Intern. J. Fuzzy Mathematical Archive
Vol. 14, No. 1, 2017, 147-154
ISSN: 2320 –3242 (P), 2320 –3250 (online)
Published on 11 December 2017
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/ijfma.v14n1a18

International Journal of
Fuzzy Mathematical Archive

Uniform Integrability for Fuzzy Random Variables

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Received 5 November 2017; accepted 4 December 2017

Abstract. In this paper the uniform integrability of fuzzy random variable is introduced. An attempt is made to study the uniformly absolutely continuous fuzzy valued functions and an equivalent condition for uniform integrable and uniformly absolutely continuous is determined.

Keywords: Fuzzy random variables, uniformly integrable, fuzzy valued functions.

AMS Mathematics Subject Classification (2010): 37H10

1. Introduction

The notion of fuzzy random variables was introduced as a natural generalization of random set in order to represent associations between the outcomes of random experiment and non-statistical in exact data. Kwakernaak [3], and Puri and Ralescu [5], have introduced the notions of fuzzy random variables and its expectations. Limit theorems for random sets and fuzzy random variables have received much attention in recent years because of the application in several applied fields such as Mathematical Economics System Analysis and Stochastic Control Theory. Fuzzy random variables have been designed to deal with situations in which both random performance and fuzzy perception must be considered.

In [1], Diestel studied the of notion uniform integrability in detail. Then Laurent Piccinniet.al., [4], have introduced uniform integrability and measure for a random variable and they have established characterization theorem for uniform integrability and some related results. The purpose of this chapter is to generalize the convergence theorems of the basic probability theory to fuzzy random variables as well as some fundamental theorems in the light of uniform integrability. Also uniformly absolutely continuous fuzzy valued functions is defined and an equivalent condition for uniformly integrable and uniformly integrable absolutely continuous is established.

The concepts of a fuzzy random variables and its expectation were introduced by Puri and Ralescu [5].

Definition 1.1. Let (Ω, A, P) be a complete probability space, and fuzzy random variable is a Borel measurable function. If $X : \Omega \rightarrow F(R)$ is a fuzzy number valued function, where F(R) is a family of all fuzzy numbers, and B is the subset of R then $X^{-1}(B)$ denotes the fuzzy subset of Ω defined by

$$X^{-1}(B)(W) = \text{Sup}_{x \in B} X(w)(x) \text{for each } w \in \Omega$$

The function $X : \Omega \to F(R)$ is called a fuzzy random variable. If for every closed subset B of R, the fuzzy set $X^{-1}(B)$ is measurable when considered as a function from Ω to [0, 1]. If we denote

$X(w) = \{(X_{\propto}^{-}(w), X_{\propto}^{+}(w))\,|\, 0 \le \propto \le 1\}$ then it is well known that X is a fuzzy random variable if and only if for each $\propto \in [0,1]$ $X_{\propto}^{-}$ and $X_{\propto}^{+}$ are random variables.

Definition 1.2. The sequence of random variables { X_1, X_2, X_3, …} is said to converge in probability to a random variable X denoted by $X_n \xrightarrow{P} X$ if $\lim_{n\to\infty} P\{|X_n - X| > \varepsilon\} = 0$, for every $\varepsilon > 0$.

2. Uniform integrability for fuzzy random variables

Definition 2.1. [2] Let $\{X_n, n \ge 1\}$ and X are fuzzy random variables on a probability space (Ω, $\mathcal{A}$, P). A sequence of fuzzy random variables $\{X_n, n \ge 1\}$ is called uniformly integrable if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{n} \int_A \alpha\,[(X_n)_\alpha^{-} \vee (X_n)_\alpha^{+}]\,dp < \epsilon \text{ whenever } P(A) < \delta \text{ and}$$

$$\sup_{n\ge1} E\left(\alpha\left[(X_n)_\alpha^{-} \vee (X_n)_\alpha^{+}\right]\right) \le C < \infty \text{ are satisfied, where } A \in \mathcal{A}.$$

Theorem 2.2. [2] (Equivalence relation of uniform integrability.)

Let $\{X_n\}$ be a sequence of a fuzzy random variables. Then $\{X_n\}$ is uniformly integrable if and only if

$$\lim_{b\to\infty} \int_{(|(X_n)_\alpha^{-} \vee (X_n)_\alpha^{+}|)\ge b} \alpha(|(X_n)_\alpha^{-} \vee (X_n)_\alpha^{+}|)\,dp = 0$$

uniformly in n.

Theorem 2.3. [2] (Mean convergence theorem)

Let $\{X_n\}$ be a sequence of fuzzy random variables (integrable) and $X_n \xrightarrow{P} X$. Then $\{X_n\}$ converges in mean if and only if $\{X_n\}$ is uniformly integrable.

Corollary 2.4. [2] (Lebesgue dominated convergence theorem)

Let $\{X_n\}$ be a sequence of fuzzy random variables and $X_n \xrightarrow{P} X$ and

$$E\left(\sup_{n\ge1} \alpha(|(X_n)_\alpha^{-} \vee (X_n)_\alpha^{+}|\right) < \infty. \text{ Then}$$

$$E(\alpha|[(X_n)_\alpha^{-} - (X_\alpha)^{-}] \vee [(X_n)_\alpha^{+} - (X_\alpha)^{+}]|\ ) \to 0 \text{ as } n \to \infty.$$

Theorem 2.5. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of fuzzy random variables on a probability space (Ω, F, P). Let $(X_n)_\alpha^{-}$ and $(X_n)_\alpha^{+} \in L^1$; $n \ge 1$. Then $\{X_n\}_{n=1}^{\infty}$ converges to the fuzzy random variable X if and only if

(i) $X_n \xrightarrow{P} X$ as $n \to \infty$.

(ii) $\{X_n\}_{n=1}^{\infty}$ is uniformly integrable.

Proof: If $\{X_n\}$ converges to X, then by Markov's Inequality, we get the converges in probability of the α-level fuzzy random variable.

Thus (i) holds. Take $\Omega = [|(X_n)_\alpha^- \vee (X_n)_\alpha^+| \geq \lambda]$

Also, for each $\lambda > 0$,

Then, we can write,

$$\int_{(|(X_n)_\alpha^- \vee (X_n)_\alpha^+|) \geq \lambda} \alpha |(X_n)_\alpha^- \vee (X_n)_\alpha^+| \, dp$$

$$\leq \int_{(|(X_n)_\alpha^- \vee (X_n)_\alpha^+|) \geq \lambda} \alpha |[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]| \, dp + \int_{(|(X_n)_\alpha^- \vee (X_n)_\alpha^+|) \geq \lambda} \alpha |(X)_\alpha^- \vee (X)_\alpha^+| \, dp$$

$$\leq \int_\Omega \alpha |[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]| \, dp + \int_{(|(X_n)_\alpha^- \vee (X_n)_\alpha^+|) \geq \frac{\lambda}{2}} \alpha |(X)_\alpha^- \vee (X)_\alpha^+| \, dp + \int_{\substack{(|(X_n)_\alpha^- \vee (X_n)_\alpha^+|) \geq \lambda, \\ |[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]| \geq \frac{\lambda}{2})}} \alpha |[(X)_\alpha^- \vee (X)_\alpha^+]| \, dp$$

As $n \to \infty$, by stipulation, we obtain

$$\int_\Omega \alpha |[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]| \, dp \to 0.$$

By Dominated Convergence Theorem,
as $n \to \infty$ and for each $\lambda > 0$,

$$\int_{\substack{(|(X_n)_\alpha^- \vee (X_n)_\alpha^+|) \geq \frac{\lambda}{2}, \\ |[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]| \geq \frac{\lambda}{2})}} \alpha |(X)_\alpha^- \vee (X)_\alpha^+| \, dp \to 0.$$

As $n \to \infty$, and by Dominated Convergence Theorem, we get

$$\int_{(|[(X)_\alpha^- \vee (X)_\alpha^+]|) \geq \frac{\lambda}{2}} \alpha |(X)_\alpha^- \vee (X)_\alpha^+| \, dp \to 0$$

Thus, there exists numbers $n(\epsilon)$ and $\lambda(\epsilon)$ such that for all $\lambda \geq \lambda(\epsilon)$,

$$\sup_{n\geq n(\varepsilon)} \int_{(|(X_n)_\alpha^- \vee (X_n)_\alpha^+|)\geq\lambda} \alpha\left(\left|(X)_\alpha^- \vee (X)_\alpha^+\right|\right) dp < \epsilon \tag{2.1}$$

Since a finite sequence of integrable random variables $\{X_n : 1 \leq n \leq n(\epsilon)\}$ is always uniformly integrable, it follows that the full sequence $\{X_n\}$ is uniformly integrable.

Thus condition (ii) holds

Conversely, by (ii), Since $\{X_n\}_{n=1}^{\infty}$ is Uniformly Integrable,

$$\int_\Omega \alpha \left|\left[(X_n)_\alpha^- \vee (X_n)_\alpha^+\right]\right| dp \leq \int_{(|[(X_n)_\alpha^- \vee (X_n)_\alpha^+]|)\geq\lambda} \alpha|(X_n)_\alpha^- \vee (X_n)_\alpha^+|\, dp + \lambda$$

$$\leq \epsilon + \lambda(\epsilon). \text{ for sufficiently large } \lambda(\epsilon). \tag{2.2}$$

Moreover $\{\int_\Omega \alpha|(X_n)_\alpha^- \vee (X_n)_\alpha^+|dp\}_{n=1}^{\infty}$ is a bounded sequence.

For each α ∈ (0, 1], $(X_n)_\alpha^-$, $(X_n)_\alpha^+$ are crisp random variables. Then

By Fatous Lemma, $\int_\Omega \alpha|(X)_\alpha^- \vee (X)_\alpha^+|dp < \infty.$

Now

$$\int_{|[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]|\geq\lambda} \alpha|[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]|dp$$

$$= \int_{\substack{(|[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]|\geq\lambda, \\ |(X_n)_\alpha^- V (X_n)_\alpha^+|\geq\frac{\lambda}{2})}} \alpha|[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]|\, dp$$

$$+ \int_{\substack{(|(X_n)_\alpha^- V (X_n)_\alpha^+|<\frac{\lambda}{2}, \\ |[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]|\geq\lambda)}} \alpha|[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]|\, dp$$

$$\leq \int_{(|(X_n)_\alpha^- V (X_n)_\alpha^+|\geq\frac{\lambda}{2})} \alpha|(X_n)_\alpha^- \vee (X_n)_\alpha^+|\, dp$$

$$+ \int_{(|[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]|\geq\lambda)} \alpha|(X)_\alpha^- \vee (X)_\alpha^+|\, dp$$

$$+ \int_{\substack{(|(X_n)_\alpha^- V (X_n)_\alpha^+|)<\frac{\lambda}{2}, \\ |[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]|\geq\frac{\lambda}{2})}} \frac{\lambda}{2} + \alpha|(X)_\alpha^- \vee (X)_\alpha^+|\, dp \tag{2.3}$$

Using (ii) given $\epsilon > 0$ we choose λ = λ(ϵ) > 0 so large, we obtain

$$\int_{(|(X_n)_\alpha^- \vee (X_n)_\alpha^+|) \geq \frac{\lambda}{2}} \alpha|(X_n)_\alpha^- \vee (X_n)_\alpha^+|\, dp < \epsilon$$

By Lebesgue's Dominated Convergence theorem and using (i), we have

$$\lim_{n\to\infty} \int_{(|[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]| \geq \lambda)} \alpha|(X)_\alpha^- \vee (X)_\alpha^+|\, dp \to 0 \text{ and}$$

$$\lim_{n\to\infty} \int_{\substack{(|(X_n)_\alpha^- \vee (X_n)_\alpha^+|) < \frac{\lambda}{2}, \\ |[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]| \geq \frac{\lambda}{2}}} \frac{\lambda}{2} + \alpha|(X)_\alpha^- \vee (X)_\alpha^+|\, dp \to 0 \tag{2.4}$$

Thus

$$\lim_{n\to\infty} \sup \int_{(|[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]| \geq \lambda(\epsilon))} \alpha|[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]|\}\, dp \leq \epsilon \tag{2.5}$$

Again applying the Dominated Convergence theorem one also has

$$\lim_{n\to\infty} sup \int_{(|[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]| \geq \lambda(\epsilon))} \{\alpha|[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]|\, dp\} \to 0 \tag{2.6}$$

$$\lim_{n\to\infty} sup \int_{(|[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]| \geq \lambda(\epsilon))} \bigcup_{\alpha\in(0,1]} \alpha|[(X_n)_\alpha^- - (X)_\alpha^-] \vee [(X_n)_\alpha^+ - (X)_\alpha^+]|\, dp = 0 \tag{2.7}$$

$$\text{i.e.) } \lim \ \sup_{n\to\infty} \int_{|X_n - X| < \lambda(\epsilon)} |X_n - X| dp = 0 \tag{2.8}$$

Thus $\{X_n\}_{n=1}^{\infty}$converges in L^1 to the fuzzy random variable X, which completes the proof.

3. Uniformly absolutely continuous fuzzy valued function

Definition 3.1. Let G be a collection of integrable fuzzy valued functions. We say G is uniformly absolutely continuous if given ϵ< 0 there exist a δ > 0 such that for E ∈ A, μ(E) < δ then

$$\int_E |(\tilde{g})_\alpha^- \vee (\tilde{g})_\alpha^+| d\mu < \epsilon \quad \text{for all g } \in \text{G.}$$

Definition 3.2. A collection G of fuzzy random variables on (X, A, P) is said to be uniformly integrable if

$$\lim_{t\to\infty} \sup_{X\in G} \int_{[|\omega;|X_\alpha^-(\omega) V X_\alpha^+(\omega)|\geq t]} X(\omega)dp(\omega) = 0$$

Theorem 3.3. Let G be a collection of fuzzy random variables on (X, F, P) where P(X) < ∞. Then the following are equivalent.

(i) G is Uniformly Integrable.

$$(ii) \quad \sup_{X\in G} \int_{[|\omega;|X_\alpha^-(\omega) V X_\alpha^-(\omega)|\geq t]} \alpha[X(\omega)]dp(\omega) < \infty$$

and G is uniformly absolutely continuous.

Proof. (i) ⇒ (ii):

Suppose (i) holds. Then given $\epsilon = 1$ we can choose ‘t’ large enough so that

$$\sup_{X\in G} \int_{[|\omega;|X_\alpha^-(\omega) V X_\alpha^+(\omega)|\geq t]} \alpha|X_\alpha^-(\omega) V X_\alpha^+(\omega)|dp(\omega) < 1;\ \alpha \in (0,1]$$

Thus for each X ∈ G,

$$\int \alpha|X_\alpha^-(\omega) \text{ V } X_\alpha^+(\omega)|dp(\omega) = \int_{[|\omega;|X_\alpha^-(\omega) V X_\alpha^+(\omega)|\geq t]} \alpha|X_\alpha^-(\omega) V X_\alpha^+(\omega)|dp(\omega)$$

$$+ \int_{[|\omega;|X_\alpha^-(\omega) V X_\alpha^+(\omega)|<t]} \alpha|X_\alpha^-(\omega) \text{ V } X_\alpha^+(\omega)|dp(\omega)$$

$$\leq 1 + tP(X).$$

Hence G is integrable and

$$\sup_{X\in G} \int \alpha|X_\alpha^-(\omega) \text{ V } X_\alpha^+(\omega)|dp(\omega) < \infty.$$

Let $\epsilon > 0$ be given. Since G is uniformly integrable we can find t, sufficiently large so that for all X ∈ G

$$\int_{[\omega;|X_\alpha^-(\omega) V X_\alpha^+(\omega)|]\geq t} \alpha|X_\alpha^-(\omega) \text{ V } X_\alpha^+(\omega)|dp(\omega) < \frac{\epsilon}{2}$$

Then for all E ∈ A

$$\int_E \alpha|X_\alpha^-(\omega) \text{ V } X_\alpha^+(\omega)|dp = \int_{([\omega;|X_\alpha^-(\omega) V X_\alpha^+(\omega)|]\geq t)\cap E} \alpha|X_\alpha^-(\omega) \text{ V } X_\alpha^+(\omega)|dp(\omega)$$

$$+ \int_{([\omega;|X_\alpha^-(\omega) V X_\alpha^+(\omega)|]\geq t)\cap E} \alpha|X_\alpha^-(\omega) \text{ V } X_\alpha^+(\omega)|dp(\omega)$$

$$< \frac{\epsilon}{2} + \text{ tP(E)}.$$

$$\int_E \bigcup_{\alpha\in(0,1]} \alpha|X_\alpha^-(\omega) \text{ V } X_\alpha^+(\omega)|dp(\omega) < \frac{\epsilon}{2} + \text{ tP(E)}$$

$$\int_E \alpha(X(\omega))dp(\omega) < \frac{\epsilon}{2} + \text{ tP(E)}$$

If we choose $\delta(\in) = \frac{\epsilon}{2t}$ the for each E ∈ A, P(E) < δ(E) implies

$$\int_E \alpha(X(\omega))dp(\omega) < \epsilon forall X \in \text{G}$$

This proves (i) ⇒ (ii).

Conversely suppose that (ii) holds.

Let K = $\sup_{X\in G} \int \alpha|X_\alpha^-(\omega) \vee X_\alpha^+(\omega)|dp$,then for all t > 0 and $X \in$ G ,

$$\int \alpha|X_\alpha^- \vee X_\alpha^+|\text{dp} = \int_{[\omega;|X_\alpha^-(\omega)\vee X_\alpha^+(\omega)|]\geq t} \alpha|X_\alpha^-(\omega) \vee X_\alpha^+(\omega)|dp(\omega)$$

$$+ \int_{[\omega;|X_\alpha^-(\omega)\vee X_\alpha^+(\omega)|]<t} \alpha|X_\alpha^-(\omega) \vee X_\alpha^+(\omega)|dp(\omega)$$

$$\geq \int_{[\omega;|X_\alpha^-(\omega)\vee X_\alpha^+(\omega)|]\geq t} \alpha|X_\alpha^-(\omega) \vee X_\alpha^+(\omega)|dp(\omega)$$

$$\geq \text{ t Pr}(\{\omega; |X_\alpha^-(\omega) \vee X_\alpha^+(\omega)| \geq t\})$$

$$\leq \frac{1}{t}\int \alpha|X_\alpha^-(\omega) \vee X_\alpha^+(\omega)|dp$$

$$\leq \frac{K}{t}.$$

Let $\epsilon > 0$ be given. If we choose δ > 0 as given by the uniform absolute continuity of G then for $t > \frac{K}{\delta}$, we will have

$$\text{Pr}(\{\omega; |X_\alpha^-(\omega) \vee X_\alpha^+(\omega)| \geq t\}) < \frac{K}{t} < \delta$$

Thus for all $t > \frac{K}{\delta}$,

$$\int_{[\omega;|X_\alpha^-(\omega)\vee X_\alpha^+(\omega)|]\geq t} \alpha|X_\alpha^-(\omega) \vee X_\alpha^+(\omega)|dp(\omega) < \epsilon$$

$$\text{i. e.} \int_{[\omega;|X_\alpha^-(\omega)\vee X_\alpha^+(\omega)|]\geq t} \bigcup_{\alpha\in(0,1]} \alpha|X_\alpha^-(\omega) \vee X_\alpha^+(\omega)|dp(\omega) < \epsilon$$

$$\text{i. e.} \int_{[\omega;|X_\alpha^-(\omega)\vee X_\alpha^+(\omega)|]\geq t} \alpha\big(X(\omega)\big)dp(\omega) < \epsilon \, .$$

This proves (ii) ⇒ (i). Hence the Theorem.

4. Conclusion

The uniform integrability of fuzzy random variable is studied and good numbers of results are established. An attempt is made to study the uniformly absolutely continuous fuzzy valued functions and an equivalent condition for uniform integrable and uniformly absolutely continuous is determined.

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