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# Fuzzy Almost e*-continuous Mappings and Fuzzy e*-compactness in Smooth Topological Spaces

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***Abstract.*** In this paper, we introduce the concept of fuzzy almost $e^*$-continuous, fuzzy weakly $e^*$-continuous, $r$-fuzzy $e^*$-compact, $r$-fuzzy almost $e^*$-compactness and $r$-fuzzy near $e^*$-compactness in $\breve{S}$ostak's fuzzy topological spaces. We study some properties of them under several types of fuzzy $e^*$-continuous mappings and fuzzy $e^*$-regular spaces.

***Keywords:*** fuzzy (almost, weakly) $e^*$-continuous, fuzzy $e^*$-regular spaces, $r$-fuzzy (almost, near) $e^*$-compactness.

***AMS Mathematics Subject Classification (2010):*** 54A40, 54C05, 03E72

## 1. Introduction

Chang [1] introduced and studied the notion of fuzzy topology. Hutton [18], Lowen [20] and others discussed respectively various aspects of fuzzy topology. In these authors's papers, a fuzzy topology is defined as a classical subset of the fuzzy power set of a nonempty classical set. It is easy to see that they have always investigated fuzzy objects with crisp methods. However, fuzziness in the concept of openness of a fuzzy set has not been considered, which seems to be a drawback in the process of fuzzification of the concept of topological spaces. Therefore, $\breve{S}$ostak [27] introduced a new definition of fuzzy topology as an extension of both crisp topology and Chang's fuzzy topology. $\breve{S}$ostak [27-29] gave some rules and showed how such an extension can be realized. It has been developed in many direction [3-16,19,23-26, 30]. The concept of the degree of compactness of a fuzzy set was introduced by $\breve{S}$ostak [27]. Es[12] developed the several types of degree of fuzzy compactness and fuzzy continuous maps. In 2014, the concept of fuzzy $e$-open sets and fuzzy $e$-continuity and separation axioms and other properties were defined by Seenivasan and Kamala [31]. Sobana et al. [33] introduced the concept of $r$-fuzzy $e$-open sets and $r$-fuzzy $e$-continuity in $\breve{S}$os tak's fuzzy topological spaces. Vadivel et.al [35, 36] introduced the concept of fuzzy almost $e$-continuity, fuzzy $e$-compactness, $r$-fuzzy $e^*$-open and $r$-fuzzy $e^*$-closed sets in a fuzzy topological space in the sense of $\hat{S}$ostak [27]. In this paper, we introduce the concept of fuzzy almost $e^*$-continuous, $r$-fuzzy $e^*$-compact, $r$-fuzzy almost $e^*$-compactness and $r$-fuzzy near $e^*$-compactness in $\breve{S}$ostak's fuzzy topological spaces in a different view point [12,27]. We study some properties of them under several types of fuzzy $e$-continuous mappings of fuzzy topological spaces.

## 2. Preliminaries

Throughout this paper, nonempty sets will be denoted by $X, Y$ etc., $I = [0,1]$ and $I_0 = (0,1]$. For $\alpha \in I, \overline{\alpha}(x) = \alpha$ for all $x \in X$. A fuzzy point $x_t$ for $t \in I_0$ is an element of $I^X$ such that $x_t(y) = \{t$ y=x ; 0if y $\neq$ x.

The set of all fuzzy points in $X$ is denoted by $P_t(X)$. A fuzzy point $x_t \in \lambda$ iff $t < \lambda(x)$. A fuzzy set $\lambda$ is quasi-coincident with $\mu$, denoted by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. If $\lambda$ is not quasi-coincident with $\mu$, we denoted $\lambda \overline{q} \mu$. If $A \subset X$, we define the characteristic function $\chi_A$ on $X$ by $\chi_A(x) = \{1$ x in A , 0if x not in A.

Now, we introduce some basic notions and definitions that are used in the sequel.

**Definition 2.1.** *[27] A function $\tau : I^X \to I$ is called a smooth topology on $X$ if it satisfies the following conditions:*

1. $\tau(\overline{0}) = \tau(\overline{1}) = 1$, 2. $\tau(\bigvee_{i \in \Gamma} \mu_i) \geq \bigwedge_{i \in \Gamma} \tau(\mu_i)$, for any $\{\mu_i\}_{i \in \Gamma} \subset I^X$,

3. $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, for any $\mu_1, \mu_2 \in I^X$.

The pair $(X, \tau)$ is called a smooth topological space or fuzzy topological space( for short, fts).

**Definition 2.2.** *[24] Let $(X, \tau)$ be a fuzzy topological space. Then for each $r \in I_0$,* $\tau_r = \{\mu \in I^X : \tau(\mu) \geq r\}$ is a Chang's fuzzy topology on $X$.

**Theorem 2.1.** *[3] Let $(X, \tau)$ be a fts. Then for each $\lambda \in I^X, r \in I_0$ we define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows: $C_\tau(\lambda, r) = \bigwedge\{\mu \in I^X : \lambda \leq \mu, \tau(\overline{1} - \mu) \geq r\}$.*

For $\lambda, \mu \in I^X$ and $r, s \in I_0$, the operator $C_\tau$ satisfies the following conditions:

1. $C_\tau(\overline{0}, r) = \overline{0}$, 2. $\lambda \leq C_\tau(\lambda, r)$, 3. $C_\tau(\lambda, r) \vee C_\tau(\mu, r) = C_\tau(\lambda \vee \mu, r)$,

4. $C_\tau(\lambda, r) \leq C_\tau(\lambda, s)$ if $r \leq s$, 5. $C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

**Theorem 2.2.** *[17, 19] Let* $(X,\tau)$ *be a fts. Then for each* $r \in I_0, \lambda \in I^X$ *we define an operator* $I_\tau : I^X \times I_0 \to I^X$ *as follows* $I_\tau(\lambda, r) = \bigvee\{\mu \in I^X : \lambda \geq \mu, \tau(\mu) \geq r\}$.
For $\lambda, \mu \in I^X$ and $r, s \in I_0$, the operator $I_\tau$ satisfies the following conditions:
1. $I_\tau(\bar{1}, r) = \bar{1}$, 2. $\lambda \geq I_\tau(\lambda, r)$, 3. $I_\tau(\lambda, r) \wedge I_\tau(\mu, r) = I_\tau(\lambda \wedge \mu, r)$,
4. $I_\tau(\lambda, r) \leq I_\tau(\lambda, s)$ if $s \leq r$, 5. $I_\tau(I_\tau(\lambda, r), r) = I_\tau(\lambda, r)$,
6. $I_\tau(\bar{1} - \lambda, r) = \bar{1} - C_\tau(\lambda, r)$ and $C_\tau(\bar{1} - \lambda, r) = \bar{1} - I_\tau(\lambda, r)$.
7. If $S = \bigvee\{r \in I_0 : I_\tau(\mu, r) = \mu\}$, then $I_\tau(\mu, s) = \mu$.
8. $I_\tau(C_\tau(I_\tau(C_\tau(\lambda, r), r), r), r) = I_\tau(C_\tau(\lambda, r), r)$.

**Definition 2.3.** *[32] Let* $(X,\tau)$ *be a fts. For* $\lambda, \mu \in I^X$ *and* $r \in I_0$, $\lambda$ *is called* $r$*-fuzzy regular open (for short,* $r$*-fro) (resp.* $r$*-fuzzy regular closed (for short,* $r$*-frc)) if* $\lambda = I_\tau(C_\tau(\lambda, r), r)$ *(resp.* $\lambda = C_\tau(I_\tau(\lambda, r), r)$*).*

**Definition 2.4.** *[32] Let* $(X,\tau)$ *be a fts. Then for each* $\mu \in I^X$, $x_t \in P_t(X)$ *and* $r \in I_0$,

1. $\mu$ is called r-open $Q_\tau$-neighborhood of $x_t$ if $x_t q \mu$ with $\tau(\mu) \geq r$.
2. $\mu$ is called r-open $R_\tau$-neighborhood of $x_t$ if $x_t q \mu$ with $\mu = I_\tau(C_\tau(\lambda, r), r)$. We

denote $Q_\tau(x_t, r) = \{\mu \in I^X : x_t q \mu, \tau(\mu) \geq r\}$,
$R_\tau(x_t, r) = \{\mu \in I^X : x_t q \mu = I_\tau(C_\tau(\lambda, r), r)\}$.

**Definition 2.5.** *[32] Let* $(X,\tau)$ *be a fts. Then for each* $\lambda \in I^X$, $x_t \in P_t(X)$ *and* $r \in I_0$,
1. $x_t$ is called $r$-$\tau$ cluster point of $\lambda$ if for every $\mu \in Q_\tau(x_t, r)$, we have $\mu q \lambda$.
2. $x_t$ is called $r$-$\delta$ cluster point of $\lambda$ if for every $\mu \in R_\tau(x_t, r)$, we have $\mu q \lambda$.
3. An $\delta$-closure operator is a mapping $D_\tau : I^X \times I \to I^X$ defined as follows: $\delta C_\tau(\lambda, r)$ or $D_\tau(\lambda, r) = \bigvee\{x_t \in P_t(X) : x_t$ is r-$\delta$-cluster point of $\lambda\}$.

Equivalently, $\delta C_\tau(\lambda, r) = \bigwedge\{\mu \in I^X : \mu \geq \lambda, \mu$ isa $r-$frcset$\}$ and
$\delta I_\tau(\lambda, r) = \bigvee\{\mu \in I^X : \mu \leq \lambda, \mu$ isa $r-$froset$\}$.

**Definition 2.6.** *[19] Let* $(X,\tau)$ *be a fuzzy topological space. For* $\lambda \in I^X$ *and* $r \in I_0$,
1. $\lambda$ is called $r$-fuzzy $\delta$-closed iff $\lambda = D_\tau(\lambda, r)$.
2. The complement of $r$-fuzzy $\delta$-closed is $r$-fuzzy $\delta$-open.

**Definition 2.7.** *[36] Let* $(X, \tau)$ *be a fuzzy topological space. For* $\lambda, \mu \in I^X$ *and* $r \in I_0$*,* $\lambda$ *is called an* $r$*-fuzzy* $e^*$*-open (resp.* $r$*-fuzzy* $e^*$*-closed) set if* $\lambda \leq C_\tau(I_\tau(\delta\text{-}C_\tau(\lambda, r), r), r)$ *(resp.* $I_\tau(C_\tau(\delta\text{-}I_\tau(\lambda, r), r), r) \leq \lambda$).

**Definition 2.8.** *[36] Let* $(X, \tau)$ *be a fuzzy topological space.* $\lambda, \mu \in I^X$ *and* $r \in I_0$*,*
1. $e^* I_\tau(\lambda, r) = \bigvee\{\mu \in I^X : \mu \leq \lambda, \mu$ is a $r$-$fe^*o$ set $\}$ is called the $r$-fuzzy $e^*$-interior of $\lambda$ 2. $e^* C_\tau(\lambda, r) = \bigwedge\{\mu \in I^X : \mu \geq \lambda, \mu$ is a $r$-$fe^*c$ set $\}$ is called the $r$-fuzzy $e^*$-closure of $\lambda$.

**Definition 2.9.** *[37] Let* $(X, \tau)$ *and* $(Y, \eta)$ *be fts's. A mapping* $f : X \to Y$ *is called a fuzzy* $e^*$*-continuous iff* $f^{-1}(\mu)$ *is* $r$-fe$^*$o *set of* $X$ *for each* $\mu \in I^Y$ *with* $\eta(\mu) \geq r$.

**Theorem 2.3.** *[36] Let* $(X, \tau)$ *be a fts. For* $\lambda \in I^X$ *and* $r \in I_0$ *we have*
1. $e^* I_\tau(1-\lambda, r) = 1-(e^* C_\tau(\lambda, r))$. 2. $e^* C_\tau(1-\lambda, r) = 1-(e^* I_\tau(\lambda, r))$.

**Theorem 2.4.** *[34] Let* $(X, \tau)$ *be a smooth topological space. Then for each* $\lambda \in I^X$ *and* $r \in I_0$
1. $e^* I_\tau(e^* C_\tau(e^* I_\tau(e^* C_\tau(\lambda, r), r), r), r) = e^* I_\tau(e^* C_\tau(\lambda, r), r)$.
2. $e^* C_\tau(e^* I_\tau(e^* C_\tau(e^* I_\tau(\lambda, r), r), r), r) = e^* C_\tau(e^* I_\tau(\lambda, r), r)$.

## 3. Fuzzy almost $e^*$-continuous mappings

Now, we introduce the following definitions.

**Definition 3.1.** *Let* $(X, \tau)$ *and* $(Y, \eta)$ *be a fuzzy topological spaces and* $r \in I_0$*. A mapping* $f : X \to Y$ *is called*

1. fuzzy almost $e^*$-continuous if $f^{-1}(\lambda)$ is a $r$-fe$^*$o set in $X$ for each $\lambda \in I^Y$ with $\lambda = I_\eta(C_\eta(\lambda, r), r)$ or equivalently, $f^{-1}(\lambda)$ is a $r$-fe$^*$c set in $X$ for each $\lambda \in I^Y$ with $\lambda = C_\eta(I_\eta(\lambda, r), r)$.
2. fuzzy almost $e^*$-open map if $f(\lambda)$ is a $r$-fe$^*$o set in $Y$ for each $r$-fro set $\lambda$ in $X$.
3. fuzzy almost $e^*$-closed map if $f(\lambda)$ is a $r$-fe$^*$c set in $Y$ for each $r$-frc set $\lambda$ in $X$.

**Remark 3.1.** *Clearly, a fuzzy* $e$*-continuous mapping is fuzzy almost* $e$*-continuous.*

**Example 3.1.** *Let* $\lambda$, $\mu$, $\gamma$, $\delta$ *and* $\alpha$ *be fuzzy subsets of* $X = Y = \{a, b, c\}$ *defined as follows:* $\lambda(a) = 0.5$, $\lambda(b) = 0.3$, $\lambda(c) = 0.2$; $\mu(a) = 0.4$, $\mu(b) = 0.6$, $\mu(c) = 0.1$;
$\gamma(a) = 0.5$, $\gamma(b) = 0.6$, $\gamma(c) = 0.2$; $\delta(a) = 0.4$, $\delta(b) = 0.3$, $\delta(c) = 0.1$;
$\alpha(a) = 0.2$, $\alpha(b) = 0.4$, $\alpha(c) = 0.5$.

Then $\tau, \eta : I^X \to I$ defined as

$$\tau(\lambda)=\begin{cases}1, & \text{if } \lambda=\bar{0} \text{ or } \bar{1},\\ \frac{1}{2}, & \text{if } \lambda=\lambda,\mu,\gamma,\delta,\\ 0, & \text{otherwise},\end{cases}\quad \eta(\lambda)=\begin{cases}1, & \text{if } \lambda=\bar{0} \text{ or } \bar{1},\\ \frac{1}{2}, & \text{if } \lambda=\alpha,\\ 0, & \text{otherwise}.\end{cases}$$

are fuzzy topologies on $X$. For $r=\frac{1}{2}$, then for any fuzzy regular open set $\alpha$ in $Y$, the identity mapping $f:(X,\tau)\to(Y,\eta)$ is fuzzy almost $e^*$-continuous.

**Example 3.2.** *Let $\lambda$, $\mu$, $\gamma$ and $\delta$ be fuzzy subsets of $X=Y=\{a,b,c\}$ defined as follows:*

$\lambda(a)=0.4,\ \lambda(b)=0.5,\ \lambda(c)=0.2$; $\mu(a)=0.2,\ \mu(b)=0.3,\ \mu(c)=0.2$;
$\gamma(a)=0.4,\ \gamma(b)=0.6,\ \gamma(c)=0.5$; $\delta(a)=0.3,\ \delta(b)=0.5,\ \delta(c)=0.3$.

Then $\tau,\eta:I^X\to I$ defined as

$$\tau(\lambda)=\begin{cases}1, & \text{if } \lambda=\bar{0} \text{ or } \bar{1},\\ \frac{1}{2}, & \text{if } \lambda=\lambda,\mu,\\ 0, & \text{otherwise},\end{cases}\quad \eta(\lambda)=\begin{cases}1, & \text{if } \lambda=\bar{0} \text{ or } \bar{1},\\ \frac{1}{2}, & \text{if } \lambda=\gamma,\delta,\\ 0, & \text{otherwise}.\end{cases}$$

are fuzzy topologies on $X$. For $r=\frac{1}{2}$, then for any fuzzy regular open set $\lambda$ in $X$, the identity mapping $f:(X,\tau)\to(Y,\eta)$ is fuzzy almost $e^*$-open map.

**Theorem 3.1.** *Let $(X,\tau)$ and $(Y,\eta)$ be fts's. Let $f:(X,\tau)\to(Y,\eta)$ be a mapping. The following statements are equivalent:*

1. $f$ is fuzzy almost $e^*$-continuous.
2. $f^{-1}(\mu)$ is $r$-fe$^*$o set, for each $\mu\in I^Y$, $r\in I_0$ with $\mu=I_\eta(C_\eta(\mu,r),r)$.
3. $f^{-1}(\mu)\le e^*I_\tau(f^{-1}(I_\eta(C_\eta(\mu,r),r)),r)$, for each $\mu\in I^Y$, $r\in I_0$ with $\eta(\mu)\ge r$.
4. $e^*C_\tau(f^{-1}(C_\eta(I_\eta(\mu,r),r)),r)\le f^{-1}(\mu)$, for each $\mu\in I^Y$, $r\in I_0$ with $\eta(\bar{1}-\mu)\ge r$.

**Proof:** (1) $\Leftrightarrow$ (2): Follows from Definition 2.1

(1) $\Rightarrow$ (3): Let $f$ be almost $e^*$-continuous and $\mu$ be any $r$-fuzzy open set in $Y$. Then $\mu=I_\eta(\mu,r)\le I_\eta(C_\eta(\mu,r),r)$.

By Definition 1.3, $I_\eta(C_\eta(\mu,r),r)$ is a $r$-fuzzy regular open set in $Y$. Since $f$ is fuzzy almost $e^*$-continuous, $f^{-1}(I_\eta(C_\eta(\mu,r),r))$ is a $r$-fuzzy $e^*$-open set in X.

Hence $f^{-1}(\mu)\le f^{-1}(I_\eta(C_\eta(\mu,r),r))=e^*I_\tau(f^{-1}(I_\eta(C_\eta(\mu,r),r)),r)$.

(3) $\Rightarrow$ (4): Let $\mu$ be a $r$-fuzzy closed set of $Y$. Then $\mu^c$ is a $r$-fuzzy open set in $Y$.

By (3), $f^{-1}(\mu^c) \le e^* I_\tau(f^{-1}(I_\eta(C_\eta(\mu^c, r), r)), r)$. Hence,

$f^{-1}(\mu) = (f^{-1}(\mu^c))^c \ge (e^* I_\tau(f^{-1}(I_\eta(C_\eta(\mu^c, r), r)), r))^c = e^* C_\tau(f^{-1}(C_\eta(I_\eta(\mu, r), r)), r)$.

(4) $\Rightarrow$ (1): Let $\mu$ be a $r$-fuzzy regular closed set in $Y$. Then $\mu$ is a $r$-fuzzy closed set in $Y$ and hence $f^{-1}(\mu) \le e^* I_\tau(f^{-1}(I_\eta(C_\eta(\mu, r), r)), r) = e^* I_\tau(f^{-1}(\mu), r)$.

Thus $f^{-1}(\mu) = e^* C_\tau(f^{-1}(\mu), r)$ and hence $f^{-1}(\mu)$ is a $r$-fuzzy $e^*$-closed set in $X$. Therefore, $f$ is a fuzzy almost $e^*$-continuous map.

**Definition 3.2.** *A fts is called $r$-fuzzy $e^*$-regular iff for each $\tau(\mu) \ge r$,*

$$\mu = \bigvee \{\rho \in I^X \mid \tau(\rho) \ge r, I_\tau(C_\tau(\delta I_\tau(\rho, r), r), r) \le \mu\}.$$

A fuzzy topological space $(X, \tau)$ is called fuzzy $e^*$-regular iff it is $r$-fuzzy $e^*$-regular for each $r \in I_0$.

**Example 3.3.** *Let $\lambda$, $\mu$, $\gamma$ and $\delta$ be fuzzy subsets of $X = \{a, b, c\}$ defined as follows:*

$\lambda(a) = 0.8$, $\lambda(b) = 0.6$, $\lambda(c) = 0.6$;

$\mu(b) = 0.6$, $\mu(c) = 0.5$;

$\gamma(a) = 0.2$, $\gamma(b) = 0.4$, $\gamma(c) = 0.4$;

$\delta(b) = 0.6$, $\delta(c) = 0.5$.

Then $\tau : I^X \to I$ defined as

$$\tau(\lambda) = \begin{cases} 1, & \text{if} \lambda = \bar{0} \text{or} \bar{1}, \\ \frac{1}{2}, & \text{if} \lambda = \lambda, \mu, \gamma, \delta, \\ 0, & \text{otherwise}, \end{cases}$$

is a fuzzy topology on $X$. For $r = \frac{1}{2}$, then the topological space $(X, \tau)$ is both $\frac{1}{2}$-fuzzy $e^*$-regular and $\frac{1}{2}$-fuzzy regular.

**Remark 3.2.** *Clearly, $r$-fuzzy regular space is $r$-fuzzy $e^*$-regular space, but the converse is false, in general, is shown by the following example.*

**Example 3.4.** *Let $\lambda$, $\mu$, $\gamma$ and $\delta$ be fuzzy subsets of $X = \{a, b, c\}$ defined as follows:*

$\lambda(a) = 0.5$, $\lambda(b) = 0.6$, $\lambda(c) = 0.5$;

$\mu(b) = 0.4$, $\mu(c) = 0.4$;

$\gamma(a) = 0.2$, $\gamma(b) = 0.6$, $\gamma(c) = 0.5$;

$\delta(b) = 0.7$, $\delta(c) = 0.6$.

Then $\tau : I^X \to I$ defined as
$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \bar{0} \text{ or } \bar{1}, \\ \frac{1}{2}, & \text{if } \lambda = \lambda, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ \frac{1}{2}, & \text{if } \lambda = \gamma, \\ \frac{1}{2}, & \text{if } \lambda = \delta, \\ 0, & \text{otherwise}, \end{cases}$$

is a fuzzy topology on $X$. For $r = \frac{1}{2}$, then the topological space $(X, \tau)$ is $\frac{1}{2}$-fuzzy $e^*$-regular but not $\frac{1}{2}$-fuzzy regular. Since for any $\omega = (0.8_a, 0.7_b, 0.6_c) \in \tau$, $C_\tau(\lambda, \frac{1}{2}) = 1 > \omega.$

**Theorem 3.2.** *Let $(X, \tau)$ and $(Y, \eta)$ be fts's. Let $f : (X, \tau) \to (Y, \eta)$ be a mapping and $(Y, \eta)$ be a fuzzy $e^*$-regular space. Then $f$ is fuzzy almost $e^*$-continuous iff $f$ is fuzzy $e^*$-continuous.*

**Proof:** $(\Leftarrow)$ It is trivial from Remark 2.1.

$(\Rightarrow)$ Let $\lambda \in I^Y$. By the fuzzy $e^*$-regularity of $(Y, \eta)$, for we have $\lambda = \bigvee \lambda_\alpha$, where $\eta(\lambda_\alpha) \geq r$ and $I_\eta(C_\eta(\delta I_\eta(\lambda_\alpha, r), r), r) \leq \lambda.$

By Theorem 2.1(3),
$$\begin{aligned} f^{-1}(\lambda) &= \bigvee f^{-1}(\lambda_\alpha) \\ &\leq \bigvee e^* I_\tau(f^{-1}(I_\eta(C_\eta(\delta I_\eta(\lambda_\alpha, r), r), r)), r) \\ &\leq e^* I_\tau(f^{-1}(\lambda), r). \end{aligned}$$

But $e^* I_\tau(f^{-1}(\lambda), r) \leq f^{-1}(\lambda)$. Thus, $f^{-1}(\lambda) = e^* I_\tau(f^{-1}(\lambda), r)$. Hence f is fuzzy $e^*$-continuous.

**Definition 3.3.** *Let $(X, \tau)$ and $(Y, \eta)$ be fts's. A mapping $f : (X, \tau) \to (Y, \eta)$ is called fuzzy weakly $e^*$-continuous if for each $\mu \in I^Y$ and $r \in I_0$ with $\eta(\mu) \geq r$, we have $f^{-1}(\mu) \leq e^* I_\tau(f^{-1}(e^* C_\eta(\mu, r)), r)$.*

**Example 3.5.** *Let $\alpha$, $\beta$, $\gamma$, $\delta$ and $\omega$ be fuzzy subsets of $X = Y = \{a, b, c, d\}$ defined as follows:*
$\alpha(a) = 0.5,\ \alpha(b) = 0.4,\ \alpha(c) = 0.6,\ \alpha(d) = 0.4$;
$\beta(a) = 0.4,\ \beta(b) = 0.6,\ \beta(c) = 0.4,\ \beta(d) = 0.5$;
$\gamma(a) = 0.5,\ \gamma(b) = 0.6,\ \gamma(c) = 0.6,\ \gamma(d) = 0.5$;

$\delta(a)=0.4,\ \delta(b)=0.4,\ \delta(c)=0.4,\ \delta(d)=0.4$;

$\omega(a)=0.3,\ \omega(b)=0.4,\ \omega(c)=0.5,\ \omega(d)=0.5.$

Then $\tau,\eta: I^X \to I$ defined as

$$\tau(\lambda)=\begin{cases}1, & \text{if}\lambda=\bar{0}\text{or}\bar{1},\\ \frac{1}{2}, & \text{if}\lambda=\alpha,\beta,\gamma,\delta,\\ 0, & \text{otherwise,}\end{cases}\quad \eta(\lambda)=\begin{cases}1, & \text{if}\lambda=\bar{0}\text{or}\bar{1},\\ \frac{1}{2}, & \text{if}\lambda=\omega,\\ 0, & \text{otherwise.}\end{cases}$$

are fuzzy topologies on $X$. For $r=\frac{1}{2}$, then the identity mapping $f:(X,\tau)\to(Y,\eta)$ is fuzzy weakly $e^*$-continuous since, for any $r$-fuzzy open set $\omega$ in $Y$, $\omega$ is $r$-fe$^*$c in $Y$ and the inverse image of $\omega$ is $r$-fe$^*$o set in $X$. Also
$f^{-1}(\omega)\le e^*I_\tau(f^{-1}(e^*C_\eta(\omega,r)),r).$

**Theorem 3.3.** *Let* $f:(X,\tau)\to(Y,\eta)$ *be a mapping. Then the following are equivalent:* 1. $f$ is fuzzy weakly $e^*$-continuous.

2. $e^*C_\tau(f^{-1}(e^*I_\eta(\mu,r)),r)\le f^{-1}(\mu)$ for each $\mu\in I^Y$, $r\in I_0$ with $\eta(\bar{1}-\mu)\ge r$.

3. $f^{-1}(I_\eta(\mu,r))\le e^*I_\tau(f^{-1}(e^*C_\eta(\mu,r)),r)$, for each $\mu\in I^Y$, $r\in I_0$.

4. $e^*C_\tau(f^{-1}(e^*I_\eta(\mu,r)),r)\le f^{-1}(C_\eta(\mu,r))$, for each $\mu\in I^Y$, $r\in I_0$.

**Proof:** $(1)\Rightarrow(2)$: Let $\mu\in I^Y$, $r\in I_0$ with $\eta(\bar{1}-\mu)\ge r$. By (1) and Theorem 1.3,

$$\begin{aligned}\bar{1}-f^{-1}(\mu)=f^{-1}(\bar{1}-\mu)&\le e^*I_\tau(f^{-1}(e^*C_\eta(\bar{1}-\mu)),r)\\ &=e^*I_\tau(f^{-1}(\bar{1}-e^*I_\eta(\mu,r)),r)=e^*I_\tau(\bar{1}-f^{-1}(e^*I_\eta(\mu,r)),r)\\ &=\bar{1}-e^*C_\tau(f^{-1}(e^*I_\eta(\mu,r)),r)\end{aligned}$$

Thus, $e^*C_\tau(f^{-1}(e^*I_\eta(\mu,r)),r)\le f^{-1}(\mu)$.

$(2)\Rightarrow(1)$: It is analogous to the proof of $(1)\Rightarrow(2)$.

$(1)\Rightarrow(3)$: Let $\mu\in I^Y$ and $r\in I_0$. Since $\eta(I_\eta(\mu,r))\ge r$, by (1),

$$f^{-1}(I_\tau(\mu,r))\le e^*I_\tau(f^{-1}(e^*C_\eta(I_\eta(\mu,r),r)),r)\le e^*I_\tau(f^{-1}(e^*C_\eta(\mu,r)),r).$$

$(3)\Rightarrow(4)$ and $(4)\Rightarrow(2)$: Obvious.

$(2)\Rightarrow(4)$: It is analogous to the proof of $(1)\Rightarrow(3)$.

**Theorem 3.4.** *A mapping* $f:(X,\tau)\to(Y,\eta)$ *is fuzzy weakly* $e^*$*-continuous iff* $e^*C_\tau(f^{-1}(\mu),r)\le f^{-1}(e^*C_\eta(\mu,r))$ *for each* $\mu\in I^Y$, $r\in I_0$ *with* $\eta(\mu)\ge r$.

**Proof:** Let $\mu\in I^Y$, $r\in I_0$ with $\eta(\mu)\ge r$. Then, by the definition of $e^*C_\eta$, $\eta(\bar{1}-e^*C_\eta(\mu,r))\ge r$. By the fuzzy weakly $e^*$-continuity of $f$ we have,

$f^{-1}(\bar{1}-e^*C_\eta(\mu,r))\le e^*I_\tau(f^{-1}(e^*C_\eta(\bar{1}-e^*C_\eta(\mu,r),r)),r)$

Since $e^*I_\tau(f^{-1}(e^*C_\eta(\bar{1}-e^*C_\eta(\mu,r),r)),r)=\bar{1}-e^*C_\tau(f^{-1}(e^*I_\eta(e^*C_\eta(\mu,r),r)),r).$

We have, $\bar{1}-f^{-1}(e^*C_\eta(\mu,r))\le\bar{1}-e^*C_\tau(f^{-1}(e^*I_\eta(e^*C_\eta(\mu,r),r)),r).$

It implies, $e^*C_\tau(f^{-1}(e^*I_\eta(e^*C_\eta(\mu,r),r)),r)\le f^{-1}(e^*C_\eta(\mu,r)).$

Since $\mu\le e^*I_\eta(e^*C_\eta(\mu,r),r),$ we have, $e^*C_\tau(f^{-1}(\mu),r)\le f^{-1}(e^*C_\eta(\mu,r)).$

Conversely, let $\mu\in I^Y$, $r\in I_0$ with $\eta(\mu)\ge r.$ Then $\eta(\bar{1}-e^*C_\eta(\mu,r))\ge r.$

By hypothesis, $e^*C_\tau(f^{-1}(\bar{1}-e^*C_\eta(\mu,r)),r)\le f^{-1}(e^*C_\eta(\bar{1}-e^*C_\eta(\mu,r),r))$

$\Rightarrow e^*C_\tau(\bar{1}-f^{-1}(e^*C_\eta(\mu,r)),r)\le f^{-1}(\bar{1}-e^*I_\eta(e^*C_\eta(\mu,r),r))$

$\Rightarrow \bar{1}-e^*I_\tau(f^{-1}(e^*C_\eta(\mu,r)),r)\le\bar{1}-f^{-1}(e^*I_\eta(e^*C_\eta(\mu,r),r))$

$\Rightarrow f^{-1}(e^*I_\eta(e^*C_\eta(\mu,r),r))\le e^*I_\tau(f^{-1}(e^*C_\eta(\mu,r)),r)$

$\Rightarrow f^{-1}(\mu)\le e^*I_\tau(f^{-1}(e^*C_\eta(\mu,r)),r).$

Hence $f$ is fuzzy weakly $e^*$-continuous.

### 3.1. Fuzzy almost $e^*$-compactness

The study of fuzzy almost compact spaces was initiated in [2]. The same concept was taken up for further study in [21] and in [22]. The notion was extended to arbitrary fuzzy sets in a fts. Now we introduce the following definitions.

**Definition 3.1.1.** *Let $(X,\tau)$ be a fts and $r\in I_0$. A fuzzy set $\mu\in I^X$ is called $r$-fuzzy $e^*$-compact in $(X,\tau)$ iff for each family $\{\lambda_i\in I^X \mid \lambda_i$ is $r$-$fe^*o$, $i\in\Gamma\}$ such that $\mu\le\bigvee_{i\in\Gamma}\lambda_i$ there exists a finite index set $\Gamma_0\subset\Gamma$ such that $\mu\le\bigvee_{i\in\Gamma_0}\lambda_i$. $(X,\tau)$ is called $r$-fuzzy $e^*$-compact iff $\bar{1}$ is $r$-fuzzy $e^*$-compact in $(X,\tau)$.*

**Definition 3.1.2.** *Let $(X,\tau)$ be a fts and $r\in I_0$. A fuzzy set $\mu\in I^X$ is called $r$-fuzzy almost $e^*$-compact in $(X,\tau)$ iff for each family $\{\lambda_i\in I^X \mid \lambda_i$ is $r$-$fe^*o$, $i\in\Gamma\}$ such that $\mu\le\bigvee_{i\in\Gamma}\lambda_i$ there exists a finite index set $\Gamma_0\subset\Gamma$ such that $\mu\le\bigvee_{i\in\Gamma_0}e^*C_\tau(\lambda_i,r)$. $(X,\tau)$ is called $r$-fuzzy almost $e^*$-compact iff $\bar{1}$ is r-fuzzy almost $e^*$-compact in $(X,\tau)$.*

**Theorem 3.1.1.** *Let $(X,\tau)$ be a fts and $r\in I_0$. Then the following statements are equivalent:* 1. $X$ is $r$-fuzzy almost $e^*$-compact.

2. For every collection $\{\lambda_i \in I^X \mid e^*C_\tau(e^*I_\tau(\lambda_i, r), r) = \lambda_i, i \in \Gamma\}$ with $\bigwedge_{i\in\Gamma} \lambda_i = \bar{0},$ there exists a finite subset $\Gamma_0 \subseteq \Gamma$ with $\bigwedge_{i\in\Gamma_0} e^*I_\tau(\lambda_i, r) = \bar{0}.$

3. $\bigwedge_{i\in\Gamma} e^*C_\tau(\lambda_i, r) \neq \bar{0}$ holds for every collection of $\{\lambda_i \in I^X \mid$ $\lambda_i = e^*I_\tau(e^*C_\tau(\lambda_i, r), r), i \in \Gamma\}$ with the finite intersection property (FIP, for short), that is, for each finite subset $\Gamma_0 \subset \Gamma$ with $\bigwedge_{i\in\Gamma} \lambda_i \neq \bar{0}.$

4. Every family of $\{\lambda_i \in I^X \mid e^*I_\tau(e^*C_\tau(\lambda_i, r), r) = \lambda_i, i \in \Gamma\}$ such that $\bigvee_{i\in\Gamma} \lambda_i = \bar{1},$ there exists a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\bigvee_{i\in\Gamma} e^*C_\tau(\lambda_i, r) = \bar{1}.$

**Proof:** $(1) \Rightarrow (2):$ Let $(X, \tau)$ be $r$-fuzzy almost $e^*$-compact and the collection $\{\lambda_i \in I^X \mid e^*C_\tau(e^*I_\tau(\lambda_i, r), r) = \lambda_i, i \in \Gamma\}$ with $\bigwedge_{i\in\Gamma} \lambda_i = \bar{0}.$ Then $\bigvee_{i\in\Gamma} (\bar{1} - \lambda_i) = \bar{1}.$ Since $\bar{1} - \lambda_i = e^*I_\tau(e^*C_\tau(\bar{1} - \lambda_i, r), r),$ we have $\bigvee_{i\in\Gamma} e^*I_\tau(e^*C_\tau(\bar{1} - \lambda_i, r), r) = \bar{1}$ and $e^*I_\tau(e^*C_\tau(\bar{1} - \lambda_i, r), r)$ is $r$-fe$^*$o. From the $r$-fuzzy almost $e^*$-compactness, it follows that there exists a finite subset $\Gamma_0$ of $\Gamma$ such that $\bigvee_{i\in\Gamma} e^*C_\tau(e^*I_\tau(e^*C_\tau(\bar{1} - \lambda_i, r), r), r) = \bar{1}$ and then

$$\bar{0} = \bar{1} - \bigvee_{i\in\Gamma} e^*C_\tau(e^*I_\tau(e^*C_\tau(\bar{1} - \lambda_i, r), r), r) = \bigwedge_{i\in\Gamma} \bar{1} - e^*C_\tau(e^*I_\tau(e^*C_\tau(\bar{1} - \lambda_i, r), r), r)$$
$$= \bigwedge_{i\in\Gamma} e^*I_\tau(e^*C_\tau(e^*I_\tau(\lambda_i, r), r), r) = \bigwedge_{i\in\Gamma} e^*I_\tau(\lambda_i, r).$$

$(2) \Rightarrow (3):$ Let $\{\lambda_i \in I^X \mid e^*I_\tau(e^*C_\tau(\lambda_i, r), r) = \lambda_i,\ i \in \Gamma\}$ with FIP. Suppose that $\bigwedge_{i\in\Gamma} e^*C_\tau(\lambda_i, r) = \bar{0}.$ Now $e^*C_\tau(\lambda_i, r) = e^*C_\tau(e^*I_\tau(e^*C_\tau(\lambda_i, r), r), r)$ and by assumption, there exists a finite subset $\Gamma_0$ of $\Gamma$ such that

$\bigwedge_{i\in\Gamma_0} \lambda_i = \bigwedge_{i\in\Gamma_0} e^*I_\tau(e^*C_\tau(\lambda_i, r), r) = \bar{0}.$ It is a contradiction.

$(3) \Rightarrow (4):$ Let $\{\lambda_i \in I^X \mid e^*I_\tau(e^*C_\tau(\lambda_i, r), r) = \lambda_i, i \in \Gamma\}$ such that $\bigvee_{i\in\Gamma} \lambda_i = \bar{1}.$ Suppose $\bigvee_{i\in\Gamma_0} e^*C_\tau(\lambda_i, r) \neq \bar{1}$ for every finite subset $\Gamma_0$ of $\Gamma.$

Then $e^*I_\tau(e^*C_\tau(\bar{1} - e^*C_\tau(\lambda_i, r), r), r) = \bar{1} - e^*C_\tau(\lambda_i, r),\ i \in \Gamma$ with FIP,

Since $e^*I_\tau(e^*C_\tau(\bar{1} - e^*C_\tau(\lambda_i, r), r), r) = e^*I_\tau(\bar{1} - e^*I_\tau(e^*C_\tau\lambda_i, r), r), r)$
$$= e^*I_\tau(\bar{1} - \lambda_i, r) = \bar{1} - e^*C_\tau(\lambda_i, r)$$

By(3), we obtain $\bigwedge_{i\in\Gamma} e^*C_\tau(\bar{1}-e^*C_\tau(\lambda_i,r),r)\neq\bar{0}\Rightarrow\bigwedge_{i\in\Gamma}\bar{1}-e^*I_\tau(e^*C_\tau(\lambda_i,r),r)\neq\bar{0}.\Rightarrow$ $\bigvee_{i\in\Gamma}\lambda_i=\bigvee_{i\in\Gamma} e^*I_\tau(e^*C_\tau(\lambda_i,r),r)\neq\bar{1}.$ It is a contradiction.

$(4)\Rightarrow(1)$: Let $\{\lambda_i\in I^X:\lambda_i$ is $r$-fe$^*$o, $i\in\Gamma\}$ with $\bigvee_{i\in\Gamma}\lambda_i=\bar{1}.$ It implies $\bigvee_{i\in\Gamma_0} e^*I_\tau(e^*C_\tau(\lambda_i,r),r)=\bar{1}$ with by Theorem 1.4,

$$e^*I_\tau(e^*C_\tau(e^*I_\tau(e^*C_\tau(\lambda_i,r),r),r),r)=e^*I_\tau(e^*C_\tau(\lambda_i,r),r)$$

From the hypothesis, it follows that there exists a finite subset $\Gamma_0$ of $\Gamma$ with $\bigvee_{i\in\Gamma_0} e^*C_\tau(e^*I_\tau(e^*C_\tau(\lambda_i,r),r),r)=\bar{1}.$ Since $\lambda_i\leq e^*I_\tau(e^*C_\tau(\lambda_i,r),r)\leq e^*C_\tau(\lambda_i,r),$ then $e^*C_\tau(e^*I_\tau(e^*C_\tau(\lambda_i,r),r),r)=e^*C_\tau(\lambda_i,r).$ Hence $\bigvee_{i\in\Gamma_0} e^*C_\tau(\lambda_i,r)=\bar{1},$ i.e $X$ is $r$-fuzzy almost $e^*$-compact.

**Theorem 3.1.2.** *A fuzzy weakly $e^*$-continuous image of $r$-fuzzy $e^*$-compact set is $r$-fuzzy almost $e^*$-compact.*

**Proof:** Let $\mu$ be $r$-fuzzy $e^*$-compact in $(X,\tau)$ and $f:(X,\tau)\to(Y,\eta)$ be a fuzzy weakly $e^*$-continuous mapping. If $\{\lambda_i\in I^Y\mid\lambda_i$ is $r$-fe$^*$o, i $\in\Gamma\}$ with $f(\mu)\leq\bigvee_{i\in\Gamma}\lambda_i,$ then $\mu\leq\bigvee_{i\in\Gamma} f^{-1}(\lambda_i).$ Since $f$ is fuzzy weakly $e^*$-continuous, then for each $i\in\Gamma,$ $f^{-1}(\lambda_i)\leq e^*I_\tau(f^{-1}(e^*C_\eta(\lambda_i,r)),r).$ Hence $\mu\leq\bigvee_{i\in\Gamma} e^*I_\tau(f^{-1}(e^*C_\eta(\lambda_i,r)),r),$ $e^*I_\tau(f^{-1}(e^*C_\eta(\lambda_i,r)),r)$ is $r$-fe$^*$o. Since $\mu$ is $r$-fuzzy $e^*$-compact in $(X,\tau),$ there exists a finite subset $\Gamma_0$ of $\Gamma$ with $\mu\leq\bigvee_{i\in\Gamma_0} e^*I_\tau(f^{-1}(e^*C_\eta(\lambda_i,r)),r).$ It follows that

$$\begin{aligned} f(\mu) &\leq f\left(\bigvee_{i\in\Gamma_0} e^*I_\tau(f^{-1}(e^*C_\eta(\lambda_i,r)),r)\right)\\ &=\bigvee_{i\in\Gamma_0} f\left(e^*I_\tau(f^{-1}(e^*C_\eta(\lambda_i,r)),r)\right)\\ &\leq\bigvee_{i\in\Gamma_0} f\left(f^{-1}(e^*C_\eta(\lambda_i,r))\right)\\ &\leq\bigvee_{i\in\Gamma_0} e^*C_\eta(\lambda_i,r)\end{aligned}$$

Hence $f(\mu)$ is $r$-fuzzy almost $e^*$-compact in $(Y,\eta).$

**Theorem 3.1.3.** *A fuzzy almost $e^*$-continuous image of $r$-fuzzy almost $e^*$-compact set is $r$-fuzzy almost $e^*$-compact.*

**Proof:** Similar to the proof of Theorem 3.2.

**Definition 3.1.3.** *A mapping* $f:(X,\tau)\to(Y,\eta)$ *is said to be fuzzy strongly* $e^*$*-continuous iff* $f(e^*C_\tau(\lambda,r))\le f(\lambda)$*, for all* $\lambda\in I^X$ *and* $r\in I_0$.

**Theorem 3.1.4.** *A fuzzy strongly* $e^*$*-continuous image of* $r$*-fuzzy almost* $e^*$*-compact set is* $r$*-fuzzy* $e^*$*-compact.*

**Proof:** Let $\mu\in I^X$ be $r$-fuzzy almost $e^*$-compact in $(X,\tau)$ and $f:(X,\tau)\to(Y,\eta)$ a fuzzy strongly $e^*$-continuous mapping. If $\{\lambda_i\in I^Y \mid \lambda_i$ is $r$-fe$^*$o, $i\in\Gamma\}$ with $f(\mu)\le\bigvee_{i\in\Gamma}\lambda_i$, then $\mu\le\bigvee_{i\in\Gamma} f^{-1}(\lambda_i)$. Since $f$ is fuzzy strongly $e^*$-continuous and hence $e^*$-continuous, $f^{-1}(\lambda_i)$ is $r$-fe$^*$o. Since $\mu$ is $r$-fuzzy almost $e^*$-compact, there exists a finite subset $\Gamma_0$ of $\Gamma$ with $\mu\le\bigvee_{i\in\Gamma_0} e^*C_\tau(f^{-1}(\lambda_i),r)$. From the fuzzy strong $e^*$-continuity of $f$, we deduce

$$\begin{aligned}
f(\mu) &\le f\left(\bigvee_{i\in\Gamma_0} e^*C_\tau(f^{-1}(\lambda_i),r)\right)\\
&= \bigvee_{i\in\Gamma_0} f\left(e^*C_\tau(f^{-1}(\lambda_i),r)\right)\\
&\le \bigvee_{i\in\Gamma_0} f\left(f^{-1}(\lambda_i)\right)\\
&\le \bigvee_{i\in\Gamma_0} \lambda_i
\end{aligned}$$

Hence $f(\mu)$ is $r$-fuzzy $e^*$-compact in $(Y,\eta)$.

### 3.2. Fuzzy near $e^*$-compactness

In this section, we introduce fuzzy nearly $e^*$-compactness and investigate the properties and some of their characterizations.

**Definition 3.2.1.** *Let* $(X,\tau)$ *be a fts and* $r\in I_0$*. A fuzzy set* $\mu\in I^X$ *is called* $r$*-fuzzy nearly* $e^*$*-compact in* $(X,\tau)$ *iff for each family* $\{\lambda_i\in I^X \mid \lambda_i$ *is* $r$-fe$^*$o*,* $i\in\Gamma\}$ *such that* $\mu\le\bigvee_{i\in\Gamma}\lambda_i$*, there exists a finite index set* $\Gamma_0\subset\Gamma$ *such that*

$\mu\le\bigvee_{i\in\Gamma_0} e^*I_\tau(e^*C_\tau(\lambda_i,r),r)$. $(X,\tau)$ *is called* $r$*-fuzzy nearly* $e^*$*-compact iff* $\bar{1}$ *is* $r$*-fuzzy nearly* $e^*$*-compact in* $(X,\tau)$.

Clearly, we have the following implications:

$r$-fuzzy $e^*$-compactness $\Rightarrow r$-fuzzy nearly $e^*$-compactness $\Rightarrow r$-fuzzy almost $e^*$-compactness.

**Theorem 3.2.1.** *Let $(X,\tau)$ be $r$-fuzzy $e^*$-regular. It is $r$-fuzzy almost $e^*$-compact iff it is $r$-fuzzy $e^*$-compact.*

**Proof:** Let $\{\lambda_i \in I^X \mid \lambda_i$ is $r$-fe$^*$o, $i \in \Gamma\}$ be a family such that with $\bigvee_{i\in\Gamma} \lambda_i = \bar{1}$. Since $(X,\tau)$ is $r$-fuzzy $e^*$-regular, for each $\tau(\lambda_i) \geq r$,

$$\lambda_i = \bigvee_{i_k \in K_i} \{\lambda_{i_k} \mid \tau(\lambda_{i_k}) \geq r, I_\tau(C_\tau(\delta I_\tau(\lambda_{i_k}, r), r), r) \leq \lambda_i\}$$

Hence $\bigvee_{i\in\Gamma}\left(\bigvee_{i_k\in K_i} \lambda_{i_k}\right) = \bar{1}$. Since $(X,\tau)$ is $r$-fuzzy almost $e^*$-compact, there exists a finite index $J \in K_J$ such that $\bar{1} = \bigvee_{j\in J}\left(\bigvee_{j_k\in K_J} e^*C_\tau(\lambda_{j_k}, r)\right)$.

For $j \in J$, since $\bigvee_{j_k\in K_J} e^*C_\tau(\lambda_{j_k}, r) \leq \lambda_j$, we have $\bigvee_{j\in J} \lambda_j = \bar{1}$. Hence $(X,\tau)$ is $r$-fuzzy $e^*$-compact.

**Corollary 3.2.1.** *Let $(X,\tau)$ be $r$-fuzzy $e^*$-regular. It is $r$-fuzzy nearly $e^*$-compact iff it is $r$-fuzzy $e^*$-compact.*

**Proof:** It is straightforward.

**Theorem 3.2.2.** *A fts $(X,\tau)$ is $r$-fuzzy nearly $e^*$-compact iff every collection*

$$\{\lambda_i \in I^X \mid e^*C_\tau(e^*I_\tau(\lambda_i, r), r) = \lambda_i, i \in \Gamma\}$$

*having the FIP, we have $\bigwedge_{i\in\Gamma} \lambda_i \neq \bar{0}$.*

**Proof:** Let $\{\lambda_i \in I^X \mid e^*C_\tau(e^*I_\tau(\lambda_i, r), r) = \lambda_i,\ i \in \Gamma\}$ with FIP. If $\bigwedge_{i\in\Gamma} \lambda_i = \bar{0}$, then $\bigvee_{i\in\Gamma} (\bar{1} - \lambda_i) = \bar{1}$ and $\lambda_i$ is $r$-fe$^*$o set. From the $r$-fuzzy near $e^*$-compactness of $(X,\tau)$, there exists a finite subset $\Gamma_0$ of $\Gamma$ with $\bigvee_{i\in\Gamma_0} e^*I_\tau(e^*C_\tau(\bar{1} - \lambda_i, r), r) = \bar{1}$. Since

$$e^*I_\tau(e^*C_\tau(\bar{1} - \lambda_i, r), r) = e^*I_\tau(\bar{1} - e^*I_\tau(\lambda_i, r), r) = \bar{1} - e^*C_\tau(e^*I_\tau(\lambda_i, r), r) = \bar{1} - \lambda_i.$$

It follows that $\bigvee_{i\in\Gamma_0} \bar{1} - \lambda_i = \bar{1}$. Hence $\bigwedge_{i\in\Gamma_0} \lambda_i = \bar{0}$. It is a contradiction.

Conversely, let $\{\lambda_i \in I^X : \lambda_i$ is $r$-fe$^*$o, $i \in \Gamma\}$ with $\bigvee_{i\in\Gamma} \lambda_i = \bar{1}$. Suppose that $\bigvee_{i\in\Gamma_0} e^* I_\tau(e^* C_\tau(\lambda_i, r), r) \neq \bar{1}$, for every finite subset $\Gamma_0$ of $\Gamma$. Then by Theorem 1.4

$$e^* C_\tau(e^* I_\tau(\bar{1} - e^* I_\tau(e^* C_\tau(\lambda_i, r), r), r), r) = \bar{1} - e^* I_\tau(e^* C_\tau(\lambda_i, r), r),$$

for each $i \in \Gamma$ with the FIP. From the hypothesis, it follows that

$\bigwedge_{i\in\Gamma} \bar{1} - e^* I_\tau(e^* C_\tau(\lambda_i, r), r) \neq 0 \Rightarrow \bigvee_{i\in\Gamma_0} e^* I_\tau(e^* C_\tau(\lambda_i, r), r) \neq \bar{1}$.

Hence $\bigvee_{i\in\Gamma} \lambda_i \leq \bigvee_{i\in\Gamma} e^* I_\tau(e^* C_\tau(\lambda_i, r), r) \neq \bar{1}$. It is a contradiction.

**Theorem 3.2.3.** *The image of a $r$-fuzzy nearly $e^*$-compact set under a fuzzy almost $e^*$-continuous and fuzzy almost $e^*$-open mapping is $r$-fuzzy nearly $e^*$-compact.*

**Proof:** Let $\mu \in I^X$ be $r$-fuzzy nearly $e^*$-compact in $(X, \tau)$ and $f : (X, \tau) \to (Y, \eta)$ be a fuzzy almost $e^*$-continuous and fuzzy almost $e^*$-open mapping. If $\{\lambda_i \in I^X \mid \lambda_i$ is $r$-fe$^*$o, $i \in \Gamma\}$ with $f(\mu) \leq \bigvee_{i\in\Gamma} \lambda_i$, then for each $i \in \Gamma$, $e^* I_\eta(e^* C_\eta(\lambda_i, r), r)$ is $r$-fe$^*$o set. From the fuzzy almost $e^*$-continuity of $f$, it follows that

$$\mu \leq \bigvee_{i\in\Gamma} (f^{-1}(e^* I_\eta(e^* C_\eta(\lambda_i, r), r)),\ f^{-1}(e^* I_\eta(e^* C_\eta(\lambda_i, r), r))$$

is $r$-fe$^*$o in $X$. Since $\mu$ is $r$-fuzzy nearly $e^*$-compact in $(X, \tau)$, there exists a finite subset $\Gamma_0$ of $\Gamma$ such that $\mu \leq \bigvee_{i\in\Gamma_0} e^* I_\tau(e^* C_\tau(f^{-1}(e^* I_\eta(e^* C_\eta(\lambda_i, r)), r), r), r)$.

$$\text{Thus } f(\mu) \leq f\left(\bigvee_{i\in\Gamma_0} e^* I_\tau(e^* C_\tau(f^{-1}(e^* I_\eta(e^* C_\eta(\lambda_i, r), r)), r), r)\right)$$

$$= \bigvee_{i\in\Gamma_0} f\left(e^* I_\tau(e^* C_\tau(f^{-1}(e^* I_\eta(e^* C_\eta(\lambda_i, r), r)), r), r)\right)$$

From $e^* I_\eta(e^* C_\eta(\lambda_i, r), r) \leq e^* C_\eta(\lambda_i, r)$ for every $i \in \Gamma$ and the almost $e^*$ continuity of $f$, it follows that $f^{-1}(e^* I_\eta(e^* C_\eta(\lambda_i, r), r)) \leq f^{-1}(e^* C_\eta(\lambda_i, r)), (\bar{1} - f^{-1}(e^* C_\eta(\lambda_i, r)))$ is $r$-fe$^*$o in $X$.

$\Rightarrow e^* C_\tau(f^{-1}(e^* I_\eta(e^* C_\eta(\lambda_i, r), r)), r) \leq f^{-1}(e^* C_\eta(\lambda_i, r)),$

$\Rightarrow e^* I_\tau(e^* C_\tau(f^{-1}(e^* I_\eta(e^* C_\eta(\lambda_i, r), r)), r), r) \leq f^{-1}(e^* C_\eta(\lambda_i, r)).$

Since $f$ is fuzzy almost $e^*$-open and

$$e^* I_\tau(e^* C_\tau(f^{-1}(e^* I_\eta(e C_\eta(\lambda_i, r), r)), r), r) = e^* I_\tau(e^* C_\tau(e^* I_\tau(e^* C_\tau(f^{-1}(e^* I_\eta(e^* C_\eta(\lambda_i, r), r)), r), r), r), r).$$

We have $f\left(e^* I_\tau(e^* C_\tau(f^{-1}(e^* I_\eta(e^* C_\eta(\lambda_i, r), r)), r), r)\right)$ is $r$-fe$^*$o in $Y$.

Now from

$$f\left(e^{*}I_{\tau}(e^{*}C_{\tau}(f^{-1}(e^{*}I_{\eta}(e^{*}C_{\eta}(\lambda_i,r),r)),r),r)\right)\leq f(f^{-1}(e^{*}C_{\eta}(\lambda_i,r)))\leq e^{*}C_{\eta}(\lambda_i,r).$$

We deduce for each $i\in\Gamma$,

$$f\left(e^{*}I_{\tau}(e^{*}C_{\tau}(f^{-1}(e^{*}I_{\eta}(e^{*}C_{\eta}(\lambda_i,r),r)),r),r)\right)\leq e^{*}I_{\eta}(e^{*}C_{\eta}(\lambda_i,r),r).$$

Hence

$$f(\mu)\leq\bigvee_{i\in\Gamma_0} f\left(e^{*}I_{\tau}(e^{*}C_{\tau}(f^{-1}(e^{*}I_{\eta}(e^{*}C_{\eta}(\lambda_i,r),r)),r),r)\right)\leq\bigvee_{i\in\Gamma_0} e^{*}I_{\eta}(e^{*}C_{\eta}(\lambda_i,r),r).$$

Thus $f(\mu)$ is $r$-fuzzy nearly $e^{*}$-compact in $(Y,\eta)$.

**Theorem 3.2.4.** *The image of a $r$-fuzzy $e^{*}$-compact set under a fuzzy almost $e$-continuous mapping is $r$-fuzzy nearly $e^{*}$-compact.*

**Proof:** Let $\mu\in I^{X}$ be $r$-fuzzy $e^{*}$-compact in $(X,\tau)$ and $f:(X,\tau)\to(Y,\eta)$ be a fuzzy almost $e^{*}$-continuous mapping. If $\{\lambda_i\in I^{X}\mid\lambda_i$ is $r$-$\text{f}e^{*}\text{o}$, $i\in\Gamma\}$ with $f(\mu)\leq\bigvee_{i\in\Gamma}\lambda_i$, then $f(\mu)\leq\bigvee_{i\in\Gamma} e^{*}I_{\eta}(e^{*}C_{\eta}(\lambda_i,r),r)$ such that, by Theorem 1.4, for each $i\in\Gamma$, $e^{*}I_{\eta}(e^{*}C_{\eta}(e^{*}I_{\eta}(e^{*}C_{\eta}(\lambda_i,r),r),r),r)=e^{*}I_{\eta}(e^{*}C_{\eta}(\lambda_i,r),r)$.
From the almost $e^{*}$-continuity of $f$, it follows that $\mu\leq\bigvee_{i\in\Gamma} f^{-1}(e^{*}I_{\eta}(e^{*}C_{\eta}(\lambda_i,r),r))$, $f^{-1}(e^{*}I_{\eta}(e^{*}C_{\eta}(\lambda_i,r),r))$ is $r$-$\text{f}e^{*}\text{o}$ in $X$. Since $\mu$ is $r$-fuzzy $e^{*}$-compact in $(X,\tau)$, there exists a finite subset $\Gamma_0$ of $\Gamma$ such that

$$\mu\leq\bigvee_{i\in\Gamma_0} e^{*}I_{\tau}(e^{*}C_{\tau}(f^{-1}(e^{*}I_{\eta}(e^{*}C_{\eta}(\lambda_i,r),r)),r),r).$$

Thus, $f(\mu)\leq f\left(\bigvee_{i\in\Gamma_0} f^{-1}(e^{*}I_{\eta}(e^{*}C_{\eta}(\lambda_i,r),r))\right)=\bigvee_{i\in\Gamma_0} f\left(f^{-1}(e^{*}I_{\eta}(e^{*}C_{\eta}(\lambda_i,r),r))\right)$
$\leq\bigvee_{i\in\Gamma_0} e^{*}I_{\eta}(e^{*}C_{\eta}(\lambda_i,r),r)$. Thus $f(\mu)$ is $r$-fuzzy nearly $e^{*}$-compact in $(Y,\eta)$.

## 4. Conclusion

$\breve{S}$ostak's fuzzy topology has been recently of major interest among fuzzy topologies. In this paper, we have introduced fuzzy almost $e^{*}$-continuous, fuzzy weakly $e^{*}$-continuous, $r$-fuzzy $e^{*}$-compact, $r$-fuzzy almost $e^{*}$-compactness and $r$-fuzzy near compactness in $\breve{S}$ostak's fuzzy topological spaces. Also, we have studied some properties of them under several types of fuzzy $e^{*}$-continuous mappings and fuzzy $e^{*}$-regular spaces. Finally, we have given some counter examples to show that these types of mappings are not equivalent. These results will help to extend some generalized continuous mappings, compactness and hence it will help to improve smooth topological and bi-topological spaces.

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