication is defined such that

(i) Y forms a semi-group under multiplication

(ii) Multiplication is right distributive over addition, that is $(a+b)c = ac + bc$ for every a,b,c∈ Y and

(iii) $\lambda(ab) = (\lambda a)b$ for every $a, b \in Y$ and $\lambda \in X$.

Definition 2.2. A subset D of a near algebra Y over a field X is said to be a sub near algebra of Y if
(i) D is a linear subspace of Y
(ii) (D, .) is a semi-group.

Definition 2.3. A nonempty subset I of a near algebra Y is called a near algebra ideal of Y if
(i) I is a linear subspace of the linear space Y
(ii) ix ∈I, for every x ∈ Y, i∈ I
(iii) y (x +i) – y x ∈ I for every x, y ∈ Y, i∈ I.
If I satisfies conditions (i) and (ii), then I is called a right ideal of Y.
If I satisfies conditions (i) and (iii), then I is called a left ideal of Y.

Definition 2.4. Let Y be a near-algebra over a field X. Then the set $Y_0 = \{a \in Y/a0 = 0\}$is called the zero-symmetric part of Y, $Y_c = \{a \in Y/a0 = a\}$ is called the constant part of Y. Y is called zero-symmetric near-algebra if $Y = Y_0$, Y is called constant near-algebra if $Y = Y_c$.

Definition 2.5. Let X be a non empty set. A fuzzy subset η of X is a function η: X → [0,1]. The number η(x) is the degree of membership of x ∈ X.

Definition 2.6. Let μ be a fuzzy subset of N and $t \in [0,1]$. Then the set
$\mu_t = \{x \in N \ / \ \mu(x) \geq t\}$ is called a level set (or level sub set) of μ. It is denoted by μ_t.

Definition 2.7. A fuzzy subset F of a field X is called a fuzzy field of X, if it satisfies the following four conditions for every x, y ∈ Y:
(i) F(x+y) ≥ min(F(x), F(y)),
(ii) F(- x) ≥ F(x),
(iii) F(x + y) ≥ min(F(x), F(y)),
(iv) F(x^{-1}) ≥ F(x) for every non-zero x∈X.
A fuzzy field F of X is denoted by (F,X).

Definition 2.8. Let Y be a near algebra over a field X and (F,X) be a fuzzy field. A fuzzy subset μ of Y is called a fuzzy near algebra of Y over a fuzzy field (F,X) if
(i) μ(x+y) ≥ min(μ(x), μ(y)),
(ii) μ(λx) ≥ min(F(λ), μ(x)),
(iii) μ(xy) ≥ min(μ(x), μ(y)),
(iv) F(1) ≥ μ(x) for every x, y ∈ Y, λ ∈ X.
A fuzzy near algebra μ of Y is denoted by (μ, Y).

Definition 2.9. Let (μ,Y) be a fuzzy near algebra over a fuzzy field (F,X). Then μ is a fuzzy ideal of Y if
(i) μ(xy) ≥ μ(x) for every x, y ∈ Y
(ii) μ(y(x + i) – y x) ≥ μ(i) for every x, y, i ∈ Y.
A is a fuzzy right ideal of Y if μ(x y) ≥ μ(x) for every x, y ∈ Y.

A is a fuzzy left ideal of Y if μ(y(x + i) – y x) ≥ μ(i) for every x, y, i ∈ Y.

Definition 2.10. A fuzzy subset μ of a near-algebra Y over a fuzzy field (F,X) is called a fuzzy bi-ideal of Y, if $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$, $\mu(\lambda x) \geq \min\{F(\lambda), \mu(x)\}$and $(\mu o \chi o \mu) \cap ((\mu o \chi) * \mu) \subseteq \mu$ for every x, y ∈ Y, λ ∈ X.

Throughout this article Y denotes a right near-algebra and X denotes a field.

3. Irreducible and strongly irreducible bi-ideals

In this section we introduce irreducible bi-ideal of a near-algebra Y and strongly irreducible bi-ideal of a near-algebra Y. Further provided some properties along with proofs.

Definition 3.1. A bi-ideal I of a near algebra Y is called an irreducible bi-ideal of Y, if for any bi-ideals I_1, I_2 of $Y, I_1 \cap I_2 = I$ implies $I_1 = I$ or $I_2 = I$.

Definition 3.2. A bi-ideal I of a near algebra Y is called strongly irreducible bi-ideal of Y, if for any bi-ideals I_1, I_2 of Y, $I_1 \cap I_2 \subseteq I$ implies either $I_1 \subseteq I$ or $I_2 \subseteq I$.

Definition 3.3. A bi-ideal I of a near algebra Y is called strongly irreducible semiprime bi-ideal of Y, if for any bi-ideals I_1, I_2 of Y, $(I_1 \cap I_2)^2 \subseteq I$ implies either $I_1 \subseteq I$ or $I_2 \subseteq I$.

Definition 3.4. A bi-ideal D of a near algebra Y is called a prime bi-ideal of Y, if $P_1P_2 \subseteq D$ implies $P_1 \subseteq D$ or $P_2 \subseteq D$ for any bi-ideals P_1, P_2 of Y.

Definition 3.5. A bi-ideal P of a near algebra Y is a semiprime bi-ideal of Y, if for any bi-ideal $P_1$of Y, ${P_1}^2 \subseteq P$ implies $P_1 \subseteq P$.

Definition 3.6. A bi-ideal P of a near algebra Y is called a strongly prime bi-ideal of Y, if $P_1P_2 \cap P_2P_1 \subseteq P$ implies $P_1 \subseteq P$ or $P_2 \subseteq P$ for any bi-ideals of P_1, P_2 of Y.

Theorem 3.7. Every strongly irreducible semiprime bi-ideal of near algebra Y is a strongly prime bi-ideal of Y.

Proof. Let I be strongly irreducible semiprime bi-ideal of near algebra Y. Let $I_1$and I_2 be two bi-ideals of Y such that $I_1I_2 \cap I_2I_1 \subseteq I$. Then we have to show that either $I_1 \subseteq I$ or $I_2 \subseteq I$. Since $I_1 \cap I_2 \subseteq I_1$ and $I_1 \cap I_2 \subseteq I_2$,
Implies $(I_1 \cap I_2)^2 \subseteq I_1I_2$ and $(I_1 \cap I_2)^2 \subseteq I_2I_1$. Thus $(I_1 \cap I_2)^2 \subseteq I_1I_2 \cap I_2I_1 \subseteq I$. It is clear that $(I_1 \cap I_2)$ is also a bi-ideal of Y. So $(I_1 \cap I_2)^2 \subseteq I$. This implies $(I_1 \cap I_2) \subseteq I$, because I is a semiprime bi-ideal of Y. Also $I_1 \subseteq I$ or $I_2 \subseteq I$, because I is strongly irreducible bi-ideal of Y. Hence, I is a strongly prime bi-ideal of Y.

Proposition 3.8. Let D be a bi-ideal of a near-algebra Y and x ∈ Y such that x ∉ D. Then there exists an irreducible bi-ideal I of Y such that $D \subseteq I$ and x ∉ I.

Theorem 3.9. For a near-algebra Y, the following statements are equivalent:

(a) $D^2 = D$ for every bi-ideal D of Y.
(b) $D_1D_2 \cap D_2D_1 = D_1 \cap D_2$ for any bi-ideas D_1, D_2 of Y.
(c) Each bi-ideal of Y is semiprime.
(d) Each proper bi-ideal of Y is the intersection of irreducible semiprime bi-ideals of Y which contain it.

Proof. (a) ⇒ (b): Let D_1 and D_2 be any two bi-ideals of Y. Then $D_1 \cap D_2$ is also a bi-ideal of Y. By hypothesis we have $(D_1 \cap D_2) = (D_1 \cap D_2)^2 = (D_1 \cap D_2)(D_1 \cap D_2) \subseteq D_1D_2$. Similarly,
$D_1 \cap D_2 \subseteq D_2D_1$. So $D_1 \cap D_2 \subseteq D_1D_2 \cap D_2D_1$. Since D_1D_2, D_2D_1, $D_1D_2 \cap D_2D_1$ bi-ideals of Y, then $D_1D_2 \cap D_2D_1 = (D_1D_2 \cap D_2D_1)(D_1D_2 \cap D_2D_1) \subseteq D_1D_2 . D_2D_1 = D_1{D_2}^2D_1 \subseteq D_1D_2D_1 \subseteq D_1YD_1 \subseteq D_1$ Similarly $D_1D_2 \cap D_2D_1 \subseteq D_2$. Thus $D_1D_2 \cap D_2D_1 \subseteq D_1 \cap D_2$. Hence $D_1D_2 \cap D_2D_1 = D_1 \cap D_2$.

(b) ⇒ (c): Let D be a bi-ideal of Y such that ${D_1}^2 \subseteq D$ for any bi-ideal D_1 of Y. Then $D_1 = D_1 \cap D_1 = D_1D_1 \cap D_1D_1 = {D_1}^2 \subseteq D$. This shows that B is a semiprime bi-ideal of Y. Hence every bi-ideal of Y is a semiprime bi-ideal of Y.

(c) ⇒ (d): Suppose that each bi-ideal of Y is semiprime. Let D be a proper bi-ideal of Y. Then there exists an irreducible bi-ideal of Y containing D. If $\bigcap_\alpha I_\alpha$ be the intersection of all irreducible bi-ideals of Y containing D, then $D \subseteq \bigcap_\alpha I_\alpha$, as $D \subseteq I_\alpha$ for all α. If this inclusion is proper, then there exists $x \in \bigcap_\alpha I_\alpha$ such that $x \notin D$. This implies $a \in I_\alpha$ for all α. As $x \notin D$, then there exist an irreducible bi-ideal I of Y such that $D \subseteq I$ and $x \notin I$. This is a contradiction to the fact that $x \in I_\alpha$ for all α. So $D = \bigcap_\alpha I_\alpha$. Thus each proper bi-ideal of Y is the intersection of irreducible semiprime bi-ideals of Y which contains it.

(d) ⇒ (a): Let each proper bi-ideal of Y is the intersection of irreducible semiprime bi-ideals of Y which contain it. Let D be a bi-ideal of Y, then $DD = D^2$. If $D^2 = Y$ is an improper bi-ideal, then $Y \subseteq D^2$, implies $D \subseteq Y \subseteq D^2$. Also $D^2 \subseteq D$. So $D^2 = D$ for each bi-ideal of D of Y. Now if D^2 is a proper bi-ideal of Y that is $D^2 \neq Y$, then $D^2 = \bigcap_\alpha \{D_\alpha : D_\alpha$ is an irreducible semiprime bi-ideal of Y such that $D^2 \subseteq D_\alpha$ for all $\alpha\}$ implies $D \subseteq D_\alpha$ for all α, because each D_α is a semiprime bi-ideal of Y. Thus $D \subseteq \cap D_\alpha = D^2$. Also $D^2 \subseteq D$. Hence $D^2 = D$ for each bi-ideal D of Y.

Theorem 3.10. If every bi-ideal of near algebra Y is an idempotent then the following assertions are equivalent: (a) D is strongly irreducible (b) D is strongly prime.

Proof. (a) ⇒(b): Let D_1 and D_2 be any two bi-ideals of Y such that $D_1D_2 \cap D_2D_1 \subseteq D$. Each bi-ideal of Y is idempotent, then we have $D_1 \cap D_2 = D_1D_2 \cap D_2D_1 \subseteq D$, this implies $D_1 \cap D_2 \subseteq D$. But D is strongly irreducible bi-ideal of Y. Thus $D_1 \subseteq D$ or $D_2 \subseteq D$. Hence D is strongly prime bi-ideal of Y.

(b) ⇒(a): Let D_1 and D_2 be any two bi-ideals of Y such that $D_1 \cap D_2 \subseteq D$. Then $D_1D_2 \cap D_2D_1 \subseteq D_1 \cap D_2 \subseteq D$, this implies $D_1D_2 \cap D_2D_1 \subseteq D$. But D is strongly prime bi-ideals of Y. Thus we have $D_1 \subseteq D$ or $D_2 \subseteq D$. Hence D is strongly irreducible bi-ideal of Y.

4. Irreducible and Strongly Irreducible Fuzzy Bi-ideal

Definition 4.1. A fuzzy bi-ideal λ of a near algebra Y is said to be an irreducible fuzzy bi-ideal of Y if for any fuzzy bi-ideals μ and γ of Y, $\mu \cap \gamma = \lambda \Longrightarrow \mu = \lambda$ or $\gamma = \lambda$.

Definition 4.2. A fuzzy bi-ideal λ of a near algebra Y is said to be strongly irreducible fuzzy bi-ideal of Y if for any fuzzy bi-ideals μ and γ of Y, $\mu \cap \gamma \leq \lambda \Longrightarrow \mu \leq \lambda$ or $\gamma \leq \lambda$.

Theorem 4.3. Every strongly irreducible semiprime fuzzy bi-ideal of near algebra Y is a strongly prime fuzzy bi-ideal of Y.

Proof: Let λ be a strongly irreducible semiprime fuzzy bi-ideals of near algebra Y. Let μ and γ be two fuzzy bi-ideals of Y such that $(\mu o \gamma) \cap (\gamma o \mu) \leq \lambda$. As $(\mu \cap \gamma) o (\mu \cap \gamma) = (\mu \cap \gamma)^2 \leq (\mu o \gamma)$ and $(\mu \cap \gamma) o (\mu \cap \gamma) = (\mu \cap \gamma)^2 \leq (\gamma o \mu) \Longrightarrow (\mu \cap \gamma)^2 \leq (\mu o \gamma) \cap (\gamma o \mu) \leq \lambda$, so $\mu \cap \gamma \leq \lambda$.

Since λ is a semiprime fuzzy bi-ideal of Y, then $\mu \leq \lambda$ or $\gamma \leq \lambda$. Hence λ is a strongly prime fuzzy bi-ideal of near algebra Y.

Theorem 4.4. For a Near algebra Y, the following statements are equivalent:

(a) $\lambda o \lambda = \lambda$ for each bi-ideal of Y.
(b) $(\mu \cap \gamma) = (\mu o \gamma) \cap (\gamma o \mu)$ for all fuzzy bi-ideals μ and γ of Y.
(c) Every fuzzy bi-ideal of Y is fuzzy semiprime.
(d) Each proper fuzzy bi-ideal of Y is the intersection of irreducible semiprime fuzzy bi-ideal of Y which contain it.

Theorem 4.5. If every fuzzy bi-ideal λ of a near-algebra Y is idempotent, then the following statements are equivalent: (a) λ is strongly irreducible (b) λ is strongly prime.

Proof. (a) ⇒ (b): Let $(\mu o \gamma) \cap (\gamma o \mu) \leq \lambda$ for any two fuzzy bi-ideals μ and γ of Y. Then $\mu \cap \gamma = (\mu o \gamma) \cap (\gamma o \mu) \leq \lambda \implies \mu \leq \lambda or \gamma \leq \lambda$, as λ is strongly irreducible. So λ is strongly prime fuzzy bi-ideal of Y.

(b) ⇒ (a): Let μ and γ be two fuzzy bi-ideals of Y such that $\mu \cap \gamma \subseteq \lambda$. Then $(\mu o \gamma) \cap (\gamma o \mu) = \mu \cap \gamma \subseteq \lambda$. So $(\mu o \gamma) \cap (\gamma o \mu) \subseteq \lambda$. This implies that either $\mu \subseteq \lambda$ or $\gamma \subseteq \lambda$, as λ is strongly prime fuzzy bi-ideal of Y. Thus λ is strongly irreducible.

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