Intern. J. Fuzzy Mathematical Archive
Vol. 14, No. 2, 2017, 179-189
ISSN: 2320 –3242 (P), 2320 –3250 (online)
Published on 11 December 2017
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/ijfma.v14n2a2

International Journal of
Fuzzy Mathematical Archive

Hesitancy Fuzzy Graph Coloring

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Received 2 November 2017; accepted 10 December 2017

Abstract. In this paper, the concept of coloring the hesitancy fuzzy graph, strong hesitancy fuzzy graph and complete hesitancy fuzzy graph are introduced with illustrative examples.

Keywords: Chromatic number, Vertex coloring, Edge coloring, Total coloring, Hesitancy fuzzy graph.

AMS Mathematics Subject Classification (2010): 05C72, 05C15

1. Introduction

Graph coloring dates back to 1852, when Francis Guthrie come up with the four color conjecture. Gary Chartrand and Ping Zhang [3] discussed various colorings of graph and its properties in their book entitled Chromatic Graph Theory. A graph coloring is the assignment of a color to each of the vertices or edges or both in such a way that no two adjacent vertices and incident edges share the same color. Graph coloring has been applied to many real world problems like scheduling, allocation, telecommunications and bioinformatics, etc.

The concept of fuzzy sets and fuzzy relations were introduced by Zadeh in 1965 [22]. Rosenfeld who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975 [19]. The concept of chromatic number of fuzzy graph was introduced by Munoz et.al. [20]. Later Eslahchi and Onagh introduced fuzzy graph coloring of fuzzy graph [2]. Lavanya and Sattanathan discussed total fuzzy coloring [10]. Kishore and Sunitha discussed chromatic number of fuzzy graph [1]. Hussain and Kanzul Fathima conferred about fuzzy coloring of fuzzy graph, using strong arcs and dominator coloring of fuzzy graph [5, 6, 7]. Nagoor Gani and Fathima Kani deliberated about Fuzzy vertex order colouring [11].

Intuitionistic fuzzy sets [9] and Intuitionistic fuzzy graph [8] were introduced by Atanassov in 1986 and 1999 respectively. Parvathi et.al. discussed the intuitionistic fuzzy graph and its properties [16, 17]. Mohideen et.al. introduced coloring of intuitionistic fuzzy graph using (α, β)-cuts [4] and strong intuitionistic fuzzy graph coloring [18].

Hesitant fuzzy sets introduced by Torra in 2010 [21]. Pathinathan et.al. introduced Hesitancy fuzzy graph in 2015 [12] and discussed various properties in [13, 14, 15]. Hesitancy Fuzzy Graphs (HFGs) has been applied to capture the common intricacy that occurs during a selection of membership degree of an element from some possible values that makes one to hesitate.

In this paper the attempt has been made to focus the hesitancy fuzzy graph (HFG), strong hesitancy fuzzy graph and complete hesitancy fuzzy graph. In addition to the above graphs, it is also proposed to define the vertex coloring, edge coloring and total coloring of hesitancy fuzzy graphs in terms of a family of hesitancy fuzzy sets satisfying certain conditions and the chromatic number is the least value of k such that k-coloring exists.

2. Preliminaries

Definition 2.1. Let X be a non-empty set. Then a fuzzy set A in X (i.e., a fuzzy subset A of X) is characterized by a function of the form $\mu_A: X \to [0,1]$, such a function μ_A is called the membership function and for each $x \in X, \mu_A(x)$ is the degree of membership of x (membership grade of x) in the fuzzy set A.
In other words, $A = \{(x, \mu_A(x)) / \, x \in X\}$ where $\mu_A: X \to [0,1]$.

Definition 2.2. A fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma: V \to [0, 1]$and $\mu: V \times V \to [0, 1]$, where for all $u\,, v \in V$, we have $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$.

Definition 2.3. An Intuitionistic Fuzzy set A in a set X is defined as an object of the form
$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / \, x \in X\}$where$\mu_A: X \to [0,1]$ and $\nu_A: X \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ respectively and for every $x \in X$; $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Definition 2.4. Intuitionistic Fuzzy Graph (IFG) is of the form $G = (V, E)$, where

(i) $V = \{v_1, v_2, \dots, v_n\}$ such that $\mu_1: V \to [0,1]$ and $\nu_1: V \to [0,1]$ denote the degrees of membership and non-membership of the element $v_i \in V$ respectively and $0 \leq \mu_1(v_i) + \nu_1(v_i) \leq 1$, for every $v_i \in V$, $(i = 1,2, \dots, n)$.

(ii) $E \subset V \times V$ where $\mu_2: V \times V \to [0,1]$ and $\nu_2: V \times V \to [0,1]$ are such that

$$\mu_2(v_i, v_j) \leq \min[\mu_1(v_i), \mu_1(v_j)]$$
$$\nu_2(v_i, v_j) \leq \max[\nu_1(v_i), \nu_1(v_j)]$$

And $0 \leq \mu_2(v_i, v_j) + \nu_2(v_i, v_j) \leq 1$ for every $(v_i, v_j) \in E$.

Definition 2.5. Hesitancy Fuzzy Graph (HFG) is of the form $G = (V, E)$, where

(i) $V = \{v_1, v_2, \dots, v_n\}$ such that $\mu_1: V \to [0,1]$, $\gamma_1: V \to [0,1]$ and $\beta_1: V \to [0,1]$ denote the degrees of membership, non-membership and hesitancy of the element $v_i \in V$ respectively and $\mu_1(v_i) + \gamma_1(v_i) + \beta_1(v_i) = 1$, for every $v_i \in V$, $(i = 1,2, \dots, n)$, where $\beta_1(v_i) = 1 - [\mu_1(v_i) + \gamma_1(v_i)]$, and $0 \leq \mu_1(v_i) + \gamma_1(v_i) \leq 1$.

(ii) $E \subseteq V \times V$where $\mu_2: V \times V \to [0,1]$, $\gamma_2: V \times V \to [0,1]$ and $\beta_2: V \times V \to [0,1]$ are such that,

$$\mu_2(v_i, v_j) \leq \min[\mu_1(v_i), \mu_1(v_j)]$$
$$\gamma_2(v_i, v_j) \leq \max[\gamma_1(v_i), \gamma_1(v_j)]$$
$$\beta_2(v_i, v_j) \leq \min[\beta_1(v_i), \beta_1(v_j)]$$

And $0 \leq \mu_2(v_i, v_j) + \nu_2(v_i, v_j) + \beta_2(v_i, v_j) \leq 1$ for every $(v_i, v_j) \in E$.

Here the $(v_i, \mu_{1i,}\gamma_{1i}, \beta_{1i})$ denote the vertex, the degree of membership, degree of non-membership and hesitancy of the vertex v_i. And the $(e_{ij,}\mu_{2ij,}\gamma_{2ij}, \beta_{2ij})$ denote the edge, the degree of membership, degree of non-membership and hesitancy of the edge relation $e_{ij} = (v_i, v_j)$ on $V \times V$.

Definition 2.6. If $G = (V, \mu)$ is such a fuzzy graph where $V = \{1,2,3, \dots, n\}$ and μ is a fuzzy number on the set of all subsets of $V \times V$. Assume $I = A \cup \{0\}$ where $A = \{\alpha_1 < \alpha_2 < \cdots < \alpha_k\}$ is the fundamental set (level set) of G. For each $\alpha \in I, G_\alpha$ denote the crisp graph $G_\alpha = (V, E_\alpha)$ where $E_\alpha = \{(i,j)/\ 1 \le i < j \le n, \mu(i,j) \ge \alpha\}$ and $\chi_\alpha = \chi(G_\alpha)$ denote the chromatic number of crisp graph G_α. By this definition the chromatic number of the fuzzy graph G is the fuzzy number $\chi(G) = \{(i, v(i))/i \in X\}$ where $v(i) = \max\{\alpha \in I/i \in A_\alpha\}$ and $A_\alpha = \{1, \dots, \chi_\alpha\}$.

Definition 2.7. A family $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ of fuzzy sets on V is called a k-fuzzy coloring of $G = (V, \sigma, \mu)$ if

a) $\vee\ \Gamma = \sigma$,
b) $\gamma_i \wedge \gamma_j = 0$,
c) For every strong edge $xy$of G, min $\{\gamma_i(x), \gamma_i(y)\} = 0 (1 \le i \le k)$.

Definition 2.8. A family $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ of fuzzy sets on $V \cup E$ is called a k-fuzzy total coloring of $G = (V, \sigma, \mu)$ if

a) $max_i\gamma_i(v) = \sigma(v)$ for all $v \in V$ and $max_i\gamma_i(uv) = \mu(uv)$ for all edge $uv \in E$
b) $\gamma_i \wedge \gamma_j = 0$,
c) For every adjacent vertices u, v of min $\{\gamma_i(u), \gamma_i(v)\} = 0$ and for every incident edges

$$\min\left\{\begin{matrix}\gamma_i(v_j, v_k)\ /v_j, v_k\ are\ set\ of\ incident\ edges\ from\ the\ vertex\ \ v_j, \\ j = 1, \dots, |v|\end{matrix}\right\}$$

Definition 2.9. A family $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ of intuitionistic fuzzy sets on a set V is called a k-vertex coloring of $\hat{G} = (V, E)$ if

a) $\vee\ \gamma_i(x) = V$, for all $x \in V$
b) $\gamma_i \wedge \gamma_j = 0$
c) For every edge xy of $\hat{G}$, $\min\{\gamma_i(\mu_1(x)), \gamma_i(\mu_1(y))\} = 0$ and $\max\{\gamma_i(\nu_1(x)), \gamma_i(\nu_1(y))\} = 1, (1 \le i \le k)$

The least value of k for which the $\hat{G}$ has a k-vertex coloring denoted by $\chi(\hat{G})$, is called the chromatic number of the intuitionistic fuzzy graph $\hat{G}$.

Definition 2.10. A hesitancy fuzzy graph, $G = (V, E)$is said to be strong hesitancy fuzzy graph if

$\mu_{2ij} = \min(\mu_{1i}, \mu_{1j})$, $\gamma_{2ij} = \max(\gamma_{1i}, \gamma_{1j})$ and $\beta_{2ij} = \min(\beta_{1i}, \beta_{1j})$for all $(v_i, v_j) \in E$.

Definition 2.11. A hesitancy fuzzy graph, $G = (V, E)$ is said to be complete hesitancy fuzzy graph if $\mu_{2ij} = \min(\mu_{1i}, \mu_{1j})$, $\gamma_{2ij} = \max(\gamma_{1i}, \gamma_{1j})$ and $\beta_{2ij} = \min(\beta_{1i}, \beta_{1j})$ for every $v_i, v_j \in V$.

3. Hesitancy fuzzy graph coloring

Definition 3.1.

1. The arc (u, v) in hesitancy fuzzy graph G is said to be a strong arc if $\frac{1}{2}\min\{\mu_1(u), \mu_1(v)\} \leq \mu_2(u, v)$, $\frac{1}{2}\max\{\gamma_1(u), \gamma_1(v)\} \leq \gamma_2(u, v)$ and $\frac{1}{2}\min\{\beta_1(u), \beta_1(v)\} \leq \beta_2(u, v)$.
2. Two vertices u and v in hesitancy fuzzy graph G are called adjacent if (u, v) is strong arc in G otherwise weakly adjacent.
3. If two distinct edges (u, v) and (v, w) in hesitancy fuzzy graph G are incident with a common vertex v, then they are called incident edges.

Definition 3.2. (Vertex coloring)

A family $C = \{c_1, \dots, c_k\}$ of hesitancy fuzzy sets on a set V is called a k-vertex coloring of $G = (V, E)$ if

(i) $\vee\ c_i(x) = V$, for all $x \in V$

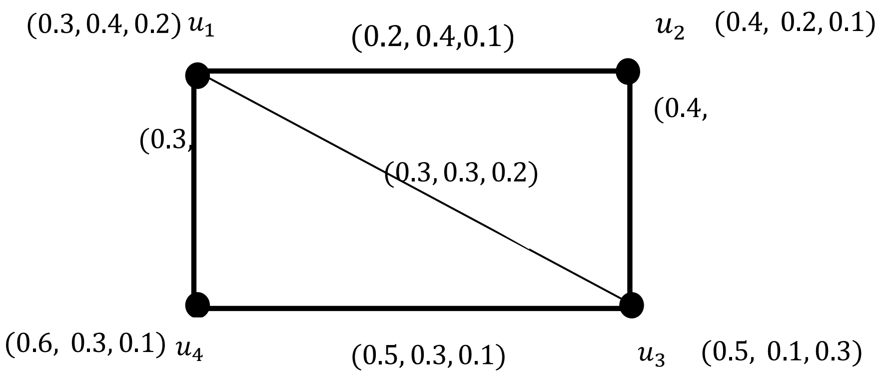
(ii) $c_i \wedge c_j = 0$

(iii) For every strong edge xy of G, $\min\{c_i(\mu_1(x)), c_i(\mu_1(y))\} = 0$, $\max\{c_i(\gamma_1(x)), c_i(\gamma_1(y))\} = 1$ and $\min\{c_i(\beta_1(x)), c_i(\beta_1(y))\} = 0, (1 \leq i \leq k)$

The least value of k for which the G has a k-vertex coloring denoted by $\chi(G)$, is called the chromatic number of the hesitancy fuzzy graph G.

Example 3.3. (Hesitancy fuzzy graph vertex coloring)

Consider the hesitancy fuzzy graph $G = (V, E)$ in figure 3.1, with four vertices and five edges.



HFG G

Figure 3.1

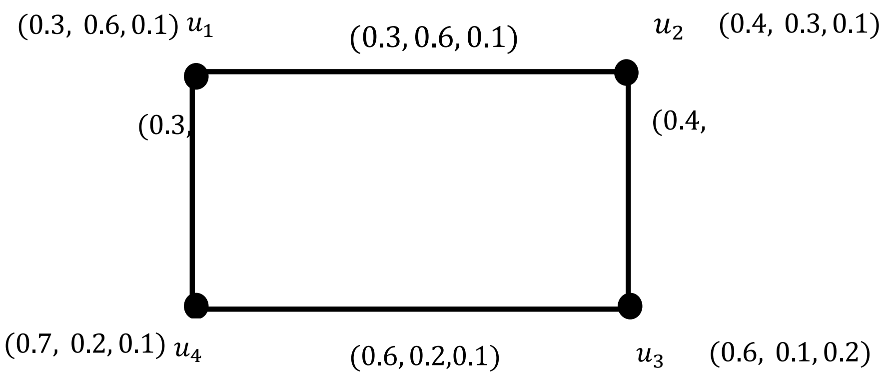
Let $C = \{c_1, c_2, c_3\}$ be a family of hesitancy fuzzy sets defined on V as follows

$$c_1(u_i) = \begin{cases} (0.3,0.4,0.2), & i = 1 \\ (0,1,0), & otherwise \end{cases} \quad c_2(u_i) = \begin{cases} (0.4,0.2,0.1), & i = 2 \\ (0.6,0.3,0.1), & i = 4 \\ (0,1,0), & otherwise \end{cases}$$

$$c_3(u_i) = \begin{cases} (0.5,0.1,0.3), & i = 3 \\ (0,1,0), & otherwise \end{cases}$$

Hence the family $C = \{c_1, c_2, c_3\}$ satisfies our definition of vertex coloring of hesitancy fuzzy graph. We find that any family of hesitancy fuzzy sets having less than three members could not satisfy our definition. Hence in this case the chromatic number $\chi(G)$ is 3.

Example 3.2. (Strong hesitancy fuzzy graph vertex coloring)
Consider the strong hesitancy fuzzy graph $G = (V, E)$ in figure 3.2, with four vertices and four edges.



Strong HFG G

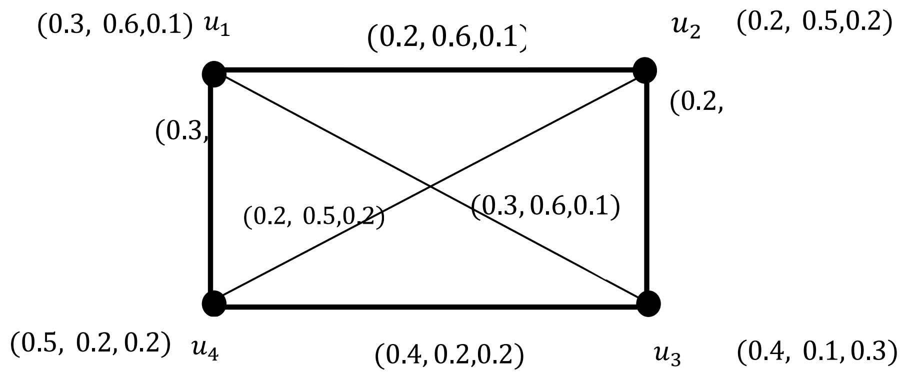
Figure 3.2

Let $C = \{c_1, c_2\}$ be a family of hesitancy fuzzy sets defined on V as follows

$$c_1(u_i) = \begin{cases} (0.3,0.6,0.1), & i = 1 \\ (0.6,0.1,0.2), & i = 3 \\ (0,1,0), & otherwise \end{cases} \quad c_2(u_i) = \begin{cases} (0.4,0.3,0.1), & i = 2 \\ (0.7,0.2,0.1), & i = 4 \\ (0,1,0), & otherwise \end{cases}$$

Hence the family $C = \{c_1, c_2\}$ satisfies our definition of vertex coloring of strong hesitancy fuzzy graph. We find that any family of hesitancy fuzzy sets having less than two members could not satisfy our definition. Hence in this case the chromatic number $\chi(G)$ is 2.

Example 3.5. (Complete hesitancy fuzzy graph vertex coloring)
Consider the complete hesitancy fuzzy graph $G = (V, E)$ in figure 3.3, with four vertices and six edges.



Complete HFG

Figure 3.3

Let $C = \{c_1, c_2, c_3, c_4\}$ be a family of hesitancy fuzzy sets defined on V as follows

$$c_1(u_i) = \begin{cases} (0.3,0.6,0.1), & i = 1 \\ (0,1,0), & otherwise \end{cases} \quad c_2(u_i) = \begin{cases} (0.2,0.5,0.2), & i = 2 \\ (0,1,0), & otherwise \end{cases}$$

$$c_3(u_i) = \begin{cases} (0.4,0.1,0.3), & i = 3 \\ (0,1,0), & otherwise \end{cases} \quad c_4(u_i) = \begin{cases} (0.5,0.2,0.2), & i = 4 \\ (0,1,0), & otherwise \end{cases}$$

Hence the family $C = \{c_1, c_2, c_3, c_4\}$ satisfies our definition of vertex coloring of complete hesitancy fuzzy graph. We find that any family of hesitancy fuzzy sets having less than four members could not satisfy our definition. Hence in this case the chromatic number $\chi(G)$ is 4.

3.6. Bound for chromatic number of HFG

Proposition 3.6.1. For any HFG, the chromatic number $\chi \quad (G) \leq \Delta(G) + 1$ where $\Delta(G)$ is the maximum number of edges incident to a vertex of G.

Proposition 3.6.2. The chromatic number of complete HFG is n where n is the number of vertices of G, i.e., $\chi(G) = n$.
(Since $\Delta(G) = n - 1$ is the maximum vertex degree of the complete HFG G, chromatic number $\chi \quad (G) = \Delta(G) + 1$)

Proof: Since $\mu_{2ij} = \min(\mu_{1i}, \mu_{1j})\, \gamma_{2ij} = \max(\gamma_{1i}, \gamma_{1j})$ and $\beta_{2ij} = \min(\beta_{1i}, \beta_{1j})$for every $v_i, v_j \in V$. Every pair of vertices are adjacent and degree of each vertex is $n - 1$. By (iii) of definition [3.2], $\min\{c_i(\mu_1(x)), c_i(\mu_1(y))\} = 0$,
$\max\{c_i(\gamma_1(x)), c_i(\gamma_1(y))\} = 1$ and $\min\{c_i(\beta_1(x)), c_i(\beta_1(y))\} = 0$,
for adjacent vertices x, y. Since all vertices are adjacent, every member of the family defining hesitancy fuzzy graph coloring have value for only one vertex and (0,1,0) for all other vertices. By (ii) of definition [3.2], $c_i \wedge c_j = 0$, so $\vee\ c_i(x_i) = V$ for all $x \in Vi = 1,2, \ldots, n$ where $n = |V|$. Thus, $\chi(G) = n$.

Definition 3.7. (Edge coloring)

A family $C = \{c_1, \dots, c_k\}$ of hesitancy fuzzy sets on E is called a k-edge coloring of $G = (V, E)$ if

a) $\vee\ c_i(xy) = E$, for all edge $xy \in E$
b) $c_i \wedge c_j = 0$
c) For every incident edges xy on vertex $x \in V$ of G, $\min\{c_i(\mu_2(xy))\} = 0$, $\max\{c_i(\gamma_2(xy))\} = 1,\ and\ \min\{c_i(\beta_2(xy))\} = 0\ \ (1 \le i \le k)$

The least value of k for which the G has a k- edge coloring denoted by $\chi'(G)$, is called the edge chromatic number of the hesitancy fuzzy graph G.

Example 3.8. (Hesitancy fuzzy graph edge coloring)
Consider the HFG $G = (V, E)$ given in Example 3.3.
Let $c = \{c_1, c_2, c_3\}$ be a family of hesitancy fuzzy sets defined on E as follows

$$c_1(u_iu_j) = \begin{cases} (0.2,0.4,0.1), & ij = 12 \\ (0.5,0.3,0.1), & ij = 34 \\ (0,1,0), & otherwise \end{cases} \quad c_2(u_iu_j) = \begin{cases} (0.3,0.4,0.1), & ij = 14 \\ (0.4,0.2,0.1), & ij = 23 \\ (0,1,0), & otherwise \end{cases}$$

$$c_3(u_iu_j) = \begin{cases} (0.3,0.3,0.2), & ij = 13 \\ (0,1,0), & otherwise \end{cases}$$

Hence the family $C = \{c_1, c_2, c_3\}$ satisfies our definition of edge coloring of hesitancy fuzzy graph. We find that any family of hesitancy fuzzy sets having less than three members could not satisfy our definition. Hence in this case the edge chromatic number $\chi'(G)$ is 3.

Example 3.9. (Strong hesitancy fuzzy graph edge coloring)
Consider the strong HFG $G = (V, E)$ given in Example 3.4.
Let $C = \{c_1, c_2\}$ be a family of hesitancy fuzzy sets defined on E as follows

$$c_1(u_iu_j) = \begin{cases} (0.3,0.6,0.1), & ij = 12 \\ (0.6,0.2,0.1), & ij = 34 \\ (0,1,0), & otherwise \end{cases} \quad c_2(u_iu_j) = \begin{cases} (0.3,0.6,0.1), & ij = 14 \\ (0.4,0.3,0.1), & ij = 23 \\ (0,1,0), & otherwise \end{cases}$$

Hence the family $C = \{c_1, c_2\}$ satisfies our definition of edge coloring of strong hesitancy fuzzy graph. We find that any family of hesitancy fuzzy sets having less than two members could not satisfy our definition. Hence in this case the edge chromatic number $\chi'(G)$ is 2.

Example 3.10. (Complete hesitancy fuzzy graph edge coloring)
Consider the complete HFG $G = (V, E)$ given in Example 3.5.
Let $C = \{c_1, c_2, c_3\}$ be a family of hesitancy fuzzy sets defined on E as follows

$$c_1(u_iu_j) = \begin{cases} (0.2,0.6,0.1), & ij = 12 \\ (0.4,0.2,0.2), & ij = 34 \\ (0,1,0), & otherwise \end{cases} \quad c_2(u_iu_j) = \begin{cases} (0.2,0.5,0.2), & ij = 23 \\ (0.3,0.6,0.1), & ij = 41 \\ (0,1,0), & otherwise \end{cases}$$

$$c_3(u_iu_j) = \begin{cases} (0.3,0.6,0.1), & ij = 13 \\ (0.2,0.5,0.2), & ij = 24 \\ (0,1,0), & otherwise \end{cases}$$

Hence the family $C = \{c_1, c_2, c_3\}$ satisfies our definition of edge coloring of complete hesitancy fuzzy graph. We find that any family of hesitancy fuzzy sets having less than three members could not satisfy our definition. Hence in this case the edge chromatic number $\chi'(G)$ is 3.

3.11. Bound for edge chromatic number of HFG

Proposition 3.11.1. For any HFG, the edge chromatic number $\chi'(G) \leq \Delta(G) + 1$ where $\Delta(G)$ is the maximum number of edges incident to a vertex of G.

Proposition 3.11.2. The edge chromatic number of complete HFG on n vertices is n if n is odd and $n - 1$ if n is even whatever may be the membership, non-membership and hesitancy functions.

Definition 3.12. (Total coloring)

A family $C = \{c_1, \dots, c_k\}$ of hesitancy fuzzy sets on $V \cup E$ is called a k- total coloring of $G = (V, E)$ if

a) $\vee\ c_i(x) = V$, for all $x \in V$ and $\vee\ c_i(xy) = E$, for all edge $xy \in E$
b) $c_i \wedge c_j = 0$,
d) For every strong edge $xy$of G, $\min\{c_i(\mu_1(x)), c_i(\mu_1(y))\} = 0$, $\max\{c_i(\gamma_1(x)), c_i(\gamma_1(y))\} = 1$ and $\min\{c_i(\beta_1(x)), c_i(\beta_1(y))\} = 0$. And for every incident edges xy on vertex $x \in V$of G, min $\{c_i(\mu_2(xy))\} = 0$, $\max\{c_i(\gamma_2(xy))\} = 1, \min\{c_i(\beta_2(xy))\} = 0 \quad (1 \leq i \leq k)$.

The least value of k for which the G has a k- total coloring denoted by $\chi^T(G)$, is called the total chromatic number of the hesitancy fuzzy graph G.

Example 3.13. (Hesitancy fuzzy graph total coloring)

Consider the HFG $G = (V, E)$ given in Example 3.3.

Let $C = \{c_1, c_2, c_3, c_4\}$ be a family of hesitancy fuzzy sets defined on $V \cup E$ as follows

$$c_1(u_i) = \begin{cases} (0.3,0.4,0.2), & i = 1 \\ (0,1,0), & otherwise \end{cases} \quad c_2(u_i) = \begin{cases} (0.4,0.2,0.1), & i = 2 \\ (0.6,0.3,0.1), & i = 4 \\ (0,1,0), & otherwise \end{cases}$$

$$c_3(u_i) = \begin{cases} (0.5,0.1,0.3), & i = 3 \\ (0,1,0), & otherwise \end{cases}$$

$$c_1(u_i u_j) = \begin{cases} (0.4,0.2,0.1), & ij = 23 \\ (0,1,0), & otherwise \end{cases} \quad c_2(u_i u_j) = \begin{cases} (0.3,0.3,0.2), & ij = 13 \\ (0,1,0), & otherwise \end{cases}$$

$$c_3(u_i u_j) = \begin{cases} (0.3,0.4,0.1), & ij = 41 \\ (0,1,0), & otherwise \end{cases} \quad c_4(u_i u_j) = \begin{cases} (0.2,0.4,0.1), & ij = 12 \\ (0.5,0.3,0.1), & ij = 34 \\ (0,1,0), & otherwise \end{cases}$$

Hence the family $C = \{c_1, c_2, c_3, c_4\}$satisfies our definition of total coloring of hesitancy fuzzy graph. We find that any family of hesitancy fuzzy sets having less than four members could not satisfy our definition. Hence in this case the total chromatic number $\chi^T(G)$ is 4.

Example 3.14. (Strong hesitancy fuzzy graph total coloring)
Consider the strong HFG $G = (V, E)$ given in Example 3.4.
Let $C = \{c_1, c_2, c_3, c_4\}$ be a family of hesitancy fuzzy sets defined on $V \cup E$ as follows

$$c_1(u_i) = \begin{cases} (0.3,0.6,0.1), & i = 1 \\ (0.6,0.1,0.2), & i = 3 \\ (0,1,0), & otherwise \end{cases} \quad c_2(u_i) = \begin{cases} (0.4,0.3,0.1), & i = 2 \\ (0.7,0.2,0.1), & i = 4 \\ (0,1,0), & otherwise \end{cases}$$

$$c_3(u_i u_j) = \begin{cases} (0.3,0.6,0.1), & ij = 12 \\ (0.6,0.2,0.1), & ij = 34 \\ (0,1,0), & otherwise \end{cases} \quad c_4(u_i u_j) = \begin{cases} (0.4,0.3,0.1), & ij = 23 \\ (0.3,0.6,0.1), & ij = 41 \\ (0,1,0), & otherwise \end{cases}$$

Hence the family $C = \{c_1, c_2, c_3, c_4\}$satisfies our definition of total coloring of strong hesitancy fuzzy graph. We find that any family of hesitancy fuzzy sets having less than four members could not satisfy our definition. Hence in this case the total chromatic number $\chi^T(G)$ is 4.

Example 3.15. (Complete hesitancy fuzzy graph total coloring)
Consider the strong HFG $G = (V, E)$ given in Example 3.5.
Let $C = \{c_1, c_2, c_3, c_4, c_5\}$ be a family of hesitancy fuzzy sets defined on $V \cup E$ as follows

$$c_1(u_i) = \begin{cases} (0.3,0.6,0.1), & i = 1 \\ (0,1,0), & otherwise \end{cases} \quad c_2(u_i) = \begin{cases} (0.2,0.5,0.2), & i = 2 \\ (0,1,0), & otherwise \end{cases}$$

$$c_3(u_i) = \begin{cases} (0.4,0.1,0.3), & i = 3 \\ (0,1,0), & otherwise \end{cases} \quad c_4(u_i) = \begin{cases} (0.5,0.2,0.2), & i = 4 \\ (0,1,0), & otherwise \end{cases}$$

$$c_1(u_i u_j) = \begin{cases} (0.2,0.5,0.2), & ij = 24 \\ (0,1,0), & otherwise \end{cases} \quad c_2(u_i u_j) = \begin{cases} (0.3,0.6,0.1), & ij = 13 \\ (0,1,0), & otherwise \end{cases}$$

$$c_3(u_i u_j) = \begin{cases} (0.3,0.6,0.1), & ij = 41 \\ (0,1,0), & otherwise \end{cases} \quad c_4(u_i u_j) = \begin{cases} (0.2,0.5,0.2), & ij = 23 \\ (0,1,0), & otherwise \end{cases}$$

$$c_5(u_i u_j) = \begin{cases} (0.2,0.6,0.1), & ij = 12 \\ (0.4,0.2,0.2), & ij = 34 \\ (0,1,0), & otherwise \end{cases}$$

Hence the family $C = \{c_1, c_2, c_3, c_4, c_5\}$ satisfies our definition of total coloring of complete hesitancy fuzzy graph. We find that any family of hesitancy fuzzy sets having less than five members could not satisfy our definition. Hence in this case the total chromatic number $\chi^T(G)$ is 5.

3.16. Bound for total chromatic number of HFG

Proposition 3.16.1. Any HFG $G = (V, E)$ can be totally colored using at most $\Delta + 2$ colors. That is, the total chromatic number of HFG $\chi^T(\hat{G}) \leq \Delta(G) + 2$ where Δ is the maximum number of edges incident from a vertex in G.

4. Conclusion

This paper tried to define the vertex, edge and total coloring for hesitancy fuzzy graph, strong hesitancy fuzzy graph and complete hesitancy fuzzy graph with elucidative examples and achieved the chromatic number as a crisp number. Moreover, it has also found bounds for that chromatic numbers.

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