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Hyers Type Fuzzy Stability of a Radical Reciprocal Quadratic Functional Equation Originating from Three Dimensional Pythagorean Means

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Abstract. The main goal of this paper is to found the generalized Ulam-Hyers stability of a radical reciprocal quadratic functional equation originating from 3 dimensional Pythagorean means

$$f\left(\sqrt{x^2+y^2}\right)+f\left(\sqrt{y^2+z^2}\right)+f\left(\sqrt{x^2+z^2}\right)=\frac{f(x)f(y)}{f(x)+f(y)}+\frac{f(y)f(z)}{f(y)+f(z)}+\frac{f(x)f(z)}{f(x)+f(z)}$$

in Fuzzy Banach space using classical Hyers method.

Keywords: Quadratic functional equation, reciprocal functional equation, generalized Ulam-Hyers stability, Fuzzy Banach space.

AMS Mathematics Subject Classification (2010): 39B52, 32B72, 32B82

1. Introduction

The revision of stability problems for functional equations is linked to a query of Ulam [29] regarding the stability of group homomorphisms and assenting counter given by D.H. Hyers [12] for Banach spaces. It was further generalized and excellent results obtained by number of authors [2, 10, 24, 25, 27].

Motived by Ger [11] and Pinales [22], in this paper we found the generalized Ulam-Hyers stability of a radical reciprocal quadratic functional equation originating from 3 dimensional Pythagorean means

$$f\left(\sqrt{x^2+y^2}\right)+f\left(\sqrt{y^2+z^2}\right)+f\left(\sqrt{x^2+z^2}\right)=\frac{f(x)f(y)}{f(x)+f(y)}+\frac{f(y)f(z)}{f(y)+f(z)}+\frac{f(x)f(z)}{f(x)+f(z)} \quad (1.1)$$

in Fuzzy Banach space using classical Hyers method. It is easy to verify that $f(x)=\frac{a}{x^2}$ is the solution of the functional equation (1.1). We demonstrate the stability results in two different ways.

2. Definitions in fuzzy Banach spaces

In this section, the authors present vital definitions and notations in Fuzzy Banach space.

Katsaras [16] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. A quiet number of mathematicians have defined fuzzy norms on a vector space from different positions of their vision in [8, 18, 30]. In particular, Bag and Samanta [4, 5] subsequentally Cheng and Mordeson [6], gave an inspiration of fuzzy norm in such a manner that the analogous fuzzy metric is of Kramosil and Michalek type [17]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [6].

We use the definition of fuzzy normed spaces given in [6,20,21,28].

Definition 2.1. *Let* X *be a real linear space. A function* $N : X \times \mathbb{R} \to [0,1]$ *is said to be a fuzzy norm on* X *if for all* $x, y \in X$ *and all* $s, t \in \mathbb{R}$*,*

$(FNS1)$ $N(x,c) = 0$ for $c \leq 0$;

$(FNS2)$ $x = 0$ *if and only if* $N(x,c) = 1$ *for all* $c > 0$;

$(FNS3)$ $N(\rho x, t) = N\left(x, \frac{t}{|\rho|}\right)$ *if* $c \neq 0$;

$(FNS4)$ $N(x+y, s+t) \geq min\{N(x,s), N(y,t)\}$;

$(FNS5)$ $N(x,\cdot)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $lim_{t\to\infty} N(x,t) = 1$;

$(FNS6)$ *for* $x \neq 0, N(x,\cdot)$ *is (upper semi) continuous on* $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space.

Example 2.2. *Let* $(X, \|\cdot\|)$ *be a normed linear space. Then*

$$N(x,t) = \begin{cases} \dfrac{t}{t + \|x\|}, & t > 0,\ x \in X, \\ 0, & t \leq 0,\ x \in X \end{cases}$$

is a fuzzy norm on X .

Definition 2.3. *Let* (X, N) *be a fuzzy normed linear space. Let* $\{x_n\}$ *be a sequence in* X *. Then* x_n *is said to be convergent if there exists* $x \in X$ *such that* $\lim_{n\to\infty} N(x_n - x, t) = 1$ *for all* $t > 0$*. In that case,* x *is called the limit of the sequence* x_n *and we denote it by* $N - \lim_{n\to\infty} x_n = x.$

Definition 2.4. *A sequence* $\{x_n\}$ *in* X *is called Cauchy if for each* $\varepsilon > 0$ *and each* $t > 0$ *there exists* n_0 *such that for all* $n \geq n_0$ *and all* $p > 0$*, we have* $N(x_{n+p} - x_n, t) > 1 - \varepsilon.$

Definition 2.5. *Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

Definition 2.6. *A mapping $f : X \to Y$ between fuzzy normed spaces X and Y is continuous at a point x_0 if for each sequence $\{x_n\}$ covering to x_0 in X, the sequence $f\{x_n\}$ converges to $f(x_0)$. If f is continuous at each point of $x_0 \in X$ then f is said to be continuous on X.*

The stability of a quiet number of functional equations in fuzzy normed spaces were investigated in [21, 22, 29].

3. Hyers type fuzzy stability results

In this section, the authors investigate the generalized Ulam-Hyers stability of the functional equation (1.1). To discuss the stability results, let as assume that $X, (Z, N')$ and (Y, N') are linear space, fuzzy normed space and fuzzy Banach space, respectively. Now, define a function $f : X \to Y$ by

$$F_{RQ}(x,y,z) = f\left(\sqrt{x^2+y^2}\right) + f\left(\sqrt{y^2+z^2}\right) + f\left(\sqrt{x^2+z^2}\right) - \frac{f(x)f(y)}{f(x)+f(y)} - \frac{f(y)f(z)}{f(y)+f(z)} - \frac{f(x)f(z)}{f(x)+f(z)}$$

for all $x, y, z \in X$.

Theorem 3.1. If a function $f : X \to Y$ satisfies the functional inequality

$$N\left(F_{RQ}(x,y,z), r\right) \geq N'(\Phi(x,y,z), r) \tag{3.1}$$

for all $x, y, z \in X$ and all $r > 0$. Let $\Phi : X^3 \to Z$ be a mapping with $0 < (2d)^{\mu} < 1$

$$N'\left(\Phi\left(\left(\sqrt{2^n}\right)^{\mu} x, \left(\sqrt{2^n}\right)^{\mu} x, \left(\sqrt{2^n}\right)^{\mu} x\right), r\right) \geq N'\left(d^{n\mu}\Phi(x,x,x), r\right) \tag{3.2}$$

for all $x \in X$ and all $r > 0$ with $d > 0$ filling the condition

$$\lim_{n\to\infty} N'\left(\Phi\left(\left(\sqrt{2^n}\right)^{\mu} x, \left(\sqrt{2^n}\right)^{\mu} x, \left(\sqrt{2^n}\right)^{\mu} x\right), 2^{\mu n} r\right) = 1 \tag{3.3}$$

for all $x, y, z \in X$ and all $r > 0$ where $\mu = \pm 1$. Then there exists one and only reciprocal quadratic mapping $Q : X \to Y$ which satisfying (1.1) and

$$N(Q(x) - f(x), r) \geq N'\left(\Phi(x,x,x), \frac{3|1-2d|r}{4}\right) \tag{3.4}$$

and

$$Q(x) = N - \lim_{n\to\infty} 2^{n\mu} f\left(\left(\sqrt{2^n}\right)^{\mu} x\right) \tag{3.5}$$

for all $x \in X$ and all $r > 0$.

Proof: First assume $\beta = 1$. Changing (x, y, z) by (x, x, x) in (3.1), we get

$$N\left(3f\left(\sqrt{2}x\right) - 3\left(\frac{f(x)}{2}\right), r\right) \geq N'(\Phi(x,x,x), r) \tag{3.6}$$

for all $x \in X$ and all $r > 0$. Using $(FNS3)$ in the above inequality, we arrive

$$N\left(2f\left(\sqrt{2}x\right)-f(x),\frac{2r}{3}\right)\geq N'\left(\Phi(x,x,x),r\right) \quad (3.7)$$

for all $x\in X$ and all $r>0$. Replacing x by $\sqrt{2^n}x$ in (3.6), we obtain

$$N\left(2f\left(\sqrt{2^{n+1}}x\right)-f\left(\sqrt{2^n}x\right),\frac{2r}{3}\right)\geq N'\left(\Phi\left(\sqrt{2^n}x,\sqrt{2^n}x,\sqrt{2^n}x\right),r\right) \quad (3.8)$$

for all $x\in X$ and all $r>0$. It follows from (3.8) and (3.2), $(FNS3)$ in (3.8), we have

$$N\left(2^{(n+1)}f\left(\sqrt{2^{n+1}}x\right)-2^n f\left(\sqrt{2^n}x\right),\frac{2\cdot 2^n r}{3}\right)\geq N'\left(\Phi(x,x,x),\frac{r}{d^n}\right) \quad (3.9)$$

holds for all $x\in X$ and all $r>0$. Replacing r by $d^n r$ in (3.9), we get

$$N\left(2^{(n+1)}f\left(\sqrt{2^{n+1}}x\right)-2^n f\left(\sqrt{2^n}x\right),\frac{2}{3}(2d)^n r\right)\geq N'\left(\Phi(x,x,x),r\right) \quad (3.10)$$

for all $x\in X$ and all $r>0$. It is easy to see that

$$2^{(n+1)}f\left(\sqrt{2^{n+1}}x\right)-f(x)=\sum_{i=0}^{n-1}2^{(i+1)}f\left(\sqrt{2^{i+1}}x\right)-2^i f\left(\sqrt{2^i}x\right) \quad (3.11)$$

for all $x\in X$. From equations (3.10), (3.11) and using $(FNS3)$, we have

$$N\left(2^n f\left(\sqrt{2^n}x\right)-f(x),\frac{2}{3}\sum_{i=0}^{n-1}(2d)^i r\right)\geq \min\bigcup_{i=0}^{n-1}\left\{N\left(2^{(i+1)}f\left(\sqrt{2^{i+1}}x\right)-2^i f\left(\sqrt{2^i}x\right),\frac{2}{3}(2d)^i r\right)\right\}$$

$$\geq min\bigcup_{i=0}^{n-1}\left\{N'\left(\Phi(x,x,x),r\right)\right\}\geq N'\left(\Phi(x,x,x),r\right) \quad (3.12)$$

for all $x\in X$ and all $r>0$. Thus the sequence $\left\{2^n f\left(\sqrt{2^n}x\right)\right\}$ is a Cauchy sequence. Indeed, replacing x by $2^m x$ in (3.12) and using (3.2), $(FNS3)$, we obtain

$$N\left(2^{n+m}f\left(\sqrt{2^{n+m}}x\right)-2^m f\left(\sqrt{2^m}x\right),\frac{2}{3}\sum_{i=0}^{n-1}(2d)^i 2^m r\right)\geq N'\left(\Phi(x,x,x),\frac{r}{d^m}\right) \quad (3.13)$$

for all $x\in X$ and all $r>0$ and all $m,n\geq 0$. Replacing r by $d^m r$ in (3.13), we arrive

$$N\left(2^{n+m}f\left(\sqrt{2^{n+m}}x\right)-2^m f\left(\sqrt{2^m}x\right),\frac{2}{3}\sum_{i=0}^{n-1}(2d)^{i+m} r\right)\geq N'\left(\Phi(x,x,x),r\right) \quad \text{or}$$

$$N\left(2^{n+m}f\left(\sqrt{2^{n+m}}x\right)-2^m f\left(\sqrt{2^m}x\right),\frac{2}{3}\sum_{i=m}^{m+n-1}(2d)^i r\right)\geq N'\left(\Phi(x,x,x),r\right) \quad (3.14)$$

for all $x\in X$ and all $r>0$ and all $m,n\geq 0$. Using $(FNS3)$ in (3.14), we obtain

$$N\left(2^{n+m}f\left(\sqrt{2^{n+m}}x\right)-2^m f\left(\sqrt{2^m}x\right),r\right)\geq N'\left(\Phi(x,x,x),\frac{r}{\frac{2}{3}\sum_{i=m}^{m+n-1}(2d)^i}\right) \quad (3.15)$$

for all $x\in X$ and all $r>0$ and all $m,n\geq 0$. Since $0<2d<1$ and $\sum_{i=0}^{n}(2d)^i<\infty$, the

Cauchy criterion for convergence and $(FNS5)$ implies that $\left\{2^n f\left(\sqrt{2^n}x\right)\right\}$ is a Cauchy sequence in (Y,N). Since (Y,N) is a fuzzy Banach space, this sequence converges to some point $Q(x)\in Y$. So one can define the mapping $Q: X\to Y$ by

$$Q(x) = N-\lim_{n\to\infty} 2^n f\left(\sqrt{2^n}x\right)$$

for all $x\in X$. Letting $m=0$ in (3.15), we get

$$N\left(2^n f\left(\sqrt{2^n}x\right)-f(x),r\right)\geq N'\left(\Phi(x,x,x),\frac{r}{\frac{2}{3}\sum_{i=0}^{n-1}(2d)^i}\right) \tag{3.16}$$

for all $x\in X$ and all $r>0$. Letting $n\to\infty$ in (3.16) and using $(FNS6)$, we arrive

$$N\left(Q(x)-f(x),r\right)\geq N'\left(\Phi(x,x,x),\frac{3(1-2d)r}{4}\right)$$

for all $x\in X$ and all $r>0$.

To prove Q satisfies the (1.1), replacing (x,y,z) by $\left(\sqrt{2^n}x,\sqrt{2^n}y,\sqrt{2^n}z\right)$ in (3.1), we obtain

$$N\left(2^n F_{RQ}\left(\sqrt{2^n}x,\sqrt{2^n}y,\sqrt{2^n}z\right),r\right)\geq N'\left(\Phi\left(\sqrt{2^n}x,\sqrt{2^n}y,\sqrt{2^n}z\right),2^n r\right) \tag{3.17}$$

for all $r>0$ and all $x,y,z\in X$. Now,

$$\begin{aligned}
&N\left(Q\left(\sqrt{x^2+y^2}\right)+Q\left(\sqrt{y^2+z^2}\right)+Q\left(\sqrt{x^2+z^2}\right)-\frac{Q(x)Q(y)}{Q(x)+Q(y)}-\frac{Q(y)Q(z)}{Q(y)+Q(z)}-\frac{Q(x)Q(z)}{Q(x)+Q(z)},\ r\right)\\
&\geq \min\Bigg\{N\left(Q\left(\sqrt{x^2+y^2}\right)-2^n f\left(\sqrt{2^n}\sqrt{x^2+y^2}\right),\frac{r}{7}\right),N\left(Q\left(\sqrt{y^2+z^2}\right)-2^n f\left(\sqrt{2^n}\sqrt{y^2+z^2}\right),\frac{r}{7}\right),\\
&\quad N\left(Q\left(\sqrt{x^2+z^2}\right)-2^n f\left(\sqrt{2^n}\sqrt{x^2+z^2}\right),\frac{r}{7}\right),N\left(-\frac{Q(x)Q(y)}{Q(x)+Q(y)}+2^n\frac{f\left(\sqrt{2^n}x\right)f\left(\sqrt{2^n}y\right)}{f\left(\sqrt{2^n}x\right)+f\left(\sqrt{2^n}y\right)},\frac{r}{7}\right),\\
&\quad N\left(-\frac{Q(y)Q(z)}{Q(y)+Q(z)}+2^n\frac{f\left(\sqrt{2^n}y\right)f\left(\sqrt{2^n}z\right)}{f\left(\sqrt{2^n}y\right)+f\left(\sqrt{2^n}z\right)},\frac{r}{7}\right),N\left(-\frac{Q(x)Q(z)}{Q(x)+Q(z)}+2^n\frac{f\left(\sqrt{2^n}x\right)f\left(\sqrt{2^n}z\right)}{f\left(\sqrt{2^n}x\right)+f\left(\sqrt{2^n}z\right)},\frac{r}{7}\right),\\
&\quad N\Bigg(2^n f\left(\sqrt{2^n}\sqrt{x^2+y^2}\right)+2^n f\left(\sqrt{2^n}\sqrt{y^2+z^2}\right)+2^n f\left(\sqrt{2^n}\sqrt{x^2+z^2}\right)-2^n\frac{f\left(\sqrt{2^n}x\right)f\left(\sqrt{2^n}y\right)}{f\left(\sqrt{2^n}x\right)+f\left(\sqrt{2^n}y\right)}\\
&\qquad -2^n\frac{f\left(\sqrt{2^n}y\right)f\left(\sqrt{2^n}z\right)}{f\left(\sqrt{2^n}y\right)+f\left(\sqrt{2^n}z\right)}-2^n\frac{f\left(\sqrt{2^n}x\right)f\left(\sqrt{2^n}z\right)}{f\left(\sqrt{2^n}x\right)+f\left(\sqrt{2^n}z\right)},\frac{r}{7}\Bigg)\Bigg\}
\end{aligned} \tag{3.18}$$

for all $x,y,z\in X$ and all $r>0$. Using (3.17) and $(FNS5)$ in (3.18), we arrive

$$N\left(Q\left(\sqrt{x^2+y^2}\right)+Q\left(\sqrt{y^2+z^2}\right)+Q\left(\sqrt{x^2+z^2}\right)-\frac{Q(x)Q(y)}{Q(x)+Q(y)}-\frac{Q(y)Q(z)}{Q(y)+Q(z)}-\frac{Q(x)Q(z)}{Q(x)+Q(z)},\ r\right)$$

$$\geq \min\left\{1,1,1,1,1,1,N'\left(\Phi\left(\sqrt{2^n}x,\sqrt{2^n}y,\sqrt{2^n}z\right),2^n r\right)\right\} \geq N'\left(\Phi\left(\sqrt{2^n}x,\sqrt{2^n}y,\sqrt{2^n}z\right),2^n r\right) \quad (3.19)$$

for all $x, y, z \in X$ and all $r > 0$. Letting $n \to \infty$ in (3.19) and using (3.3), we see that

$$N\left(Q\left(\sqrt{x^2+y^2}\right)+Q\left(\sqrt{y^2+z^2}\right)+Q\left(\sqrt{x^2+z^2}\right)-\frac{Q(x)Q(y)}{Q(x)+Q(y)}-\frac{Q(y)Q(z)}{Q(y)+Q(z)}-\frac{Q(x)Q(z)}{Q(x)+Q(z)},\ r\right)=1$$

for all $x, y, z \in X$ and all $r > 0$. Using $(FNS2)$ in the above inequality gives

$$Q\left(\sqrt{x^2+y^2}\right)+Q\left(\sqrt{y^2+z^2}\right)+Q\left(\sqrt{x^2+z^2}\right)=\frac{Q(x)Q(y)}{Q(x)+Q(y)}+\frac{Q(y)Q(z)}{Q(y)+Q(z)}+\frac{Q(x)Q(z)}{Q(x)+Q(z)}$$

for all $x, y, z \in X$. Hence Q satisfies the reciprocal quadratic functional equation (1.1). In order to prove the existance of $Q(x)$ is unique, we let $R(x)$ be another reciprocal quadratic functional equation satisfying (1.1) and (3.4). Hence,

$$N(Q(x)-R(x),r) = N\left(2^n Q\left(\sqrt{2^n}x\right)-2^n R\left(\sqrt{2^n}x\right),r\right)$$

$$\geq \min\left\{N\left(2^n Q\left(\sqrt{2^n}x\right)-2^n R\left(\sqrt{2^n}x\right),\frac{r}{2}\right),N\left(2^n Q\left(\sqrt{2^n}x\right)-2^n R\left(\sqrt{2^n}x\right),\frac{r}{2}\right)\right\}$$

$$\geq N'\left(\Phi\left(\sqrt{2^n}x,\sqrt{2^n}x,\sqrt{2^n}x\right),\frac{3(1-2d)}{4\cdot 2}r\right) \geq N'\left(\Phi(x,x,x),\frac{3(1-2d)}{4\cdot 2d^n}r\right)$$

for all $x \in X$ and all $r > 0$. Since, $\lim_{n\to\infty}\frac{3(1-2d)}{4\cdot 2d^n}r = \infty,$ we obtain

$$\lim_{n\to\infty} N'\left(\Phi(x,x,x),\frac{3(1-2d)}{4\cdot 2d^n}r\right)=1.$$

Thus

$$N(Q(x)-R(x),r)=1$$

for all $x \in X$ and all $r > 0$. Hence $Q(x) = R(x)$. Therefore $Q(x)$ is unique.

Assume $\mu = -1$. Replacing x by $\frac{x}{\sqrt{2}}$ in (3.6) and using $(FNS3)$, we arrive

$$N\left(f(x)-\frac{1}{2}f\left(\frac{x}{\sqrt{2}}\right),\frac{r}{3}\right) \geq N'\left(\Phi\left(\frac{x}{\sqrt{2}},\frac{x}{\sqrt{2}},\frac{x}{\sqrt{2}}\right),r\right) \quad (3.20)$$

for all $x \in X$ and all $r > 0$. The rest of the proof is similar to that of $\mu = 1$. This completes the proof of the theorem.

From Theorem 3.1, we prove the next corollary pertaining to the stabilities for the functional equation (1.1).

Corollary 3.2. Suppose that a function $f : X \to Y$ satisfies the inequality

$$F_{RQ}(x,y,z) \geq \begin{cases} N'(\Delta, r), \\ N'\left(\Delta\{|x|^{\alpha}+|y|^{\alpha}+|z|^{\alpha}\}, r\right), \\ N'\left(\Delta|x|^{\alpha}|y|^{\alpha}|z|^{\alpha}, r\right) \end{cases} \tag{3.21}$$

for all $r > 0$ and all $x, y, z \in X$, where Δ, α are constants. Then there exists one and only reciprocal quadratic mapping $Q: X \to Y$ which satisfying (1.1) and

$$N\left(f(x) - Q(x), r\right) \geq \begin{cases} N'\left(\Delta, \dfrac{3|1|r}{4}\right); \\ N'\left(\Delta|x|^{\alpha}, \dfrac{3\left|1-2^{\frac{\alpha}{2}+1}\right|r}{4}\right); \quad \alpha \neq -2 \\ N'\left(\Delta|x|^{3\alpha}, \dfrac{3\left|1-2^{\frac{3\alpha}{2}+1}\right|r}{4}\right); \quad 3\alpha \neq -2 \end{cases} \tag{3.22}$$

for all $x \in X$ and all $r > 0$.

Proof: If we take

$$N'(\Phi(x,y,z), r) = \begin{cases} N'(\Delta, r), \\ N'\left(\Delta\{\|x\|^{p}+\|y\|^{p}+\|z\|^{p}\}, r\right), \\ N'\left(\Delta\|x\|^{p}\|y\|^{p}\|z\|^{p}, r\right) \end{cases}$$

then the corollary is followed from Theorem 3.1 by defining

$$d = \begin{cases} \sqrt{2}^{0}; \\ \sqrt{2}^{p}; \\ \sqrt{2}^{3p}; \end{cases}$$

we arrive our result.

The next Theorem gives an another type of stability result for the functional equation (1.1).

Theorem 3.3. If a function $f: X \to Y$ satisfies the functional inequality

$$N\left(F_{RQ}(x,y,z), r\right) \geq N'\left(\Phi(x,y,z), r\right) \tag{3.23}$$

for all $x, y, z \in X$ and all $r > 0$. Let $\Phi: X^3 \to Z$ be a mapping with $0 < \left(\frac{2}{d}\right)^{\mu} < 1$

$$N'\left(\Phi\left(\left(\sqrt{2^{n}}\right)^{\mu} x, \left(\sqrt{2^{n}}\right)^{\mu} x, \left(\sqrt{2^{n}}\right)^{\mu} x\right), r\right) \geq N'\left(d^{-n\mu}\Phi(x,x,x), r\right) \tag{3.24}$$

for all $x \in X$ and all $r > 0$ with $d > 0$ filling the condition

$$\lim_{n\to\infty} N'\left(\Phi\left(\left(\sqrt{2^n}\right)^{\mu} x,\left(\sqrt{2^n}\right)^{\mu} x,\left(\sqrt{2^n}\right)^{\mu} x\right),\frac{r}{2^{\mu n}}\right)=1 \tag{3.25}$$

for all $x,y,z\in X$ and all $r>0$ where $\mu=\pm 1$. Then there exists one and only reciprocal quadratic mapping $Q:X\to Y$ which satisfying (1.1) and

$$N\left(Q(x)-f(x),r\right)\geq N'\left(\Phi(x,x,x),\frac{3|2-d|r}{4}\right) \tag{3.26}$$

and

$$Q(x) = N-\lim_{n\to\infty} 2^{n\mu} f\left(\left(\sqrt{2^n}\right)^{\mu} x\right) \tag{3.27}$$

for all $x\in X$ and all $r>0$.

Proof: First assume $\mu=1$. Changing (x,y,z) by (x,x,x) in (3.24), we get

$$N\left(3f\left(\sqrt{2}x\right)-3\left(\frac{f(x)}{2}\right),r\right)\geq N'\left(\Phi(x,x,x),r\right) \tag{3.28}$$

for all $x\in X$ and all $r>0$. Using $(FNS3)$ in the above inequality, we arrive

$$N\left(2f\left(\sqrt{2}x\right)-f(x),\frac{2r}{3}\right)\geq N'\left(\Phi(x,x,x),r\right) \tag{3.29}$$

for all $x\in X$ and all $r>0$. Replacing x by $\sqrt{2^n}x$ in (3.29), we obtain

$$N\left(2f\left(\sqrt{2^{n+1}}x\right)-f\left(\sqrt{2^n}x\right),\frac{2r}{3}\right)\geq N'\left(\Phi\left(\sqrt{2^n}x,\sqrt{2^n}x,\sqrt{2^n}x\right),r\right) \tag{3.30}$$

for all $x\in X$ and all $r>0$. It follows from (3.30) and (3.24), $(FNS3)$, we have

$$N\left(2^{(n+1)}f\left(\sqrt{2^{n+1}}x\right)-2^n f\left(\sqrt{2^n}x\right),\frac{2\cdot 2^n r}{3}\right)\geq N'\left(\Phi(x,x,x),d^n r\right) \tag{3.31}$$

holds for all $x\in X$ and all $r>0$. Replacing r by $\frac{r}{d^n}$ in (3.31), we get

$$N\left(2^{(n+1)}f\left(\sqrt{2^{n+1}}x\right)-2^n f\left(\sqrt{2^n}x\right),\frac{2}{3}\left(\frac{2}{d}\right)^n r\right)\geq N'\left(\Phi(x,x,x),r\right) \tag{3.32}$$

for all $x\in X$ and all $r>0$. The rest of the proof is similar tothat of Theorem 3.1.

From Theorem 3.3, we prove the next corollary pertaining to the stabilities for the functional equation (1.1).

Corollary 3.4. If a function $f:X\to Y$ satisfies the inequality

$$F_{RQ}(x,y,z)\geq\begin{cases} N'(\Delta,r), \\ N'\left(\Delta\{|x|^{\alpha}+|y|^{\alpha}+|z|^{\alpha}\},r\right), \\ N'\left(\Delta|x|^{\alpha}|y|^{\alpha}|z|^{\alpha},r\right)\end{cases} \tag{3.33}$$

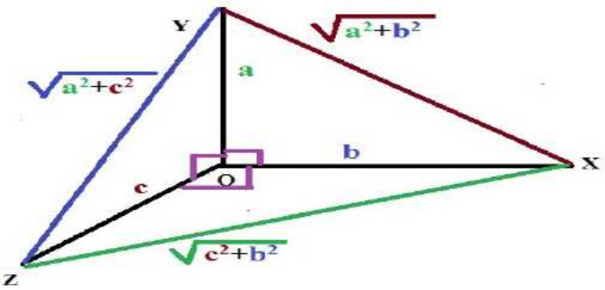
for all $r>0$ and all $x,y,z\in X$, where Δ,α are constants. Then there exists one and only reciprocal quadratic mapping $Q:X\to Y$ which satisfying (1.1) and

$$N\left(f(x)-Q(x),r\right) \geq \begin{cases} N'\left(\Delta, \dfrac{3|\sqrt{2}-1|\,r}{4}\right); & \\ N'\left(3\Delta\,|x|^{\alpha}, \dfrac{3|2-\sqrt{2}^{\alpha}|\,r}{4}\right); & \alpha \neq 2 \\ N'\left(\Delta\,|x|^{3\alpha}, \dfrac{3|2-\sqrt{2}^{3\alpha}|\,r}{4}\right); & 3\alpha \neq 2 \end{cases} \tag{3.34}$$

for all $x \in X$ and all $r > 0$.

4. Application of functional equation (1.1)

Let *O* be the center and X,Y,Z be any point on the three perpendicular axises. Also, assume that OX = b, OY= a, OZ = c.



From the Triangle XOY, YOZ and XOZ, we have by Pythagoras Theorem

$$\mathrm{YX}^2 = \mathrm{OY}^2 + \mathrm{OX}^2 = a^2 + b^2 \Rightarrow \mathrm{YX} = \sqrt{a^2+b^2}; \tag{4.1}$$

$$\mathrm{ZY}^2 = \mathrm{OY}^2 + \mathrm{OZ}^2 = a^2 + c^2 \Rightarrow \mathrm{ZY} = \sqrt{a^2+c^2}; \tag{4.2}$$

$$\mathrm{XZ}^2 = \mathrm{OZ}^2 + \mathrm{OX}^2 = a^2 + c^2 \Rightarrow \mathrm{XZ} = \sqrt{c^2+b^2}. \tag{4.3}$$

Adding (4.1), (4.2) and (4.3), we obtain

$$\mathrm{YX} + \mathrm{ZY} + \mathrm{XZ} = \sqrt{a^2+b^2} + \sqrt{a^2+c^2} + \sqrt{c^2+b^2}. \tag{4.4}$$

The above equation can be transformed into a radical reciprocal quadratic functional equation of the following form

$$f\left(\sqrt{x^2+y^2}\right) + f\left(\sqrt{y^2+z^2}\right) + f\left(\sqrt{x^2+z^2}\right) = \frac{f(x)\,f(y)}{f(x)+f(y)} + \frac{f(y)\,f(z)}{f(y)+f(z)} + \frac{f(x)\,f(z)}{f(x)+f(z)} \tag{4.5}$$

having solution

$$f(x) = \frac{a}{x^2} \tag{4.6}$$

for any constant *a.*

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