

Symmetries and Similarity Reductions of (2+1)-Dimensional Equal Width Wave Equation

T.Siva Subramania Raja¹ and K.Sathya²

¹PG & Research Department of Mathematics, National College
 Trichirapalli Tamilnadu, India. E-mail: sivasumatrix@gmail.com

²Department of Mathematics, Amman College of Arts and Science, Dindigul-02
 Tamilnadu, India. E-mail: sathyaphdmaths@gmail.com

Received 2 March 2018; accepted 18 April 2018

Abstract. We have employed the symmetries and similarity reductions of (2+1)-dimensional equal width wave equation

Keywords: The symmetry group, lie algebra, wave equation

AMS Mathematics Subject Classification (2010): 17B45

1. Introduction

A simple model equation is the Korteweg-de Vries (KdV) equation

$$v_t + 6vv_x + \delta v_{xxx} = 0 \tag{1.1}$$

which describe the long waves in shallow water. Its modified version is,

$$u_t - 6u^2u_x + u_{xxx} = 0 \tag{1.2}$$

And again there is Miura transformation

$$v = u^2 + u_x, \tag{1.3}$$

between the KdV equation (1.1) and its modified version (1.2).

In 2002, Liu and Yang studied the bifurcation properties of generalized KdV equation (GKdVE)

$$U_t + au^n u_x + u_{xxx} = 0, a \in \mathbb{R}, n \in \mathbb{Z}^+ \tag{1.4}$$

Gungor and Winternitz transformed the Generalized Kadomtsev-Petviashvili Equation (GKPE)

$$(u_t + p(t)uu_x + q(t)u_{xxx})_x + \sigma(y,t)u_{yy} + a(y,t)u_y + b(y,t)u_{xy} + c(y,t)u_{xx} + e(y,t)u_x + f(y,t)u + h(y,t) \tag{1.5}$$

to its canonical form and established conditions on the coefficient functions under which (1.5) has an infinite dimensional symmetry group having a Kac-Moody-Virasoro structure.

In this chapter, we discuss the symmetry reductions of the (2+1)-dimensional Equal Width Wave equation as,

$$u_t + uu_x - \mu(u_{xxt} + u_{yyt}). \tag{1.6}$$

2. The symmetry group and lie algebra of equal width wave equation

If (1.6) is invariant under a one parameter Lie group of Point transformations (Bluman and Kumei, Olver)

T.Siva Subramania Raja and K.Sathya

$$x^* = x + \epsilon \xi(x, y, t; u) + O(\epsilon^2), \quad (1.7)$$

$$y^* = y + \epsilon \eta(x, y, t; u) + O(\epsilon^2), \quad (1.8)$$

$$t^* = t + \epsilon \tau(x, y, t; u) + O(\epsilon^2), \quad (1.9)$$

$$u^* = u + \epsilon \phi(x, y, t; u) + O(\epsilon^2), \quad (1.10)$$

Then the third prolongation $\text{pr}^3(v)$ of the corresponding Vector field

$$V = \epsilon(x, y, t; u) \frac{\partial}{\partial x} + \eta(x, y, t; u) \frac{\partial}{\partial y} + \tau(x, y, t; u) \frac{\partial}{\partial t} + \phi(x, y, t; u) \frac{\partial}{\partial u} \quad (1.11)$$

Satisfies

$$\text{pr}^3(v)\Omega(x, y, t; u)|_{\Omega(x, y, t; u=0)} = 0 \quad (1.12)$$

The determining equations are obtained from (1.12) and solved for the infinitesimals ϵ, η, τ and ϕ . They are as follows

$$\epsilon = k_1, \quad (1.13)$$

$$\eta = k_2, \quad (1.14)$$

$$\tau = k_3 + tk_4, \quad (1.15)$$

$$\phi = -uk_4. \quad (1.16)$$

At this stage, we construct the symmetry generators corresponding to each of the constants involved.

Totally there are four generators given by

$$\begin{aligned} V_1 &= \partial_x, \\ V_2 &= \partial_y, \\ V_3 &= \partial_t, \\ V_4 &= t\partial_t - u\partial_u \end{aligned} \quad (1.17)$$

The symmetry generators found in Eq. (1.17) form a closed Lie Algebra whose commutation table is shown below.

Table 1: Commutator Table

$[v_i, v_j]$	V_1	V_2	V_3	V_4
V_1	0	0	0	0
V_2	0	0	0	0
V_3	0	0	0	V_3
V_4	0	0	0	0

The commutation relations of the Lie algebra, determined by V_1, V_2, V_3 and V_4 are shown in the above table.

3. Reductions of (2+1)-dimensional equal width wave equation by one-dimensional subalgebras

Case 1: $v_1 = \partial_x$

The characteristic equation associated with this generator is

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{0}$$

We integrate the characteristic equation to get three similarity variables,

$$y=r, t=s, \text{ and } u=W(r,s). \quad (1.18)$$

Using these similarity variables in Eq.(1.6) can be recast in the form

$$Ws - \mu(Wrrs) = 0. \quad (1.19)$$

Symmetries and Similarity Reductions Of (2+1)-Dimensional Equal Width Wave Equation

Case 2: $v_2 = \partial_{ys}$

The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dt}{0} = \frac{du}{0}$$

Following the standard procedure we integrate the characteristic equation to get three similarity variables,

$$X=r, y=s \text{ and } u=W(r,s) \quad (1.20)$$

Using these similarity variables in Eq.(1.6) can be recast in the form

$$W_s + WW_r - \mu(WW_{rs})=0 \quad (1.21)$$

Case 3: $v_3 = \partial_t$

The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0}$$

Following the standard procedure we integrate the characteristic equation to get three similarity variables,

$$X=r, y=s \text{ and } u=W(r,s) \quad (1.22)$$

Using these similarity variables in Eq.(1.6) can be recast in the form

$$WW_r=0 \quad (1.23)$$

Case 4: $v_4 = t\partial_t - u\partial_u$

The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{t} = \frac{du}{-u}$$

Following the standard procedure we integrate the characteristic equation to get three similarity variables,

$$X=r, t=s \text{ and } u=W(r,s) t^{-1}. \quad (1.24)$$

4. Reductions of (2+1)-dimensional equal width wave equation by two-dimensional abelian subalgebras

Case I: Reduction under v_1 and v_2 .

The transformed v_2 is

$$\tilde{v}_2 = \partial_r$$

The characteristic equation for \tilde{v}_2 is

$$\frac{dr}{1} = \frac{ds}{0} = \frac{dW}{0}$$

Integrating this equation as before leads to new variables

$$s = \zeta \text{ and } W = R(\zeta)$$

which reduce Eq. (1.19) to

$$R(\zeta) = 0 \quad (1.25)$$

Case II: Reduction under V_1 and V_3

The transformed V_3 is

$$\tilde{V}_3 = \partial_s$$

The characteristic equation for \tilde{V}_3 is

$$\frac{dr}{0} = \frac{ds}{1} = \frac{dW}{0}$$

Integrating this equation as before leads to new variable

$$r = \zeta \text{ and } W = R(\zeta).$$

which satisfies Eq. (1.19)

Case III: Reduction under V_1 and V_4

The transformed V_1 is

$$\tilde{V}_1 = \partial_r$$

The characteristic equation for \tilde{V}_1 is

$$\frac{dr}{1} = \frac{ds}{0} = \frac{dW}{0}$$

Integrating this equation as before leads to new variables

$$s = \zeta \text{ and } W = R(\zeta).$$

which reduce eq. (1.21) to

$$R - \mu R_{\zeta\zeta} = 0 \tag{1.26}$$

Case IV: Reduction under V_2 and V_3

The transformed V_3 is

$$\tilde{V}_3 = \partial_s$$

The characteristic equation for \tilde{V}_3 is

$$\frac{dr}{0} = \frac{ds}{1} = \frac{dW}{0}$$

Integrating this equation as before leads to new variables

$$r = \zeta \text{ and } W = R(\zeta).$$

which reduce Eq.(1.21) to

$$RR_{\zeta} = 0. \tag{1.27}$$

Case V: Reduction under V_2 and V_4

The transformed V_2 is

$$\tilde{V}_2 = \partial_s$$

The characteristic equation for \tilde{V}_2 is

$$\frac{dr}{0} = \frac{ds}{1} = \frac{dW}{0}$$

Integrating this equation as before leads to new variables

$$r = \zeta \text{ and } W = R(\zeta)$$

which reduce Eq. (1.23) to

$$R - RR_{\zeta} - \mu R_{\zeta\zeta} = 0 \tag{1.28}$$

5. Conclusions

In this paper,

- (i) A (2+1)-dimensional Equal width wave equation, $u_t + uu_x - \mu (u_{xxt} + u_{yyt}) = 0$ where $\mu \in \mathbb{R}$ is subjected to Lie's classical method.
- (ii) Equation (1.6) admits a four-dimensional symmetry group.
- (iii) It is established that the symmetry generators form a closed Lie algebra.
- (iv) Classification of symmetry algebra of (1.6) into one-and two-dimensional sub-algebras is carried out.

Symmetries and Similarity Reductions Of (2+1)-Dimensional Equal
Width Wave Equation

- (v) Systematic reduction to (1+1)-dimensional PDE and then to first order ODEs are performed using one dimensional and two dimension solvable Abelian sub-algebras.

REFERENCES

1. G.Bluman and S.Kumei, *Symmetries and Differential Equations*, Springer-Verlag, New York, (1989).
2. A.Ahmad, A.H.Bokhari, A.H.Kara and F.D.Zaman, Symmetry classifications and reductions of some classes of (2+1)-nonlinear heat equation, *J. Math. Anal. Appl.*, 339 (2008) 175-181.
3. Z.Liu and C.Yang, The application of bifurcation method to a higher-order Kdv equation, *J. Math. Anal. Appl.*, 275 (2002) 1-12.
4. R.M.Miura, Korteweg-de vries equations and generalizations: A remarkable explicit nonlinear transformation, *I. Math. Phys.*, 9 (1968) 1202-1204.