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Optimal Control of Service Rate in a Service Facility System Maintenance and Retrial Demands

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Abstract. This paper, deals with a problem of optimally controlling service rates for a retrial service facility system. We consider a finite capacity service facility system with Poisson arrivals and exponentially distributed service times. Here, the customer who sees the server busy, joins the orbit and reattempts the facility with exponential distributed time interval. The system is formulated as Markov Decision Process and we find the optimum service rates to be employed at each instant of time. Linear Programming method is implemented in the context of minimizing the long-run expected cost rate.

Keywords: Single server, Service facility, Retrial queue, Semi-Markov Decision Process, LPP method.

AMS Mathematics Subject Classification (2010): 90B05

1. Introduction

The main contribution of this paper is the determination the optimal control policy that yields the specific optimal service rate to be used for every possible state of the system (i.e. server rates are functions of the number of customer in the orbit). Queuing systems with retrials, in which customers repeat attempts to obtain service, was originally a topic of telecommunications research. More recently, these systems have served as models for particular computer networks, which may explain the current level of activity on the subject. As an example, the "customers" of this queue could be a network of computers attempting to access the same database, which may only be used by one customer at a time.

In last two decades, many researchers in the field of retrial queuing system contributed many results. For example, Elcan [8], Arivudainambi et al. [1], Dragieva [6], Dudin et al. [7] and Artalejo et al. [3,4,5] are discussed single server retrial queue with returning customers are examined by balking or Bernoulli vacations and derived the analysis part and solution technique using Matrix method or generating function or Truncation method using level dependent quasi-birth-and-death process (LDQBD).

In all these systems, arrivals of customers form a Poisson process and service times are exponentially distributed. They investigate the systems to obtain performance measures and construct suitable cost functions

The rest of the paper is organized as follows. We provide a formulation of our Semi - Markov Decision model in the next section. Analysis part of the model is given in section 3. In section 4, we present a procedure to implement long–run Expected cost rate criteria to get the optimal service levels of the systems.

2. Problem formulation

In this paper we assume the following

- customers arrive the service facility according to a Poisson process with rate λ (>0)
- An arriving customer when see server idle enters the server, gets service and leave the system
- An arriving customer who finds a server busy is obliged to leave the service area and enter an orbit, whose capacity is limited to maximum N
- Customer in the orbit reattempts for getting service often an exponential time at rate j θ(>0). (j -number of customer in the orbit)
- Service times of customers are independent of each other and have a common exponential distribution with parameter β_j .
- Each β_j $(1 \le j \le k)$ can be chosen from given set of k values $\{\beta_1 \beta_2, \beta_3 ... \beta_k\}$ where β_j depends on number of customer in orbit, β_0 ,when N(t)=0, the service rate become $\beta_0 = 0$.

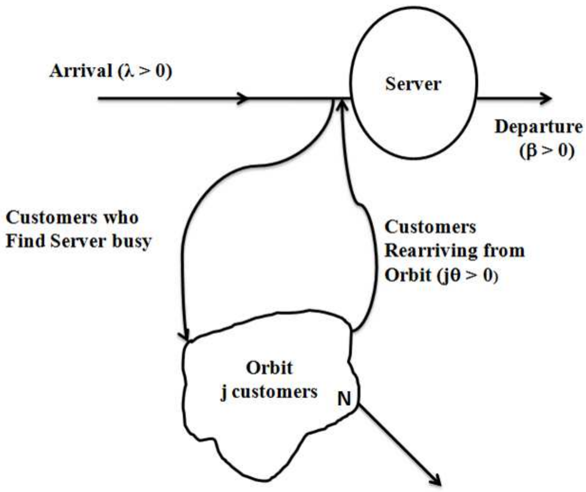

Figure 1:

3. **Analysis**

Let X(t) and N(t) denotes the status of the server and number of customers in the orbit at time t, respectively.

Then $\{(X(t),N(t)):t \ge 0\}$ is a two dimensional continuous time Markov process with state space E_1 x E_2, where, E_1= {0, 1}, (0 denotes the idle server and 1 denotes the busy server) and E_2= {0,1,2,…,N}

The infinitesimal generator A of the Markov process has entries of the form $\left(a_{(i,j)}^{(l,m)}\right)$

where $(a_{(i,j)}^{l,m})$ denote the transition rate from state (i,j) to state (l,m).

Some of the state transitions are explained as follows:

From state (0, j) only transitions into the following states are possible:

(i) (1, j) with rate λ for j=0, 1, 2, 3….N (customer arrival).

(ii) (1, j-1) with rate jθ, for j=1, 2…N (Customer arrive from an orbit).

From state (1, j) only transitions into the following states are possible:

(i) (1, j+1) with rate λ (customer arrival).

(ii) (0, j) with rate β (Service completion), where j=0, 1, 2,…N

The corresponding continuous time MDP is obtained by considering the following five components,

(i) Decision epochs: The decision epochs are random time points as time line at each service completion.

(ii) State space: E=E_1 x E_2 where , E_1= {0, 1} and E_2= {0,1,2,…,N}

(iii) Action set:

Let A_j denote the set of actions taken when the system is at state $j \in S$.

The augmented actions set is given by , $A = \bigcup_{j \in s} A_j$

where

$$A_s = \{\beta_0\}\ if\ s \in E = \{(\mathrm{j,r}) / \mathrm{j} = 0 \,\mathrm{and}\, \mathrm{r} = 0\}$$

$$A_s = \{\beta_1 \beta_2 ... \beta_k\},\ if\ s \in E = \{(\mathrm{j,r}) / \mathrm{j} \neq 0, \mathrm{r} \neq 0\}$$

Then the controlled process is a Markov process when stationary policy Π is adopted.

(iv) Transition probability :

A transition probability from state (i,j) to state (l,m) is given by $p_{(i,j)}^{(l,m)}(\mathrm{a})$.

(v) Cost: Cost accrued when action ‘a’ is taken at state (i,j) is given by

$c^{\Pi}((\mathrm{l,m}) / (\mathrm{i.j}), \mathrm{a})$

The long-run expected (average) cost rate when policy π is adopted is given by

$$C^{\Pi} = c_1 \overline{w}^{\Pi} + c_2 \overline{S}^{\Pi} \tag{1}$$

where, c_1 denotes the waiting cost/customer/unit time, $c_2$denotes the service cost /customer $\overline{w}^{\pi}$ denotes the mean waiting time for a customer and $\overline{S}^{\Pi}$ denotes the service completion rate

3.1. Steady state analysis

Let f denote the stationary policy, which is Deterministic time invariant and Markovian policy (MD). From the assumptions made in our system model, it can be seen that {X(t),N(t):t≥0} as the controlled process $\{\mathrm{x}^f(\mathrm{t}), \mathrm{N}^f(\mathrm{t}) : \mathrm{t} \geq \mathrm{o}\}$ when policy f is adopted. Since the process $\{\mathrm{x}^f(\mathrm{t}), \mathrm{N}^f(\mathrm{t}) : \mathrm{t} \geq \mathrm{o}\}$ is a Markov process with finite state space E. The process is completely Ergodic, if every stationary policy gives rise to an irreducible Markov chain. It can be seen that for every stationary policy Π the Markov process is completely Ergodic and also the optimal stationary policy Π^* exists, because the state and action spaces are finite.

Our objective is to find an optimal policy Π^* for which $C^{\Pi^*} \le C^{\Pi}$ for every MR policy in Π^{MR} For any fixed MR policy Π^{MR} and (i,j),(l,m)$\in E$,define

$p_{ij}^{\Pi}(\mathrm{l,m,t}) = \mathrm{pr}\{X^{\Pi}(t) = l, N^{\Pi}(t) = m / X^{\Pi}(0) = i, N^{\Pi}(0) = j\}; (i,j),(l,m) \in E$

Now $p_{ij}^{\Pi}(\mathrm{l,m,t})$ satisfies the Kolmogorov forward differential equation. $P'(t)=P(t)A$, where A is an infinitesimal generator of the Markov process $\{(X^{R}(t), N^{R}(t)) : t \ge 0\}$

For each MR policy Π , we get a Markov chain with state space E and action set A which are finite, $p^{\Pi}(\mathrm{l,m}) = \lim_{t\to\infty} p_{ij}^{\Pi}(\mathrm{l,m;t})$ exists and is independent of initial state (i,j)conditions.

The balance equations are obtained by using the fact that transition out of a state is equal to transition into a state.

$$\lambda P^{\Pi}(0,0) = \beta P^{\Pi}(1,0) \quad (2)$$

$$(\lambda + j\theta)P^{\Pi}(0,j) = \beta P^{\Pi}(1,j) \quad ; 0 \le j \le N \quad (3)$$

$$(\lambda + \beta) P^{\Pi}(1,0) = \lambda P^{\Pi}(0,0) + \theta P^{\Pi}(0,1) \quad (4)$$

$$(\lambda + \beta) P^{\Pi}(1,j) = \lambda P^{\Pi}(0,j) + \lambda P^{\Pi}(1,j-1) + (j+1)\theta P^{\Pi}(0,j+1); 1 \le j \le N-1 \quad (5)$$

$$\beta P^{\Pi}(1,N) = \lambda P^{\Pi}(0,N) + \lambda P^{\Pi}(1,N-1), \quad (6)$$

together with total probability condition $\sum_{(i,j)\in E} P_{ij}^{\Pi} = 1,$ (7)

we get the steady state solution $(P_{ij}^{\Pi})_{(i,j)\in E}$, uniquely.

3.2. System performance measures

Mean waiting time in the orbit is given by

$$\overline{W}^{\Pi} = \sum_{j=1}^{N} \left(\frac{j}{\beta^{\Pi}} \right) P^{\Pi}(1,j) \quad (8)$$

The service completion rate is given by

$$\overline{S}^{\Pi} = \sum_{j=1}^{N} \beta^{\Pi} P^{\Pi}(0,j) \quad (9)$$

The long run expected cost rate is given by

$$C^{\Pi} = c_1 \sum_{j=1}^{N} \left(\frac{j}{\beta^{\Pi}} \right) P^{\Pi}(1,j) + c_2 \sum_{j=1}^{N} \beta^{\Pi} P^{\Pi}(0,j) \quad (10)$$

4. Linear programming problem

4.1. Formulation of LPP

In this section we propose a LPP model within a MDP framework.

First we define the variables,D(i,j,a) as a conditional probability expression

$D(\mathrm{i,j,a}) = \Pr\left\{ decision is\ 'a' / \text{state is } (i,j) \right\}$

Since $0 \leq D(i,j,a) \leq 1$, this is compatible with Randomized time invariant Markovian policies. Here, the Semi-Markovian decision problem can be formulated as a linear programming problem.

Hence

$$0 \leq D(i,j) \leq 1 \text{ and} \sum_{a\in A} D(i,j,a) = 1, \mathrm{i} = 0,1; 0 \leq j \leq N.$$

For the reformulation of the MDP as LPP, we define another variable y(i,j,a) as follows.

$$\mathrm{y}(i,j,a) = D(\mathrm{i,j,a})P^{\Pi}(\mathrm{i,j}) \tag{11}$$

From the above definition of the transition probabilities

$$P^{\Pi}(i,j) = \sum_{a\in A} y(\mathrm{i,j,a}), (\mathrm{i,j}) \in \mathrm{E}, \mathrm{a} \in \mathrm{A} = \{0,1,2...N\} \tag{12}$$

Expressing $P^{\Pi}(i,j)$ in terms of y(i,j,a), the expected total cost rate functions(10) is Obtained and the LPP formulation is of the form

Minimize

$$C^{\Pi} = c_1 \sum_{a\in(0,1,2,3...\mathrm{k})} \sum_{j=1}^{N} \left(\frac{j}{\beta^{\Pi}}\right) P^{\Pi}(1,\mathrm{j}) + c_2(\beta) \sum_{a\in(0,1,2,3...\mathrm{k})} \sum_{j=1}^{N} \beta^{\Pi} P^{\Pi}(0,j) \tag{13}$$

Subject to the constraints,

$$\text{(i) } \mathrm{y(i,j,a)} \geq 0; (\mathrm{i,j}) \in \mathrm{E}, \mathrm{a} \in \mathrm{A}_l, l = 0,1; \mathrm{A}_0 = \{\beta_0\}, A_1 = \{\beta_1, \beta_2, \beta_3, ...\beta_k\}.$$

$$ii) \sum_{l=0,1} \sum_{(\mathrm{i,j})\in \mathrm{E}} \sum_{a\in A_l} y(\mathrm{i,j,a}) = 1,$$

And the balance equation (2)-(6) obtained by replacing

$$P^{\Pi}(\mathrm{i,j}) \text{ by} \sum_{a\in A} y(\mathrm{i,j,a}).$$

Lemma 4.3. The optimal solution of the above Linear Programming Problem yield a deterministic policy.

Proof: From the equations (11) and (12)

$$\mathrm{D(i,j,a)} = \frac{y(\mathrm{i,j,a})}{\sum_{k=0}^{N} y(\mathrm{i,j},\beta_k)}, a = \beta_k, k = 0,1,2,...N \tag{14}$$

$$P^{\Pi}(\mathrm{i,j}) = \sum_{a\in A} y(\mathrm{i,j,a}), (\mathrm{i,j}) \in \mathrm{E} \tag{15}$$

Since the decision process is completely ergodic every basic feasible solution to the above Linear Programming problem has the property that for each $(\mathrm{i,j}) \in \mathrm{E}$, $\mathrm{y(i,j,a)} > 0$ for exactly one $a \in A$

Hence, for each $(i, j) \in E$, D (i, j,a)=1, for a unique a zero values of a. Thus give the number of customers in the orbit, we have to choose the service rate β for which D (i,j.a)=1. Hence the basic feasible solution of the LPP yields a deterministic policy .

5. Numerical illustration and discussion

In this section we consider a service facility system to illustrate the method described in section 4, through numerical examples. We implemented TORA software to solve LPP by simplex algorithm.

The following table describes the solution for LPP problem by varying the arrival (Poisson) rates from 2 to 5 and an exponential service rates from 1 to 7. The expected cost is computed by taking waiting cost per customer is 0.5 and the service cost per customer is 0.8.

Arrival rate: → Service rate: ↓	λ = 2	λ = 3	λ = 4	λ = 5
β = 1	1.9498	2.07346	2.13148	2.31299
β = 2	**0.98432**	1.0723	1.103441	1.042195
β = 3	1.0263	**1.01216**	**0.986711**	**0.96865**
β = 4	1.35712	1.26805	1.16163	1.0911
β = 5	1.77853	1.68233	1.4995	1.36179
β = 6	2.2257	2.1758	1.94466	2.49231
β = 7	2.6859	2.7055	2.45772	2.19088

Table 1: The Expected total cost

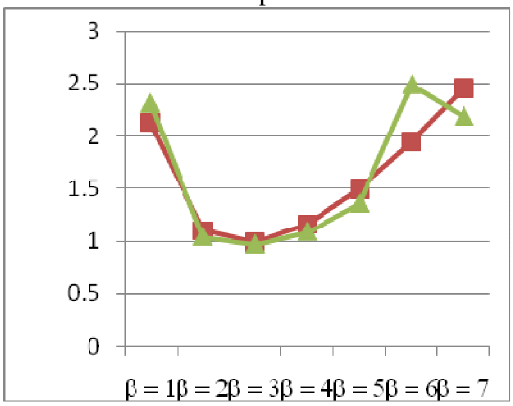

Figure 1: The Expected cost for different service rate

From the above table, (a) The minimum expected cost for arrival rate 2 will be obtained by adjusting the service rate μ=2 as per unit time. (b) The minimum expected cost for arrival rate 3, 4 and 5 will be obtained. (c) The only parameters that have appreciable effect on the service rates are the arrival rate, the waiting time cost per customer, service cost per customer, the number of customers in the system see fig.1). The following table 2 shows that to increase the waiting time cost per customer will increase the expected total cost (see figure - 2). The following table 3 shows that the service cost per customer affects the expected total cost (see figure - 3).

(waiting cost per customer, the service cost per customer) (c_1, c_2)	Expected total cost
(0.5, 0.8)	0.98432
(1, 0.8)	1.631682
(1.5, 0.8)	2.27905
(2, 0.8)	2.92642

Table 2: The Expected total cost for varying waiting cost per customer

(waiting cost per customer, the service cost per customer) (c_1, c_2)	Expected total cost
(0.5, 0.8)	0.98432
(005, 1.6)	1.32126
(0.5, 2.4)	1.65821
(0.5, 3.2)	1.99516

Table 3: The expected total cost for varying service cost per customer

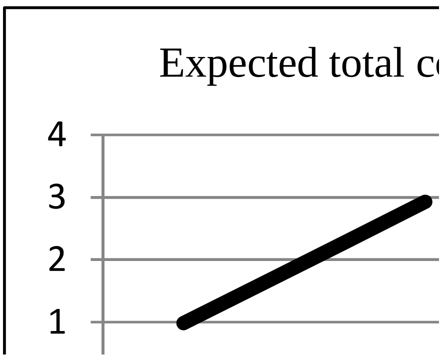

Figure 2: The expected cost for varying waiting cost per customer

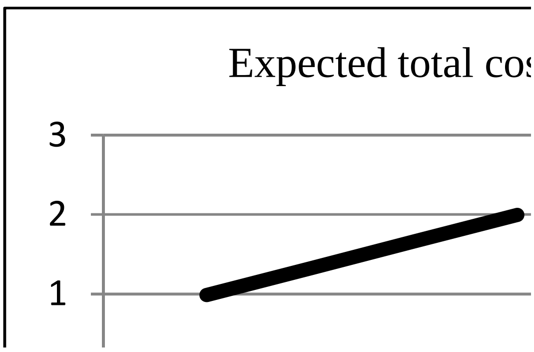

Figure 3: The expected cost for varying service cost per customer

6. Conclusions and future research

Analysis of service control at retrial service facility system is fairly recent. In most of previous works, optimal ordering policies or system performance measures are determined. We approached the problem using semi-Markov decision process tocontrol the optimum service rates in service facility system. The optimal control policy to be employed is found using linear –programming method so that the long-run expected cost rate is minimized. In future we would like to extend this model to general distribution for arrival of customers in retrial service facility system as a semi-markov decision problem.

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