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Common Fixed Point Theorem of two Self Mappings in Fuzzy Normed Spaces

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Abstract. In this paper, we prove common fixed point theorem for two self mappings defined on Fuzzy Normed Space. Our result is an extension of Cheng-Cheng Zhu et al.

Keywords: Fuzzy normed space, unique common fixed point, partially ordered set.

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1. Introduction

The well known Banach contraction mapping principle is a powerful tool in nonlinear analysis. Many mathematicians have much contributed to the improvement and generalization of this principle in many ways. Especially, some recent meaningful results have been obtained. When Zadeh [16] introduced the concept of fuzzy sets, any contributions added in different Mathematical subjects. Mihet [11] obtained some new results of modifying the notion of convergence in fuzzy metric space. The fuzzy sets were used widely in functional analysis and many authors enriched the matter, like Kramosil [10], George and Veeramni [6] are constructing the fuzzy metric spaces, Katras [9], Bag and Samanta [1] introduced and modified concept of fuzzy normed space, Goguen [7], and Sanchez [14] defined and studied fuzzy relations. Fuzzy partial ordered relations are introduced by Chon [3], while Yuan and Wu [15] introduced the concept of sub lattice. Chitra and Mordeson [2] defined fuzzy norm and thereafter the concept of fuzzy norm space has been introduced and generalized the different ways by Bag and Samanta [1]. Iterative techniques for approximating fixed point in Fuzzy normed spaces have been studied by various authors (see e.g. [4,5,8,12,13]).

2. Preliminaries

Definition 2.1. A binary operation $*:[0,1]\times[0,1]\rightarrow[0,1]$ is a continuous t-norm if satisfies the following conditions:

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a, \forall a \in [0,1]$;

(iv) $a*b \le c*d$, whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Definition 2.2. A fuzzy normed space is a triple $(X, M, *)$, where X is a vector space, $x, y \in X$ and $t, s > 0$,

(i) $M(x,t) > 0$

(ii) $M(x,t) = 1$ if and only if $x = 0$;

(iii) $M(cx,t) = M(x, \frac{t}{|c|})$ for all $c \ne 0$;

(iv) $M(x,s) * M(y,t) \le M(x+y, s+t)$;

(v) $M(x,.)$ is a continuous function of R^+ and $\lim_{t\to\infty} M(x,t) = 1$, $\lim_{t\to 0} M(x,t) = 0$

Definition 2.3. A sequence $\{x_n\}$ in a fuzzy normed space is said to be convergent if for each $r, 0 < r < 1$ and $t > 0$, there exists $n_0 \in N$ such that $M(x_n - x, t) > 1 - r$ for all $n \ge n_0$.

Definition 2.4. A sequence $\{x_n\}$ in a fuzzy normed space is said to be Cauchy if for each $r, 0 < r < 1$ and $t > 0$, there exists $n_0 \in N$ such that $M(x_n - x_m, t) < 1 - r$ for all $n, m \ge n_0$.

Definition 2.5. A fuzzy normed space is said to be complete if every Cauchy sequence is convergent.

Example 2.6: Let M be a fuzzy set on $X \times [0,\infty)$ defined by $M(x,t) = \frac{t}{t + |x|}$ for all $x \in X, t > 0$ and $*$ is a $t - norm$ defined by $a * b = ab$. Then $(X, M, *)$ is a fuzzy normed space.

3. Main results

Let $(X, °)$ be a partially ordered set, and let $(X, M, *)$ be a complete fuzzy normed space with continuous t-norm defined by $a*b = min\{a,b\}$. Let $f, g : X \to X$ be a mapping satisfying

$$M(fx - gy, kt) \ge *^2 \min\{M(fx - y, kt) * M(x - gy, kt) * M(fx - x, kt) \\ * M(y - gy, kt) * M(x - y, kt)\}$$

for which $x, y \in X$ and $t > 0$, where $0 < k < 1$. Suppose that $\{x_n\}$ is a non-decreasing sequence and $\lim_{n\to\infty} x_n = x$ and $x_n ° x$ for all $n \in N$, then f and g have a unique common fixed point in X.

Proof: Let $x_0 \in X$, construct the sequence $\{x_n\}$ by taking

$x_{n+1} = f(x_n), x_{n+2} = g(x_{n+1})$ for $n = 1,2,3...$ then we have that

$x_0 \circ x_1 \circ x_2 \circ \cdots \circ x_n \circ x_{n+1} \ldots$

Now put

$$\delta_n(t) = M(x_{n+1} - x_{n+2}, t).$$

Then by using (2.1) we have

$$\begin{aligned} M(x_{n+1} - x_{n+2}, kt) &= M(fx_n - gx_{n+1}, kt) \\ &\geq *^2 \min\{M(fx_n - x_{n+1}, kt) * M(x_n - gx_{n+1}, kt) * M(fx_n - x_n, kt) \\ &\quad * M(x_{n+1} - gx_{n+1}, kt) * M(x_n - x_{n+1}, kt)\} \\ &\geq *^2 \min\{M(x_{n+1} - x_{n+1}, kt) * M(x_n - x_{n+2}, kt) * M(x_{n+1} - x_n, kt) \\ &\quad * M(x_{n+1} - x_{n+2}, kt) * M(x_n - x_{n+1}, kt)\} \\ &\geq *^2 \min\{M(x_n - x_{n+2}, kt) * M(x_{n+1} - x_{n+2}, kt) * M(x_{n+1} - x_n, kt)\} \\ &\geq *^2 \min\{M(x_n - x_{n+1}, kt)\} \\ &= *^2 \delta_{n-1}(t) \end{aligned}$$

Thus it follows that $\delta_n(kt) \geq *^2 \delta_{n-1}(t)$, and so

$$\delta_n(t) \geq *^2 \delta_{n-1}(\frac{t}{k}) \geq *^{2^n} \delta_0(\frac{t}{k^n}).$$

On the other hand, we have

$$t(1-k)(1+kt+\cdots+k^{m-n-1}) < t, \forall m > n, 0 < k < 1.$$

By definition 2.2 we get that,

$$\begin{aligned} M(x_n - x_m, t) &\geq \min\{M(x_n - x_m, t(1-k)(1+k+\cdots+k^{m-n-1}))\} \\ &\geq \min\{M(x_n - x_{n+1}, t(1-k)) * M(x_{n+1} - x_m, t(1-k)(k+\cdots+k^{m-n-1}))\} \\ &\geq \min\{M(x_n - x_{n+1}, t(1-k)) * M(x_{n+1} - x_{n+2}, t(1-k)k) * \ldots * \\ &\quad M(x_{m-1} - x_m, t(1-k)k^{m-n-1})\}. \end{aligned}$$

It follows from (3.2) and (3.3) that,

$$\begin{aligned} M(x_n - x_m, t) &\geq \min\{M(x_n - x_{n+1}, t(1-k)) * M(x_{n+1} - x_{n+2}, t(1-k)k) * \ldots * \\ &\quad M(x_{m-1} - x_m, t(1-k)k^{m-n-1})\} \\ &\geq \min\{[*^{2^n} \delta_0(\frac{t(1-k)}{k^n})] * \ldots * [*^{2^{m-1}} \delta_0(\frac{t(1-k)}{k^n})]\} \\ &= *^{2^m - 2^n} \delta_0\left(\frac{t(1-k)}{k^n}\right) \end{aligned}$$

By the hypothesis, the $t-norm$ $*$ is defined as $a*b = min\{a,b\}$ for all $\varepsilon \in (0,1)$, there exist $\eta > 0$ such that $*^p(s) > 1-\varepsilon$ for all $\delta \in (1-\eta, 1]$ and for all p.

Note that, $\lim_{n\to\infty} \delta_0(\frac{t(1-k)}{k^n}) = 1$ for all $t > 0$ and $0 < k < 1$, we have that there exist n_0 such that $M(x_n - x_m, t) > 1-\varepsilon$, for all $m > n > n_0$. Thus $\{x_n\}$ is a

Cauchy sequence. Since X is complete, there exist $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

According to our assumption we have that $x_n \circ x$ for all $n \in N$. It follows from (3.1) that,

$$\lim_{n\to\infty} M(fx - x_{n+1}) = \lim_{n\to\infty} M(fx - gx_n, kt)$$
$$\geq *^2 \lim_{n\to\infty} \min\{M(fx - x_n, kt) * M(x - gx_n, kt) * M(fx - x, kt)$$
$$* M(x_n - gx_n, kt) * M(x - x_n, kt)\}$$
$$= 1.$$

Thus, $M(fx - x, kt) = 1$, that is $fx = x$.

Similarly,

$$\lim_{n\to\infty} M(x_{n+1} - gx, kt) = \lim_{n\to\infty} M(fx_n - gx, kt)$$
$$\geq *^2 \lim_{n\to\infty} \min\{M(fx_n - x, kt) * M(x_n - gx, kt) * M(fx_n - x_n, kt)$$
$$* M(x - gx, kt) * M(x_n - x, kt)\}$$
$$= 1.$$

Thus, $M(x - gx, kt) = 1$, that is $x = gx$.

Therefore, $x = fx = gx$.

Thus, x is the fixed point of f and g.

Uniqueness. To prove, uniqueness of x as a common fixed point of f and g, let z be another fixed point. Then by using (3.1) we have,

$$\lim_{n\to\infty} M(x_{n+1} - gz, kt) = \lim_{n\to\infty} M(fx_n - gz, kt)$$
$$\geq *^2 \min\{M(fx_n - z, kt) * M(x_n - gz, kt) * M(fx_n - x_n, kt)$$
$$* M(z - gz, kt) * M(x_n - z, kt)\}$$
$$= 1.$$

Thus, $M(x - z, kt) = 1$, that is $x = z$. This complete the proof.

Example 3.1. Let $X = R, M(x,t) = \frac{t}{t + |x|}, M(y,t) = \frac{t}{t + |y|}$ for every $x, y \in X$ and let $t > 0, a * b = \min\{a, b\}$ for all $a, b \in [0,1]$. Then $(X, M, *)$ is a complete fuzzy normed space. If X is used with the usual order $x \circ y \Leftrightarrow x - y \leq 0$, then $(X, \circ)$ is partially ordered set. Let $0 < k < 1$ and define $f(x, y) = \frac{x - y}{4}$ for any $x, y \in X$. Then we have,

$$M(fx - gy, kt) = \frac{kt}{kt + \frac{|fx - x + gy - y|}{4}}$$

$$\geq \min\{\frac{kt}{kt+|fx-x|}, \frac{kt}{kt+|gy-y|}\}$$
$$= \min\{\frac{t}{t+\frac{|fx-x|}{k}}, \frac{t}{t+\frac{|gy-y|}{k}}\}$$
$$= \min\{M(fx-x,t), M(gy-y,t)\}$$
$$= *^2 M(x-y,t).$$

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