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Hermite Interpolation with Triangular Fuzzy Numbers

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Abstract. In this paper the polynomial interpolation of triangular fuzzy number is discussed. First general form of the polynomial with fuzzy coefficients is proposed. The hermite interpolation method is studied with triangular fuzzy number an example is provided to illustrate the algorithm.

Keywords: Interpolation, polynomial ,triangular fuzzy numbers

AMS Mathematics Subject Classification (2010): 03E72, 15B15

1. Introduction

In some problem, people have to deal with massive imprecise data usually represented them as triangular fuzzy numbers. Many researchers focus on fuzzy numbers, for the convenience in dealing with uncertainty. people call this 'fuzzy mathematics'.

Interpolation method has become very popular method in engineering, economy, and computer science etc. Polynomial interpolation is one of the most common methods while some times we do not know the exact value of the measured quantities because of the inevitable measurement inaccuracy. We can describe the uncertainty by using fuzzy numbers. Chenyi Hu Anelina etc have discussed the interval polynomial interpolation problem and its lagrange form[6].

In this paper, we study the theory and algorithm of interpolatin problem of triangular fuzzy numbers by defining the fuzzy coefficients polynomial to obtain the general from solution. Then we study the hermite interpolation method, which is more convenient for practice .

We know that lagrange interpolation formula is not most convenient for practical purpose. But hermite interpolation formula gives more accurate result than obtained in lagrange interpolation formula Finally an example is provided to illustrate the method.

2. Preliminaries

Definitions 2.1. It is a fuzzy number represented with three points as follows A=(a^l, a^c, a^r) This representation is interpreted as a membership function

$$\mu_A(x) = \begin{cases} \mathbf{0}, & \mathbf{x} < a^l \\ \frac{x-a^l}{a^c-a^l}, & a^l \le \boldsymbol{x} \le a^c \\ \frac{a^r-x}{a^r-a^c}, & a^c \le \boldsymbol{x} \le a^l \\ \mathbf{0}, & \mathbf{x} > a^r \end{cases}$$

Definitions 2.2. Let $\tilde{a}= (a^l, a^c, a^r)$ $\tilde{b}=(b^l, b^c, b^r)$ be two triangular fuzzy numbers. $\tilde{a}=\tilde{b}$ iff $a^l= b^l$, $a^c= b^c$, $a^r= b^r$

Definitions 2.3. The operations of triangular fuzzy numbers are defined as follows:

(1) $\tilde{a} + \tilde{b} =(a^l + b^l, a^c + b^c, a^r + b^r)$

(2) $k\tilde{a} = \begin{cases} (ka^l, ka^c, ka^r) & k \ge 0 \\ (ka^r, ka^c, ka^l), & else \end{cases}$

Definitions 2.4. Let $\tilde{a}_0, \tilde{a}_1 \dots \dots \dots \dots \tilde{a}_m$ be triangular fuzzy number. A function $P_n(x, \tilde{a}_0, \tilde{a}_1 \dots \dots \dots \dots \tilde{a}_m)$ denoted by $\tilde{P}_n(x)$ is called the n-oreder polynomial with triangular fuzzy numbers coefficients, if it satisfies the following conditions

(1) $\tilde{P}_n(x)$ is an n-order polynomial about x;

(2) $\tilde{P}_n(x)$is a 1-order polynomial about $\tilde{a}_0, \tilde{a}_1 \dots \dots \dots \dots \tilde{a}_m$

We get an equivalent definition of the triangular polynomial $\tilde{P}_n(x)$.

Definitions 2.5. Let $Q_0(x), Q_1(x) \dots \dots .., Q_m(x)$ be m+1 polynomials, whose degree is not more than n, and at least one of them is a polynomial of degree n .The fuzzy triangular polynomials $\tilde{P}_n(x)$ has the following from

$$\tilde{P}_n(x) = \tilde{a}_0 Q_0(x) + \tilde{a}_1 Q_1(x) + \cdots \dots \dots \dots + \tilde{a}_m Q_m(x)$$

The set of all triangular fuzzy polynomials of degree m (m≤n) is denoted by$\tilde{P}_n$

3. Interpolation polynomials of fuzzy numbers

In this section the interpolation problem of triangular fuzzy numbers is formulated in detail by comparing with the hermite interpolation problem. Then we obtain the existence theorem of solutions and investigate its properties.

Definition 3.1. Let x_0, x_1,…………..x_n be n+ 1 distinct node .

Given$y_0 = f(x_0), y_1 = f(x_1), \dots \dots \dots, y_n = f(x_n)$

y_0'=$f'_{(x_0)}$,y_1'=$f'_{(x_1)}$,………………..,y_n'=$f'_{(x_n)}$,

The problem is to find a polynomial $P_n(x) \in P_{n,}$ is called interpolating polynomials such that $P_n(x_i) = Q_n(x_i) + R_n(x_i)$,

where $Q_n(x_i) = y_i R_n(x_i) = y_i'$(i=0,1,2………n)

Theorem 3.1. There exists a unique polynomial $W_{2n+1}(x) \in W_{2n+1}$ such that $H_{2n+1}(x_i)$= y_i,$T_{2n+1}(x_i)$=$y_i'$for i=0,1,2….,n

The Hermite form of $W_{2n+1}(x)$ is

$$W_{2n+1}(x) = \sum_{i=0}^{n} u_i(x) y_i + \sum_{i=0}^{n} v_i(x)\, y_i' \tag{1}$$

$$u_i(x_j) = \begin{Bmatrix} 1 & if\ i = j \\ 0 & if\ i \neq j \end{Bmatrix} \quad v_i(x) = 0\ for\ all\ i$$

$$u_i'(x) = 0 \; for\ all\ i \qquad v_i'(x_j) = \begin{Bmatrix} 1 & if & i = j \\ 0 & if & i \neq j \end{Bmatrix}$$

$$H_{2n+1}(x_i, y_0, y_1, \dots \dots \dots \dots y_n) = y_i \qquad i = 0,1,2, \dots \dots \dots n$$

$$T_{2n+1}(x_i, y_0', y_1', \dots \dots \dots \dots \dots \dots, y_n') = y_i' \qquad i = 0,1,2 \dots \dots \dots \dots n$$

Theorem 3.2. Let $x_0, x_1, \dots\dots\dots\dots x_n$ be n+1 distinct nodes. and let $\tilde{y}_i = (y_i^l, y_i^c, y_i^r)$, (i=0,1,2,…....n) be n+1 triangular fuzzy numbers. There exists at least one fuzzy polynomial such that

$$\widetilde{W}_{2n+1}(x_i) = \tilde{y}_i \; and \; \tilde{T}_{2n+1}(x_i) = \tilde{y}_i' \qquad for\ i = 0,1,2 \dots \dots \dots \dots \dots n$$

Proof: Suppose that $Q_n(x_i) = y_i R_n(x_i) = y_i'$ is an interpolation polynomial such that

$$H_{2n+1}(x_i, y_0, y_1, \dots \dots \dots \dots y_n) = y_i \qquad i = 0,1,2, \dots \dots . n$$

$$T_{2n+1}(x_i, y_0', y_1', \dots \dots \dots \dots, y_n') = y_i' \qquad i = 0,1,2 \dots \dots n$$

Define

$$\widetilde{H}_{2n+1}^l(x) = \inf_{\substack{\forall y_{ii} \in \tilde{y}_i \\ i=0,1,\dots\dots n}} H^l{}_{2n+1}(x, y_0, y_1, \dots \dots \dots \dots y_n) \tag{2}$$

$$\widetilde{H}_{2n+1}^r(x) = \sup_{\substack{\forall y_{ii} \in \tilde{y}_i \\ i=0,1,\dots\dots n}} H^r{}_{2n+1}(x, y_0, y_1, \dots \dots \dots \dots y_n) \tag{3}$$

$$\widetilde{H}_{2n+1}^C(x) = \sum_{i=0}^{n} u_i(x) y_i \tag{4}$$

and

$$\tilde{T}_{2n+1}^l(x) = \inf_{\substack{\forall \tilde{y}_{ii}' \in \tilde{y}_i' \\ i=0,1,\dots\dots n}} T^l{}_{2n+1}(x, y_0', y_1', \dots \dots \dots \dots y_n') \tag{5}$$

$$\tilde{T}_{2n+1}^r(x) = \inf_{\substack{\forall \tilde{y}_{ii}' \in \tilde{y}_i' \\ i=0,1,\dots\dots n}} T^r{}_{2n+1}(x, y_0', y_1', \dots \dots \dots \dots y_n') \tag{6}$$

$$\tilde{T}_{2n+1}^C(x) = \sum_{i=0}^{n} v_i(x) y_i' \tag{7}$$

$$\widetilde{W}_{2n+1}^l(x) = \widetilde{H}_{2n+1}^l(x) + \tilde{T}_{2n+1}^l(x)$$

$$\widetilde{W}_{2n+1}^r(x) = \widetilde{H}_{2n+1}^r(x) + \tilde{T}_{2n+1}^r(x)$$

$$\widetilde{W}_{2n+1}^c(x) = \widetilde{H}_{2n+1}^c(x) + \tilde{T}_{2n+1}^c(x)$$

Since $u_i(x) and\ v_i(x)$ are interpolating polynomials in x of degree 2n+1 using conditions

$$u_i(x_j) = \begin{Bmatrix} 1 & if\ i = j \\ 0 & if\ i \neq j \end{Bmatrix} v_i(x) = 0 \qquad for\ all\ i$$

$$u_i'(x) = 0 \quad for\ all\ i \qquad v_i'(x_j) = \begin{Bmatrix} 1 & if & i = j \\ 0 & if & i \neq j \end{Bmatrix}$$

Let $\quad u_i(x) = A_i(x)[l_i(x)]^2$

$$v_i(x) = B_i(x)[l_i(x)]^2$$

where

$$l_i(x) = \sum_{\substack{J=0 \\ J \neq I}}^{n} \frac{x - x_j}{x_j - x_i} \;, i = 0,1, \dots \dots \dots \dots \dots . n$$

and $A_i(x)$ and $B_i(x)$ are linear function in x

Define

$$A_i(x) = (a_i x + b_i)[l_i(x)]^2$$

$$B_i(x) = (c_i x + d_i)[l_i(x)]^2$$

Then

$$\left.\begin{aligned} a_i x + b_i &= 1 \\ c_i x + d_i &= 0 \\ u_i(x) &= (a_i x + b_i)[l_i(x)]^2 \\ v_i(x) &= (c_i(x) + d_i)[l_i(x)]^2 \end{aligned}\right\} \quad (8)$$

and

$$u_i(x) = (a_i x + b_i)[l_i(x)]^2$$
$$= (a_i x + b_i)\ 2l_i(x)l_i'(x) + a_i[l_i(x)]^2$$

Put x=x_i

$$u_i'(x_i)=(a_i x_i + b_i)\ 2l_i(x_i)l_i'(x_i) + a_i[l_i(x_i)]^2$$
$$0=2l_i'(x_i) + a_i$$

Therefore $a_i + 2l_i'(x_i) = 0$

$$v_i(x) = (c_i(x) + d_i)[l_i(x)]^2$$
$$= (c_i x + d_i)\ 2l_i(x)l_i'(x) + c_i[l_i(x)]^2$$

Put x=x_j

$$v_i'(x_j)=(c_i x_j + d_i)\ 2l_i(x_j)l_i'(x_j) + c_i\left[l_i(x_j)\right]^2$$
$$1= c_i$$

$$a_i x + b_i = 1$$
$$c_i x + d_i = 0 \text{ and}$$
$$a_i + 2l_i'(x_i) = 0$$
$$c_i = 1$$

Solving the above equations we get

$$a_i = -2l_i'(x_i), b_i = 1+2x_i l_i'(x_i)$$
$$c_i = 1 d_i = -x_i \quad (9)$$

Using the above values on applying in equations (8) the equation becomes

$$u_i(x) = [1 - 2(x - x_i)l_i'(x_i)][l_i(x)]^2$$
$$v_i(x) = (x - x_i)[l_i(x)]^2$$

Using the above expression for $u_i(x)\ and\ v_i(x)$ we obtain finally

$$W_{2n+1}(x) = \sum_{i=0}^{n}[1 - 2(x - x_i)l_i'(x_i)][l_i(x)]^2 y_i + \sum_{i=0}^{n}(x - x_i)[l_i(x)]^2 y_i'$$

This is the required hermite interpolation formula

Therefore

$$W^l{}_{2n+1}(x) = \sum_{i=0}^{n}[1 - 2(x - x_i)l_i'(x_i)][l_i(x)]^2 y_i + \sum_{i=0}^{n}(x - x_i)[l_i(x)]^2 y_i'$$
$$W^c{}_{2n+1}(x) = \sum_{i=0}^{n}[1 - 2(x - x_i)l_i'(x_i)][l_i(x)]^2 y_i + \sum_{i=0}^{n}(x - x_i)[l_i(x)]^2 y_i'$$
$$W^r{}_{2n+1}(x) = \sum_{i=0}^{n}[1 - 2(x - x_i)l_i'(x_i)][l_i(x)]^2 y_i + \sum_{i=0}^{n}(x - x_i)[l_i(x)]^2 y_i'$$
$$\widetilde{W}_{2n+1}(x_i) = (\widetilde{W}^l_{2n+1}(x), \widetilde{W}^c_{2n+1}(x), \widetilde{W}^r_{2n+1}(x)$$
$$=(y_i^l, y_i^c, y_i^r) = \tilde{y}_i \qquad for\ i = 0,1, \dots \dots \dots n$$

As a consequence the interpolating polynomial of triangular fuzzy number exists. More specifically, by theorem 3.2 $\widetilde{W}_{2n+1}(x)$ contains all the interpolating polynomials of $\tilde{y}_{i,}$ which are arbitrarily chosen

Theorem 3.3. The hermite interpolating polynomials of fuzzy numbers is

$$\widetilde{W}_{2n+1}(x) = \sum_{i=0}^{n} u_i\,(x)\tilde{y}_i + \sum_{i=0}^{n} v_i(x)\tilde{y}_i'$$

where

$$\widetilde{W}_{2n+1}(x) = \widetilde{H}_{2n+1}(x) + \tilde{T}_{2n+1}(x)$$

Obviously

$$H_{2n+1}(x_i) = \tilde{y}_i \quad i = 0,1,\dots\dots\dots n$$

$$u_i(x_j) = \begin{Bmatrix} 1 & if\ \ i = j \\ 0 & if\ i \neq j \end{Bmatrix} \qquad v_i(x) = 0\ for\ all\ i$$

$$u_i{}'(x) = 0\ for\ all\ i \qquad v_i{}'(x_j) = \begin{Bmatrix} 1 & if\ \ i = j \\ 0 & if\ \ i \neq j \end{Bmatrix}$$

Theorem 3.4.

Suppose that $\widetilde{W}_{2n+1}(x) = \left[W_{2n+1}^{l}(x), W_{2n+1}^{r}(x)\right]$ given by

$$W_{2n+1}^{l}(x) = \inf_{\substack{\forall y_{ii} \in \tilde{y}_i \\ i=0,1,\dots..n}} H^{l}{}_{2n+1}(x, y_0, y_1, \dots\dots\dots\dots y_n) + \inf_{\substack{\forall \tilde{y}_{ii}' \in \tilde{y}_i' \\ i=0,1,\dots..n}} T^{l}{}_{2n+1}(x, y_0', y_1', \dots\dots\dots\dots y_n')$$

$$W_{2n+1}^{r}(x) = \sup_{\substack{\forall y_{ii} \in \tilde{y}_i \\ i=0,1,\dots..n}} H^{r}{}_{2n+1}(x, y_0, y_1, \dots\dots\dots\dots y_n) + \sup_{\substack{\forall \tilde{y}_{ii}' \in \tilde{y}'_i \\ i=0,1,\dots..n}} T^{r}{}_{2n+1}(x, y_0', y_1', \dots\dots\dots\dots y_n')$$

such that $\widetilde{H}_{2n+1}(x) = \sum_{i=0}^{n} u_i(x)\tilde{y}_i$

$$\tilde{T}_{2n+1}(x) = \sum_{i=0}^{n} v_i(x)\tilde{y}_i'$$

Then we have $\widetilde{W}_{2n+1}(x) = \widetilde{H}_{2n+1}(x) + \tilde{T}_{2n+1}(x)$

Hence the hermite interpolation formula holds to replace y_i by $\tilde{y}_i'$ in (1).

We know that Lagrange interpolation formula is not convenient for a practical purpose. But hermite interpolation formula given more accurate result than obtained in lagrange interpolation formula.

4. Numerical example

1. Determine the Hermite polynomial which fits the following data and hence find an approximate value of In 4 $x_0 = 2.0 \quad y_0 = (0.67, 0.69315, 0.71) y_0' = (0.4, 0.5, 0.6)$

$x_1 = 2.5 \quad y_1 = (0.90, 0.0.91629, 0.92) y_1' = (0.3, 0.4, 0.5)$

$x_2 = 3.0 \quad y_2 = (1.08, 1.09861, 1.1) \quad y_2' = (0.32, 0.3333, 0.35)$

Solution:

Consider

$$l_i(x) = \sum_{\substack{J=0 \\ J\neq I}}^{n} \frac{x - x_j}{x_j - x_i} \quad , i = 0,1, \dots \dots \dots \dots \dots . n$$

$$l_0(x) = \frac{(x-2.5)(x-3.0)}{(2-2.5)(2-3.0)} \, l_0(x) = 2x^2 - 11x + 15$$

$$l_1(x) \; = \frac{(x-2)(x-3.0)}{(2.5-2)(2.5-3.0)} \, l_1(x) = -(4x^2 - 20x + 24)$$

$$l_2(x) = \frac{(x-2)(x-2.5)}{(3-2)(3.0-2.5)} \, l_2(x) = 2x^2 - 9x + 10$$

$$l_0'(x) = 4x - 11 \qquad l_1'(x) = -(8x - 20) l_2'(x) = 4x - 9$$

$$l_0'(x_0) = -3 \qquad l_1'(x_1) = 0, \qquad\qquad l_2'(x_2) = 3$$

$$W^{l}{}_{2n+1}(x) = \sum_{i=0}^{n} [1 - 2(x - x_i) l_i'(x_i)][l_i(x)]^2 y_i + \sum_{i=0}^{n} (x - x_i)[l_i(x)]^2 y_i'$$

$$W^{c}{}_{2n+1}(x) = \sum_{i=0}^{n} [1 - 2(x - x_i) l_i'(x_i)][l_i(x)]^2 y_i + \sum_{i=0}^{n} (x - x_i)[l_i(x)]^2 y_i'$$

$$W^{r}{}_{2n+1}(x) = \sum_{i=0}^{n} [1 - 2(x - x_i) l_i'(x_i)][l_i(x)]^2 y_i + \sum_{i=0}^{n} (x - x_i)[l_i(x)]^2 y_i'$$

$$u_i(x) = [1 - 2(x - x_i) l_i'(x_i)][l_i(x)]^2$$

$$u_0(x) = [1 - 2(x - x_0) l_0'(x_0)][l_0(x)]^2$$

$$u_0(x) = [6x - 11][2x^2 - 11x + 5]^2$$

$$u_1(x) = [4x^2 - 20x + 24]^2$$

$$u_2(x) \; = \; [-6x + 19][2x^2 - 9x + 10]$$

Consider

$$u_0(x) = [6x - 11][2x^2 - 11x + 5]^2$$

Put $x = 4$,$u_0(x)$ =117

$u_0(x) = [6x - 11][2x^2 - 11x + 5]^2$ (0.67, 0.69315, 0.71)

$u_0(x)$ = (78.39, 81.09855, 83.07)

$$u_1(x) = [4x^2 - 20x + 24]^2$$

Put $x = 4$,$u_1(x) = 64$

$u_1(x) = [4x^2 - 20x + 24]^2$(0.91,0.91629,0.4)

= 64 (57.6,58.64256,58.88)

$$u_2(x) = [-6x + 19][2x^2 - 9x + 10]$$

Put $x = 4$,$u_2(x)$ = -180

$u_2(x) = [-6x + 19][2x^2 - 9x + 10]$(1.1,1.09861,1.1)

$u_2(x) = -180$(1.1,1.09861,1.1)

$u_2(x) = (-198, -197.7498, -194.4)$

Consider

$$v_i(x) = (x - x_i)[l_i(x)]^2$$

$$v_0(x) = (x - 2)[2x^2 - 11x + 5]^2$$

Put $x = 4$ $\quad v_0(x) = 18$

$v_0(x) = (x - 2)[2x^2 - 11x + 5]^2$(0.4,0.5,0.6)

$v_0(x) = 18$(0.4,0.5,0.6)

$v_0(x) = (7.2,9,10.8)$

$v_1(x) = (x - 2.5)[4x^2 - 20x + 24]^2$

Put $x = 4$ $v_1(x) = 96$ (0.3,0.4,0.5)

$v_1(x) =$ (28.8,38.4,48)

$v_2(x) = (x - 3.0)[2x^2 - 9x + 10]$

Put $x = 4$ $v_2(x) = 36$ (0.30,0.33,0.5)

$v_2(x) =$ (11.52,11.9988,12.6)

$W_{2n+1}(x) =$ (-14.49,1.39011,18.95)

Hence In (4) ≅ 1.39011

5. Conclusion

In this paper ,we get the general form of the fuzzy interpolating polynomial problem. Then, the solution formula is given, including hermite form. Finally, we obtain several interesting conclusions and illustrate their meanings by examples. In future, we will focus on other interpolation problems with uncertainty.

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