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Fuzzy Valued Measure Based on Integral, Decomposition and Representation under Closed Intervals

***D.Rajan*[1] and *A.Beulah*[2]**

[1]Department of Mathematics, T.B.M.L College
Poraiyar -609307, Tamilnadu, India.
[1]Email: dan _rajan@rediffmail.com
[2]Corresponding author. Email: beulahsrk02@gmail.com

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Abstract. Fuzzy valued measure is a mapping which maps a σ- algebra to a set of fuzzy set with convex compact σ- level defined on banach space (β). In this paper, we investigate the notion of decomposition and representation of fuzzy valued measure (F.V.M) is given in the sequence of singular valued measure. Also we have been studied and develop the notion of integrals with the aid of F.V.M under closed intervals based on the proposed notions, we can prove relevant theorems.

Keywords: Fuzzy measure, representation, decomposition, hausdorff metric, closed fuzzy measures, fuzzy-valued measurable functions, fuzzy-valued integrals.

AMS Mathematics Subject Classification (2010): 94Dxx

1. Introduction

The concept of fuzzy set was introduced by Zadeh [5]. After that, many applications of fuzzy sets have been developed one of them is the fuzzy stochastic, fuzzy measures which utilizes the fuzziness and randomness in the analysis in fuzzy environmental. A concept of fuzzy integral (not fuzzy-valued integral) was defined by surgeon [4].Many formulations of fuzzy integrals have been developed. Sims and Wang [3] gave a good review in this area. The fuzzy-valued integral problem can be transformed as the closed interval problem.

We give the definition of the positive part and the negative part of fuzzy numbers, and then we use the Hausdorff metric to define the metric between two fuzzy numbers. Using this metric, we consider the limit of a sequence of fuzzy numbers.

Since the meaning of infinite sum of a sequence of fuzzy numbers is clear, the fuzzy-valued measure can be defined in the classical sense. We define the fuzzy-valued integral of fuzzy –valued simple measurable functions with respect to fuzzy-valued measures. Following that, we provide some Lemmas to define the fuzzy-valued integrals of nonnegative fuzzy-valued measurable functions with respect to fuzzy-valued measures and then extend to the general fuzzy-valued measurable functions. In this setting, the fuzzy-valued integral problem becomes the closed interval problem.

In fuzzy mathematics the term "fuzzy measure" is used to denote the notion which generalizes a measure - an additive function which maps σ −algebra to the Banach set. These generalizations can be made in different directions is but with one common property - they all have a grade of uncertainty. Two different approaches are dominant. The first approach treats the fuzzy measure as a non-additive function which maps σ- algebra to the set of real's. Additively is usually sub substituted by some form of continuity. The examples of this kind of fuzzy measure are possibility, plausibility, null – measures. The fuzzy valued measure played an important role in the branch of fuzzy. Measure has been studied recend in many authors. In fuzzy measure (named fuzzy valued measure) as a natural generalization of multivalued (or set valued) measure is considered. A subset $A \subset x$ can be identified by its characteristic function $k_A : x \rightarrow \{0.1\}$, and since the fuzzy set $u_A : x \rightarrow [0.1]$ is a natural generalization of the characteristic function.

Decomposition and representation of measurable and integrable fuzzy - valued mappings were considered. In [13] it was and proved that under some condition the measurable and integrable fuzzy - valued mapping can be decomposed or represented by a countable family of single valued mappings with the same properties as the original. In this paper, using results from [3], some other problem for the fuzzy - valued measure is solved.

2. Preliminaries

Definition 2.1. A fuzzy measure over a measurable space (X_λ,$\tilde{A}$) is a map :$\tilde{A}$ →[0,1] with the following properties

(i) m(∅)=0

(ii) m(x)=1

(iii) A⊆B implies m (A) ≤ m(B).

Definition 2.2. By a fuzzy valued measure $\tilde{\mu}$ on a measurable space (X_λ, M) we mean non-negative fuzzy valued set function defined for all sets of M and satisfying the following two conditions:

(i) $\tilde{\mu}(\emptyset)=\tilde{0}$

(ii) $\tilde{\mu}(\cup_{i=1}^{\infty} E_i) = \bigoplus_{i=1}^{\infty} \tilde{\mu}(E_i)$ exists for any sequence E_i of disjoint measurable sets.

$\tilde{\mu}$ is called closed (bounded or standard) fuzzy valued measurable. If $\tilde{\mu}$ is a non-negative closed (bounded or standard) fuzzy valued set function

Definition 2.3. Let X_λ is a universal set. Then a fuzzy subset $\widetilde{A}$ of X_λ is defined by its membership function

$\mu_{\tilde{A}}: x \rightarrow [0, 1]$. we can also write the fuzzy set $\tilde{A}$ as $\{(\mu_{\tilde{A}}(X_\lambda)): x \in X_\lambda\}$. We denote $\tilde{A}_\propto = \{X_\lambda : \mu_{\tilde{A}}(x) \geq \propto\}$ as the $\propto - level$ set of $\tilde{A}$.

Definition 2.4.

(i) $\tilde{A}$ is called normal fuzzy set if there exists x such that $\mu_{\tilde{A}}(x)$ =1.

(ii) $\tilde{A}$ is called convex fuzzy set if $\mu_{\tilde{A}}(\lambda x$ + (1-λ) y) ≥ min {$\mu_{\tilde{A}}(x)$,$\mu_{\tilde{A}}(y)$},

For λ ∈ [0,1](that is, $\mu_{\tilde{A}}$ is a quasi-concave function)

Definition 2.5.

(i) $\tilde{m}$ is called fuzzy number if $\widetilde{m}$ is a normal convex fuzzy set with piecewise continuous membership function $\mu_{\tilde{m}}$. The set of all fuzzy numbers is denoted by $\mathcal{F}_{\lambda}$, for all λ ,λ∈ (0,1]

(ii) $\tilde{m}$ is called closed fuzzy number if $\widetilde{m}$ is normal convex fuzzy set and its membership function $\mu_{\tilde{m}}$ is upper semi continuous and the α −level sets are Bounded $\forall\alpha \neq 0$. the set of all closed fuzzy numbers is denoted as $\mathcal{F}_{\lambda_{\alpha_{cl}}}$.

(iii) $\widetilde{m}$ is called bounded fuzzy number if $\tilde{m}$ is a normal convex fuzzy set and its membership function $\mu_{\widetilde{m}}$ has compact support. The set of all bounded fuzzy numbers is donated as $\mathcal{F}_{\lambda_b}$.

(iv) $\widetilde{m}$ is called standard fuzzy number if $\tilde{m}$ is a normal convex fuzzy set and its membership function $\mu_{\tilde{m}}$ is one-to-one. The set of all standard fuzzy numbers is denoted as $\mathcal{F}_{\lambda_s}$.

Definition 2.6. Let $\tilde{m}$ and $\tilde{n}$ be fuzzy numbers.

(i) The membership function of sum $\tilde{m}\oplus\tilde{n}$ is defined as

$$\mu_{\tilde{m}\oplus\tilde{n}}(z)= \sup_{x+y} \min\{\mu_{\tilde{m}}(x), \mu_{\tilde{n}}(y)\}.$$

(ii) The membership function of the opposite of $\tilde{m}$ is defined by

$$\mu_{\ominus\tilde{m}}(z) = \sup_{z=-x} \min \mu_{\tilde{m}}(x) = \mu_{\tilde{m}}(-z)$$

(iii) The different of $\tilde{m}$ and $\tilde{n}$ is define as $\tilde{m} \ominus \tilde{n} = \tilde{m}\oplus (\ominus \tilde{n})$.

(iv) The membership function of product $\tilde{m} \otimes \tilde{n}$ is defined by

$$\mu_{\tilde{m}\otimes\tilde{n}}(z) = \sup_{xy=z} \min\{\mu_{\tilde{m}}(x) ,\mu_{\tilde{n}}(y)\}.$$

(v) The membership function of the inverse of $\tilde{m}$ is defined by

$$\mu 1/\tilde{m}(z) = \sup_{z=1/x,x\neq0} \min \mu_{\tilde{m}}(x) = \mu_{\tilde{m}}(1/z).$$

(vi) The quotient of $\tilde{m}$ and $\tilde{n}$ is defined as $\tilde{m} \oslash \tilde{n} = \tilde{m} \otimes$(1/n).

Remark 2.1.

$$\mu_{\tilde{m}\ominus\tilde{n}}(z)=sup_{x-y=z}\min\{\mu_{\tilde{m}}(x), \mu_{\tilde{n}}(y)\} \text{and}$$

$$\mu_{\tilde{m}\oslash\tilde{n}}(z)=sup_{z=x/y,y\neq0}\min\{\mu_{\tilde{m}}(x), \mu_{\tilde{n}}(y)\}.$$

Definition 2.7. Let $\tilde{a}$ be a fuzzy number.

(i) $\tilde{a}$ is called nonnegative if $\mu_{\tilde{a}}(x)$ =0¸$\forall x < 0$.

(ii) $\tilde{a}$ is called no positive if $\mu_{\tilde{a}}(x)$ =0¸$\forall x > 0$.

(iii) $\tilde{a}$ is called positive if $\mu_{\tilde{a}}(x) = 0, \forall x \le 0$.
(iv) $\tilde{a}$ is called negative if $\mu_{\tilde{a}}(x) = 0, \forall x \ge 0$.
(iv) The membership function of $\tilde{0}$ is defined by

$$\mu_{\tilde{0}}(r)=\begin{cases}1 & if\ r = 0,\\ 0 & if\ r \ne 0.\end{cases}$$

Definition 2.8. We say that $\tilde{a}$ is a crisp number with value m if and only if its membership function is

$$\mu_{\tilde{a}}(r)=\begin{cases}1 & if\ r = m\\ 0 & otherwise\end{cases}$$

Let $\tilde{a}$ be a fuzzy number. We define the membership functions of $\tilde{a}^{+}$ and $\tilde{a}^{-}$ by

$$\mu_{\tilde{a}^{+}}(r) = \begin{cases}\mu_{\tilde{a}}(r) & if\ r>0,\\ 1 & if\ r=0\ and\ \mu_{\tilde{a}}(r)<1,\ \forall r>0,\\ \mu_{\tilde{a}}(0) & if\ r=0\ and\ \exists r>0\ such\ that\ \mu_{\tilde{a}}(r)=1,\\ 0 & otherwise\end{cases}$$

$$\mu_{\tilde{a}^{-}}(r) = \begin{cases}\mu_{\tilde{a}}(r) & if\ r<0,\\ 1 & if\ r=0\ and\ \mu_{\tilde{a}}(r)<1,\ \forall r<0\\ \mu_{\tilde{a}}(0) & if\ r=0\ and\ \exists r<0\ such\ that\ \mu_{\tilde{a}}(r)=1,\\ 0 & otherwise\end{cases}$$

Remark 2.2. (Sakawa and Yano [2]). $(\tilde{m} \odot \tilde{n})_x = \tilde{m}_x \odot \tilde{n}_x$.

Definition 2.9.

(i) Let [a¸b] and [c, d] be closed in-travel. We say that [a, b] $\succcurlyeq_{int}$ $[c, d]$ if and only if a$\ge c\ and$ b$\ge$d.
(ii) Let $\tilde{a}$ and $\tilde{b}$ be closed fuzzy numbers we say that $\tilde{a} \succcurlyeq \tilde{b}$ if and only if $\tilde{a}_x \succcurlyeq_{int} \tilde{b}_x\ \forall_x$.
(iii) Let $\tilde{a}$ and $\tilde{b}$ be closed fuzzy numbers. We say that $\tilde{a} \subseteq_{cl} \tilde{b}$ if and only if $\tilde{a}_x \subseteq \tilde{b}_x\ \forall_x$.

Definition 2.10.

(i) If $\bar{A} \subseteq \mathcal{F}_{\lambda_{\alpha cl}}$ (the set of all closed fuzzy numbers)¸ $\tilde{u} \in \mathcal{F}_{\lambda_{cl}}$ then
(ii) $\tilde{u}$ is called the least upper bound of $\tilde{A}$ if and only if the following two conditions hold:
  (a) $\tilde{u} \succcurlyeq \tilde{a}, \forall \tilde{a} \in \tilde{A}$,
  (b) If $\tilde{u} \succcurlyeq \tilde{a}$,$\forall \tilde{a} \in \tilde{A}$ then $\tilde{u} \succcurlyeq \tilde{u}$
(iii) $\tilde{l}$ is called the greatest lower bound of $\tilde{A}$ if and only if the following two conditions hold:
  (a) $\tilde{a} \succcurlyeq \tilde{l}, \forall \tilde{a} \in \tilde{A}$,
  (b) If $\tilde{a} \succcurlyeq \tilde{m}$, $\forall \tilde{a} \in \tilde{A}$ then $\tilde{l} \succcurlyeq \tilde{m}$.

We denote $\tilde{u} = \grave{sup}\ \tilde{A}$ and $\tilde{l} = \grave{inf}\ \tilde{A}$.

Definition 2.11. If $A\subseteq \mathbb{R}^n$ and $B\subseteq \mathbb{R}^n$, the Hausdorff metric is defined by

$$d_H(A,B)= max\{ \sup_{a\in A} \inf_{b\in B} \|a-b\| , \sup_{b\in B} \inf_{a\in A} \|a-b\| \}$$

Definition 2.12. We define the metric $d_{\mathcal{F}_{\lambda}cl}$ in $\mathcal{F}_{\lambda_{cl}}$ and $d_{\mathcal{F}_{\lambda}b}$ in $\mathcal{F}_{\alpha_b}$ b

(i) $d_{\mathcal{F}_{\lambda}cl}(\tilde{a},\tilde{b}) = sup_{0<\chi\leq 1}\, d_H(\tilde{a}_x,\tilde{b}_x)$ for any two fuzzy numbers (if one of them is not them is not bounded fuzzy numbers) [1].

(ii) $d_{\mathcal{F}_{\lambda_\alpha}b}(\tilde{a},\tilde{b})= sup_{0\leq x\leq 1}\; d_H(\tilde{a}_x,\tilde{b}_x)$ for any two bounded fuzzy numbers

Remark 2.4. (Pure and Relics [1]) $(\mathcal{F}_{\lambda_{cl}}, \mathcal{F}_{\lambda_b})$ is a complete metric space.

Lemma 2.1. Let $\tilde{a}$ and $\tilde{b}$ be closed fuzzy numbers. Then we have

$$d_H(\tilde{a}_x,\tilde{b}_x) = \max\{|\tilde{a}^{L}{}_x - \tilde{b}^{L}{}_x| , |\tilde{a}^{U}{}_x - \tilde{b}^{U}{}_x| \}$$

Proof: We have $d_H(\tilde{a}_x, b_x) = d_H([\tilde{a}^{L}{}_x,\tilde{a}^{U}{}_x],[\tilde{b}^{L}{}_x, \tilde{b}^{U}{}_x])$

$= \max\{ \sup_{\tilde{a}^{L}{}_x\leq x\leq \tilde{a}^{U}{}_x} \inf_{\tilde{b}^{L}{}_x\leq y\leq \tilde{b}^{U}{}_x} |x-y| , \sup_{\tilde{b}^{L}{}_x\leq y\leq \tilde{b}^{U}{}_x} \inf_{\tilde{a}^{L}{}_x\leq x\leq \tilde{a}^{U}{}_x} |x-y|\}$ (by def 2.11)

(i) If $\tilde{a}_x \succcurlyeq_{int} \tilde{b}_x$ or $\tilde{b}_x \succcurlyeq_{int} \tilde{a}_x$ then $d_H(\tilde{a}_x,\tilde{b}_x) =$ $\{|\tilde{a}^{L}{}_x-\tilde{b}^{L}{}_x|, |\tilde{a}^{U}{}_x-\tilde{b}^{U}{}_x|\}$.

(ii) If $\tilde{a}_x \subseteq \tilde{b}_x$ or $\tilde{b}_x \subseteq \tilde{a}_x$ then $d_H(\tilde{a}_x,\tilde{b}_x)$ $= \max\{0, \max\{|\tilde{a}^{L}{}_x-\tilde{b}^{L}{}_x|, |\tilde{a}^{U}{}_x-\tilde{b}^{U}{}_x|\}\}$ $= \max\{|\tilde{a}^{L}{}_x-\tilde{b}^{L}{}_x| , |\tilde{a}^{U}{}_x-\tilde{b}^{U}{}_x|$.

Hence the proof.

Lemma 2.2. Let $\tilde{a}$ and $\tilde{b}$ be closed fuzzy numbers.

(i) Let $\mathcal{F}_\lambda(X_\lambda) = d_H(\tilde{a}_x ,\tilde{b}_x)$. If $\tilde{a}$ and $\tilde{b}$ are also standard fuzzy numbers then $\mathcal{F}_\lambda(X_\lambda)$ is continuous on[0,1] .

(ii) If $d_{\mathcal{F}_{\lambda}cl}(\tilde{a},\tilde{b}) < \varepsilon$ then $|\tilde{a}^{L}{}_x - \tilde{b}^{L}{}_x| < \varepsilon$ and, $|\tilde{a}^{U}{}_x - \tilde{b}^{U}{}_x| < \varepsilon$ for $0 < x \leq 1$.

(iii) Suppose that $\tilde{a}$ and $\tilde{b}$ are bounded and standard fuzzy numbers, then $d_{\mathcal{F}_{\lambda}b}(\tilde{a},\tilde{b}) < \varepsilon$ if and only if $|\tilde{a}^{L}{}_x - \tilde{b}^{L}{}_x| < \varepsilon$ and $|\tilde{a}^{U}{}_\chi-\tilde{b}^{U}{}_\chi| < \varepsilon$ for $0\leq x \leq 1$.

Proof:

(i) Since $\tilde{a},\tilde{b} \in \mathcal{F}_{\lambda_{cl}} \cap \mathcal{F}_{\lambda_s}$, We have that $\tilde{a}^{L}{}_x$, $\tilde{a}^{U}{}_x$, $\tilde{b}^{L}{}_x$ and $\tilde{b}^{U}{}_x$ are continuous with respect to $\propto$ (by definition).The result follow from previous Lemma.

(ii) By Lemma 2.1.

(iii) By (i) and proposition 2.1, we have $d_{\mathcal{F}_{\lambda_b}}(\tilde{a},\tilde{b}) = sup_{0\leq x\leq 1}\; d_H(\tilde{a}_x,\tilde{b}_x) < \varepsilon$ if and only if $d_H(\tilde{a}_x,\tilde{b}_x) < \varepsilon$ for $0\leq x \leq 1$,By Previous Lemma, the proof is complete.

Lemma 2.3. Let $\{\tilde{a}_m\}$ be a sequence of closed fuzzy numbers. If $\grave{\lim}_{m\to\infty}\tilde{a}_m$ exists then it is unique and

$(\grave{\lim}_{m\to\infty}\tilde{a}_m)_\propto = [\lim_{m\alpha\to\infty}(\tilde{a}_m)^L{}_\alpha, \lim_{m\to\infty}(\tilde{a}_m)^U{}_\alpha]\ \forall\alpha$

(Furthermore, $\{(\tilde{a}_m)^L{}_\alpha\}$ and $\{(\tilde{a}_m)^U{}_\alpha\}$ converge uniformly with respect to α).

Proof: Let $\grave{\lim}_{m\to\infty}\tilde{a}_m = \tilde{a}$.

We know that $d_{\mathcal{F}_{\lambda_{cl}}}(\tilde{a}_n, \tilde{a}) \Leftrightarrow\ |(\tilde{a}_m)^L{}_\alpha - \tilde{a}^L{}_\alpha| < \varepsilon$ and $|(\tilde{a}_m)^U{}_\alpha - \tilde{a}^U{}_\alpha| < \varepsilon, \forall\alpha$.

i.e., $(\grave{\lim}_{m\to\infty}\tilde{a}_m)^L{}_\alpha = \breve{a}^L{}_\alpha = \lim_{m\to\infty}(\tilde{a}_m)^L{}_\alpha$ and

$(\grave{\lim}_{m\beta\to\infty}\tilde{a}_m)^U{}_\alpha = \tilde{a}^U{}_\alpha = \lim_{m\to\infty}(\tilde{a}_m)^U{}_\alpha \forall\alpha$. The uniqueness follows from the uniqueness of classical notion of limit.

For a sequence of fuzzy numbers$\{\tilde{a}_m\}$, we define the limit interior of $\{\tilde{a}_m\}$ by

$$\acute{\lim_{m\to\infty}} \inf \tilde{a}_m = \underset{m\geq 1}{\acute{sup}}\ \underset{k\geq m}{\acute{inf}}\ \tilde{a}_k$$

And by limit superior by

$$\acute{\lim_{m\to\infty}} \sup \tilde{a}_m = \underset{m\geq 1}{\acute{inf}}\ \underset{k\geq m}{\acute{sup}}\ \tilde{a}_k$$

Hence the result

Definition 2.13. Let $\{\tilde{a}_k\}$ be a sequence of fuzzy number.

If $\grave{\lim}_{k\to\infty}\overset{n}{\underset{i=1}{\oplus}}\tilde{a}_i$ exists then we define $\overset{\infty}{\underset{k=1}{\oplus}}\tilde{a}_k = \grave{\lim}_{k\to\infty}\overset{k}{\underset{i=1}{\oplus}}\tilde{a}_i$,

Otherwise the infinite (fuzzy) sum of sequence $\{\tilde{a}_k\}$ is said to diverge.

3. Fuzzy-valued measures and fuzzy-valued measurable functions

In this section, we discuss some properties of fuzzy-valued measures and fuzzy-valued measurable functions.

Definition 3.1. Let $\mathcal{F}_\lambda$ be a set of all fuzzy numbers, $\mathcal{F}_{\lambda_{\alpha_{cl}}}$ a set of all closed fuzzy numbers, $\mathcal{F}_{\lambda_b}$ a set of all bounded fuzzy numbers and $\mathcal{F}_{\lambda_s}$ a set of all bounded fuzzy numbers. We say that

(i) $\widetilde{f_\lambda}(x)$ Is a fuzzy–valued function if $\widetilde{f_\lambda}: X_\lambda \to \mathcal{F}_\lambda$.

(ii) $\widetilde{f_\lambda}x)$ Is a closed –fuzzy–valued function if $\widetilde{f_\lambda}: X_\lambda \to \mathcal{F}_{\lambda_{cl.}}$

(iii) $\widetilde{f_\lambda}(x)$ is a bounded–fuzzy–valued functioning if $\widetilde{f_\lambda}: X_\lambda \to \mathcal{F}_{\lambda_b}$

(iii) $\widetilde{f_\lambda}(x)$ Is a standard–fuzzy–valued function if $\widetilde{f_\lambda}: X_\lambda \to \mathcal{F}_{\lambda_s}$.

We denote $\widetilde{f_\lambda}^L{}_\alpha(x) = (\widetilde{f_\lambda}(x))^L{}_\alpha$ and $\widetilde{f_\lambda}^U{}_\alpha(x) = (\widetilde{f_\lambda}(x))^U{}_\alpha$.

Definition 3.2. Let $\widetilde{f_\lambda}$ and $\tilde{g}_\lambda$ be two closed-fuzzy-valued-functions.

(i) $\widetilde{f_\lambda} \succcurlyeq_{fvf} \tilde{g}_\lambda$ if and only if $\widetilde{f_\lambda}(x) \succcurlyeq \tilde{g}_\lambda(x)\ \forall x \in X_\lambda$.

(ii) $\widetilde{f_\lambda} \subseteq_{fvf} \tilde{g}_\lambda$ If and only if $\widetilde{f_\lambda}(x) \subseteq_{cl} \tilde{g}_\lambda(x)\ \forall x \in X_\lambda$.

Definition 3.3. By a fuzzy-valued measure $\tilde{\mu}$ on a measurable space (X_λ, M), we mean a nonnegative fuzzy–valued set function defined for $\mathcal{F}_\lambda$all sets of M and satisfying the following two conditions:

(i) $\mu\ (\emptyset)= \tilde{0}$.

(ii) $\mu\ (\bigcup_{i=1}^{\infty} F_{\lambda i}) = \bigoplus_{i=1}^{\infty} \tilde{\mu}(F_{\lambda i})$ exists for any sequence $F_{\lambda i}$ of disjoint measurable sets.

$\tilde{\mu}$ is called closed-(bonded-, or standard-) fuzzy-valued measure if $\tilde{\mu}$ is a non-negative closed (bounded-, or standard-) fuzzy-valued set function.

Definition 3.4. $\tilde{\infty} \succ \tilde{a}$ If and only if $\tilde{a}^{L}{}_{\alpha} < \infty$ and $\tilde{a}^{U}{}_{\alpha} < \infty\ \forall \alpha$.

Theorem 3.1. Let $\tilde{\mu}$ be a closed-fuzzy-valued measure.

(i) Let A and B be measurable sets. If A ⊆ B then $\tilde{\mu}$(B) ≥ $\tilde{\mu}$(A).

(ii) If $F_{\lambda i} \subseteq F_{\lambda i_{+1}}$ and $\lim_{k\to\infty} \tilde{\mu}\ (F_{k\lambda})$ exists then $\tilde{\mu}\ = \lim_{k\to\infty}(F_{k\lambda})$

(iii) If $F_\lambda \subseteq F_{\lambda i_,} \tilde{\infty} \succ \tilde{\mu}(F_1)$ and $\lim_{k\to\infty} \tilde{\mu}(F_k)$ exists then $\tilde{\mu} = \lim_{k\to\infty} \tilde{\mu}\ (A_k)$.

Proof:

(i) B $= A \cup (B\backslash A)$ is a disjoint union. Then

(B) = (A ∪(B/A)) = (A) ⊕ (B/A) ≥ (A)

(ii) By stipulation $F_{\lambda i}{=}F_{\lambda i_{i+1}}$.

Let $A_1 = F_{1\lambda}$.

$A_k = F_{\lambda k} – F_{\lambda k+1}$; n>1.

Then $A_k \cap A_l = \emptyset$ for k ≠ l (l=L) and

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} F_{\lambda k}$$

$$F_{\lambda n} = \bigcup_{k=1}^{n} A_k$$

$$\tilde{\mu}(F_{\lambda n}) = \tilde{\mu}\ (\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} \tilde{\mu}\ (A_k)$$

$$\tilde{\mu}\ (\bigcup_{k=1}^{\infty} F_{\lambda k}) = \tilde{\mu}\ (\bigcup_{k=1}^{\infty} A_k)$$

$$\Rightarrow (\tilde{\mu}\ (\bigcup_{i=1}^{\infty} F_{\lambda i}))^{L}{}_{\alpha} = (\tilde{\mu}\ (\bigcup_{k=1}^{\infty} A_k))^{L}{}_{\alpha}$$

$$= \sum_{k=1}^{\infty}(\tilde{\mu}\ (A_k))^{L}{}_{\alpha} = \lim_{n\to\infty} \sum_{k=1}^{n}(\tilde{\mu}\ (A_n))^{L}{}_{\alpha}$$

$$= \lim_{n\to\infty} \tilde{\mu}\ (F_{\lambda n})^{L}{}_{\alpha} = \lim_{k\to\infty}(\tilde{\mu}\ (F_{\lambda k}))^{L}{}_{\alpha}$$

i.e., $(\tilde{\mu}\ (\bigcup_{i=1}^{\infty} F_{i\lambda}))^{L}{}_{\alpha} = \lim_{k\to\infty}(\tilde{\mu}\ (F_{\lambda k})^{L}{}_{\alpha}$

Similarly, we can prove

$$(\tilde{\mu}\ (\bigcup_{i=1}^{\infty} F_{\lambda i}))^{U}{}_{\alpha} = \lim_{k\to\infty}(\tilde{\mu}\ (F_{\lambda k}))^{U}{}_{\alpha}$$

$$[\ (\tilde{\mu}\ (\bigcup_{i=1}^{\infty} F_{\lambda i}))^{L}{}_{\alpha}, (\tilde{\mu}\ (\bigcup_{i=1}^{\infty} F_{\lambda i}))^{U}{}_{\alpha}\ ] = \lim_{k\to\infty}[(\tilde{\mu}\ (F_{\lambda k}))^{L}{}_{\alpha}, (\tilde{\mu}\ (F_{\lambda k}))^{U}{}_{\alpha}]$$

$$\Rightarrow (\tilde{\mu}\ (\bigcup_{i=1}^{\infty} F_i))_{\alpha} = \lim_{k\to\infty}(\tilde{\mu}\ (F_{\lambda k}))_{\alpha}$$

$$\Rightarrow \bigcup_{\alpha\in[0,1]} \alpha(\tilde{\mu}\ (\bigcup_{i=1}^{\infty} F_{\lambda i}))_{\alpha} = \lim_{k\to\infty} \bigcup_{\alpha\in[0,1]} \alpha\ (\tilde{\mu}\ (F_{\lambda k}))_{\alpha}$$

$$\Rightarrow \tilde{\mu}\ (\bigcup_{i=1}^{\infty} F_{\lambda i}) = \lim_{k\to\infty} \tilde{\mu}\ (F_{\lambda k})$$

(iii) By stipulation $F_{\lambda i}+1 \subseteq F_{\lambda i}$

Then $(F_1 - F_k)_\lambda \subset (F_1 - F_{k+1})_\lambda$ and

$F_1 - \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} (F_1 - F_k)_\lambda$

Hence by part (2)

$\tilde{\mu}(F - \bigcup_{k=1}^{\infty} F_k)_\lambda = \tilde{\mu}(\bigcup_{k=1}^{\infty}(F_{1\lambda} - F_k))$

$= \lim_{k\to\infty} \tilde{\mu}\,(F_{1\lambda} - F_k)$ by

$= \tilde{\mu}\,(A_1) - \lim_{k\to\infty} \tilde{\mu}\,(F_{\lambda k})$

$\tilde{\mu}(F_1) - \tilde{\mu}\,(\bigcup_{k=1}^{\infty} F_k) = \tilde{\mu}\,(A_1) - \lim_{k\to\infty} \tilde{\mu}\,(F_{\lambda k})$

$\Rightarrow \tilde{\mu}\,(\bigcup_{k=1}^{\infty} F_k) = \lim_{k\to\infty} \tilde{\mu}\,(F_{\lambda k})$

If $\{\tilde{a}_k\}$ is a sequence of closed fuzzy numbers and $\lim_{k\to\infty} \bigoplus_{i=1}^{k} \tilde{a}_i$

Exists then we can write

$\left(\bigoplus_{k=1}^{\infty} \tilde{a}_k\right)\alpha = [\sum_{k=1}^{\infty} (\tilde{a}_k)^{L}{}_{\alpha}, \sum_{k=1}^{\infty} (\tilde{a}_k)^{U}{}_{\alpha}]$

Let $\tilde{\mu}$ be a closed fuzzy valued measured on a measurable space(X_λ, M). We define

$\tilde{\mu}^{L}{}_{\alpha}(F_\lambda) = (\tilde{\mu}^{L}{}_{\alpha}(F_\lambda))^{L}{}_{\alpha}$ and $\tilde{\mu}^{U}{}_{\alpha}(F_\lambda) = (\tilde{\mu}^{L}{}_{\alpha}(F_\lambda))^{U}{}_{\alpha}$

Then $\tilde{\mu}^{L}{}_{\alpha}$ and $\tilde{\mu}^{U}{}_{\alpha}$ are the classical measures and measurable space(X_λ, M).

Hence the result.

4. The fuzzy-valued integrals

In this section, we introduce the concept of fuzzy integral of a positive measurable function with respect to a fuzzy measurable. Let $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n$. If we set $A_i = \{x: \tilde{T}_\lambda(x) = \tilde{a}_i\}$ then $X = \bigcup_{i=1}^{n} A_i$ is a disjoint union.

Let, $\tilde{\tilde{T}}_{\lambda_{\tilde{a}_1}}(x) = \begin{cases} \tilde{a}_i\ ; & if\ x \in A_i \\ \tilde{0}\ ; & otherwise \end{cases}$

Then $\tilde{T}_\lambda(x) \bigoplus_{i=1}^{\infty} \tilde{T}_{\lambda_{\tilde{a}_i}}(x)$

It is easy to note that $\tilde{T}_\lambda{}^{L}{}_{\alpha}$ and $\tilde{T}_\lambda{}^{U}{}_{\alpha}$ are simple measurable Functions with

$\tilde{T}_\lambda{}^{L}{}_{\alpha}(x) = \sum_{i=1}^{n} (a_i)^{L}{}_{\alpha} 1_{A_i}(x)$ and

$\tilde{T}_\lambda{}^{U}{}_{\alpha}(x) = \sum_{i=1}^{n} (a_i)^{U}{}_{\alpha} 1_{A_i}(x)$

Definition 4.1. Let $\tilde{T}_\lambda$ be a fuzzy-valued simple measurable function of the form $\tilde{T}_\lambda(x)$ $= \bigoplus_{i=1}^{n} \tilde{T}_{\lambda_{\tilde{a}_i}}(x) \int_{F_\lambda} \tilde{T}_\lambda d\tilde{\mu}$

Then we define the fuzzy-valued integral of $\tilde{T}_\lambda(x)$ by

$\int_{F_\lambda} \tilde{T} d\tilde{\mu} = \bigoplus_{i=1}^{n} \tilde{a}_i \otimes \tilde{\mu}(A_i \cap F_\lambda)$.

Theorem 4.1. Let $\tilde{\mu}$ be a closed-fuzzy-valued measure.

(i) If $\tilde{T}_\lambda$ is a non-negative closed-fuzzy-valued simple measurable function then

$$(\int_F \tilde{T} d\tilde{\mu})_\alpha = \left[\int_{F_\lambda} \tilde{T_\lambda}^{L}{}_\alpha \, d\tilde{\mu}^{L}{}_\alpha, \int_{F_\lambda} \tilde{T_\lambda}^{U}{}_\alpha \, d\tilde{\mu}^{U}{}_\alpha\right].$$

(ii) If $\tilde{T}_\lambda$ is a non-positive closed-fuzzy-valued simple measurable function then

$$(\int_F \tilde{T} d\tilde{\mu})_\alpha = \left[\int_{F_\lambda} T^{L}{}_\alpha \, d\tilde{\mu}^{U}{}_\alpha, \int_{F_\lambda} \tilde{T}^{U}{}_\alpha \, d\tilde{\mu}^{L}{}_\alpha\right].$$

Proof:

(i) $(\int_F \tilde{T} d\tilde{\mu})^{L}{}_\alpha = \overset{n}{\underset{i=1}{\oplus}} \tilde{a}_i \otimes \tilde{\mu}(A_i \cap F_\lambda))^{L}{}_\alpha$

$= \sum_{i=1}^{n} (\tilde{a}_i \otimes \tilde{\mu}(A_i \cap F_\lambda))^{L}{}_\alpha$

$= \sum_{i=1}^{n} (\tilde{a}_i)^{L}{}_\alpha \, (\tilde{\mu}(A_i \cap F_\lambda))^{L}{}_\alpha$

$= \sum_{i=1}^{n} (\tilde{a}_i)^{L}{}_\alpha \tilde{\mu}^{L}{}_\alpha (A_i \cap F_\lambda)$

(Both $\tilde{a}_i$ and $\tilde{\mu}(A_i \cap F_\lambda)$ are non-negative)

$= \int_{F_\lambda} \tilde{T_\lambda}^{L}{}_\alpha \mathrm{d}\, \tilde{\mu}^{L}{}_\alpha$

Similarly we can prove

$\int_{\boldsymbol{F}} (\tilde{T} \mathrm{d}\tilde{\mu})^{U}{}_\alpha = \int_{F_\lambda} \tilde{T_\lambda}^{L}{}_\alpha \mathrm{d}\, \tilde{\mu}^{L}{}_\alpha$

Then $(\int_F \tilde{T} \mathrm{d}\tilde{\mu})_\alpha = [\int_{F_\lambda} (\tilde{T}_\lambda \mathrm{d}\tilde{\mu})^{L}{}_\alpha, \int_F (\tilde{T}_\lambda \mathrm{d}\tilde{\mu})^{U}{}_\alpha]$

$= [\int_{F_\lambda} \tilde{T_\lambda}^{L}{}_\alpha \mathrm{d}\, \tilde{\mu}^{L}{}_\alpha, \int_F \tilde{T_\lambda}^{U}{}_\alpha \mathrm{d}\, \tilde{\mu}^{U}{}_\alpha]$

(ii) $\int_{\boldsymbol{F}} (T \mathrm{d}\tilde{\mu})^{U}{}_\alpha = (\overset{n}{\underset{i=1}{\oplus}} \tilde{a}_i \otimes \tilde{\mu}(A_i \cap F_\lambda))^{L}{}_\alpha$

$= \sum_{i=1}^{n} (\tilde{a}_i \otimes \tilde{\mu}(A_i \cap F_\lambda))^{L}{}_\alpha$

($\tilde{a}_i$ Is non-positive and $\tilde{\mu}(A_i \cap F_\lambda))^{L}{}_\alpha$

$= \sum_{i=1}^{n} (\tilde{a}_i)^{L}{}_\alpha \tilde{\mu}^{U}{}_\alpha (A_i \cap E)$

$= \int_{F_\lambda} \tilde{T_\lambda}^{L}{}_\alpha \mathrm{d}\, \tilde{\mu}^{U}{}_\alpha$

Similarly we can prove

$\int_F (\tilde{T} d\tilde{\mu})^{U}{}_\alpha = \int_{F_\lambda} \tilde{T_\lambda}^{U}{}_\alpha \mathrm{d}\, \tilde{\mu}^{U}{}_\alpha$

Then $(\int_F \tilde{s} d\tilde{\mu})_\alpha = [\int_{F_\lambda} \tilde{T_\lambda}^{L}{}_\alpha \mathrm{d}\ \tilde{\mu}^{U}{}_\alpha, \int_{F_\lambda} \tilde{T_\lambda}^{U}{}_\alpha \mathrm{d}\tilde{\mu}^{L}{}_\alpha]$ contd……..

Hence the proof.

Definition 4.2. Let $\{\tilde{f}_n\}$ be a sequence of fuzzy valued measurable functions defined over a measurable set X_λ.Let $\tilde{f}$ be a fuzzy valued measurable function defined on X_λ.We say $\{\tilde{f}_n\}$ is said to converge in measure to the fuzzy valued measurable function $\tilde{f}$ if it satisfies the following conditions.

(i) $\tilde{\infty} > \tilde{f}\,(x)$ a.e. on the set X_λ

(ii) For $x \in X$, and for each $\in > 0$,

$\lim_{n\to\infty} (\tilde{\mu}\,\{|\tilde{f}_n(x) - \tilde{f}(x)| \geq \in\})^{L}{}_\alpha = 0$ and

$\lim_{n\to\infty}(\tilde{\mu}\{|\tilde{f}_n(x)-\tilde{f}(x)|\geq\in\})^{U}{}_{\alpha}$

We can theoreprove below the

Theorem 4.2. If {x: r=f(x)} is compact set $\forall_r$then (f $(\tilde{a}))_x = \{f(x): x \in \tilde{a}_x\}$.

Proof:

1. (⊇): If r∈ $\{(x):x\in \tilde{a}_x\}$ then there exists x such that $r=f(x)$ and $x\in \tilde{a}_{\propto}$, that is ,$\mu_{\tilde{\propto}}(x)\geq\propto$.Thus $sup_{r=f(x)}\mu_{\tilde{a}}(x)\geq\propto$ implies $r\in(f(\tilde{a}))_x$.
2. (⊆): If $r\in (f(\tilde{a}))_{\propto}$ then $sup_{r=f(x)}\mu_{\tilde{a}}(x)\geq\propto$, that is, $\exists_x$ such that $\mu_{\tilde{a}}(x)\geq\propto$ and $r=(x)$. (Since {x: r=f(x)} is a compact set and $\mu_{\tilde{a}}(x)$ is an upper semi continuous, if $\mu_{\tilde{a}}(x)<\propto\ \forall\propto$ then $sup_{r=f(x)}\mu_{\tilde{a}}(x)\geq\propto$ .Therefore,$r\in \{f(x): x\in\tilde{a}_{\propto}\}$. Let $(x)=|x|$,then $|\tilde{a}_{\propto}|=\{|r|: r\in\tilde{a}_{\propto}\}\forall\propto$.

Proposition 4.11. Let $\tilde{a}$ and $\tilde{b}$ be closed and bounded fuzzy numbers.
Given$\varepsilon > 0$, if $|\tilde{a}^L{}_X-\tilde{b}^L{}_X| < \varepsilon$ then $|(\tilde{a}^+)^L{}_X-(\tilde{b}^+)^L{}_X| < \varepsilon$, $|(\tilde{a}^-)^L{}_X-(\tilde{b}^-)^L{}_X| < \varepsilon$ and if $|\tilde{a}^U{}_X-\tilde{b}^U{}_X| < \varepsilon$ then $|(\tilde{a}^-)^U{}_X-(\tilde{b}^+)^U{}_X| < \varepsilon$, $|(\tilde{a}^-)^U{}_X-(\tilde{b}^-)^U{}_X| < \varepsilon$

Proof: We have $|((\tilde{a}^-)^L{}_x\text{-}(\tilde{b}^+)^L{}_x)+|((\tilde{a}^-)^L{}_x\text{-}(\tilde{b}^-)^L{}_x)|<\varepsilon$.
For fixed α, if $(\tilde{a}^+)^L{}_x>0$(or $(\tilde{a}^+)^U{}_x>0$) then $(\tilde{a}^-)^L{}_x=0$(or $(\tilde{a}^-)^U{}_x=0$) and if $(\tilde{a}^-)^U{}_x<0$(or $(\tilde{a}^-)^U{}_x<0$) then $(\tilde{a}^+)^L{}_x=0$(or $(\tilde{a}^+)^U{}_x=0$).
Now, it is enough to prove that $(\tilde{a}^+)^L{}_x\text{-}(\tilde{b}^+)^L{}_x$and $(\tilde{a}^-)^L{}_x\text{-}(\tilde{b}^-)^L{}_x$have the same sign or one of them is zero. Suppose that $(\tilde{a}^+)^L{}_x \geq(\tilde{b}^-)^L{}_x$.Then we have two cases

(i) If $(\tilde{b}^+)^L{}_x>0$ then $(\tilde{a}^-)^L{}_x= (\tilde{b}^-)^L{}_x=0$.

(ii) If $(\tilde{b}^+)^L{}_x=0$ then we just need to consider If $(\tilde{a}^+)^L{}_x>0$.Now If $(\tilde{a}^-)^L{}_x\text{-}(\tilde{b}^-)^L{}_x=\text{-}(\tilde{b}^-)^L{}_x \geq0$(since $(\tilde{a}^+)^L{}_x>0$), that is, $(\tilde{a}^-)^L{}_x \geq (\tilde{b}^-)^L{}_x$. Similarly, if $(\tilde{a}^+)^L{}_x \leq (\tilde{b}^+)^L{}_x$ then $(\tilde{a}^-)^L{}_x \leq (\tilde{b}^-)^L{}_x$, thus the result follows immediately. Similarly for the upper bound case.

5. Representation of fuzzy valued measure and decompose to $x = \mathbf{R}$

In this section, Q_1 will donate the set of rational in [0.1]. Let $(\Omega, \bar{\mathcal{A}}, \bar{\mu})$ a complete, finite measure space, x a separable branch space here we consider.

$\wp(x) = \{A \subseteq x$: nonempty, closed}

$\wp_{c\kappa}(x) = \{A \subseteq x$: nonempty, convex, and compact}

We defined the measurability of multi-valued mapping G : $\Omega \to \wp(x)$ is defined by [11].

By T_G we denote the set of all selections of G that belong to the Lebesgue Bochner space L (x), i.e. $T_G = \{g \in L(x): g(\omega) \in G(\omega)\ \bar{\mu}\text{- a. e.}\}$. T_G is nonempty integrably bounded, i.e. G is measurable and $\omega \to |G(\omega)| = svp_{x\in G(\omega)}\|x\|$ belongs to L(x).

The norm $\|\cdot\|_L$ in L(x) given by
$\|g\|_L = \int_\Omega \| g(\omega)\| \, d_{\bar{\mu}}$

The integral is defined by setting
$\int_\Omega G \, d_{\bar{\mu}} = \{\int_\Omega g d_{\bar{\mu}} : g d_{\bar{\mu}:} \, g \in T_G\}$
If A,B ∈ $\wp(x)$, their Hausdorff distance is defined by
H (A, B) = $\max\{{}^{\text{sur}}_{a\in A} \, {}^{\text{inf}}_{b\in B} \| a-b \| {}^{\text{sur}}_{a\in A} \, {}^{\text{inf}}_{b\in B} \| a-b \| \}$
The fuzzy set u on x is the mapping $u: x \to [0.1]$. the subset $u_\alpha \subseteq x$ defined by $u_\alpha \subseteq x$ where $u_\alpha = \{x \in x : u(x) \geq \alpha\}$, where $\alpha \in (0.1]$ is called α- level of u and the subset $u_0 \subseteq x$, $u_0 = cl\{x \in x: u(x) > 0\} = cl \bigcup_{\alpha\in(0.1]} u_\alpha$ is support of u. Let $\mathcal{F}_{ck}(x)$ denote the set of all fuzzy sets u: $x \to [0.1]$ such that for all $\alpha \in (0.1], u_\alpha$ belongs to $p_{ck}(x)$.

Definition 5.1. Let $(\Omega, \bar{\mathcal{A}})$ be a measurable space with a σ- measurable subsets of the **s**et Ω we shall call the mapping $\bar{\mu}: \bar{\mathcal{A}} \to \mathcal{F}_{ck}(x)\bar{\mathcal{A}}$ a fuzzy - valued measure such that for every sequence $\{\bar{\mathcal{A}}_1\}$ of pair wise disjoint elements of $\bar{\mathcal{A}}, \sum_{i=1}^{\infty} |\mu_\alpha(A_i)| < \infty$ For all $\alpha \in (0.1]$ and the next equality is satisfied.

$$\bar{\mu}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \bar{\mu}(A_i),$$

where

$$\left(\sum_{i=1}^{\infty} \bar{\mu}(A_i)\right)(x) = \sup\left\{\bigwedge_{i=1}^{\infty} \bar{\mu}(A_i)(x_i) \; x = \sum_{i=1}^{\infty} x_i\right.$$

Theorem 5.1. If x has the Radon – Nikodym property x is separable and $\bar{\mu}: \bar{\mathcal{A}} \to \boldsymbol{\mathcal{F}_{ck}}(x)$ .is a $\bar{\mu}$-continuous fuzzy – valued measure of bounded variation. Then there exists a sequence $\{m_{\alpha k}\}_{\substack{\alpha\in Q1 \\ k\in N}}$ $m_{\alpha k}: \bar{\mathcal{A}} \to$ of $\bar{\mu}$- continuous measures of bounded variation. such that for every A ∈ $\bar{\mathcal{A}}$ the following conditions are satisfied:

1. If $\alpha, \beta \in Q_1, \alpha < \beta$, then $\{m_{\beta k}\}_{k\in N} \subseteq \{m_{\alpha k}\}_{k\in N}$,
2. For every $\alpha \in Q_1, \mu_\alpha(A) = cl\{m_{ak}(\text{A})\}_{k\in N}$
3. For every $\alpha \in (0.1], \mu_\alpha(\text{A}) = \cap_{\substack{\beta\in Q1 \\ \beta<\alpha}} \text{cl}\{m_{\beta k}(\text{A})\}_{k\in N}$g
4. $\bar{\mu}_0(A) = cl\ \{m_{\alpha k}(\text{A})\}_{\substack{\alpha\in Q1, \\ k\in N}}$

Proof: We known that $\bar{\mu}$ has a Radon − Nikodym derivative $X_\lambda : \mathcal{F}_{ck}(x)$, *that is*
$\bar{\mu}(A) = \int_A X_\lambda \, d\bar{\mu}$ for all $A \in \bar{\mathcal{A}}$, $\forall \lambda$

Also if a is fixed, then α- level $\bar{\mu}_\alpha$ of fuzzy- valued measure $\bar{\mu}$ and α-level X_{λ_α} of radon-nikodym derivative X_λ are connected by

$\bar{\mu}_\alpha(A) = \int_A X_{\lambda_\alpha}\, d\bar{\mu}$

The representation of integrable fuzzy- valued mapping X_λ by the sequence $\{f_{\alpha k}\}_{\substack{\alpha\in Q1\\ k\in N}}$ is provide and for every $\omega \in \Omega$.

$X_{\lambda_\alpha}(\omega) = cl\{f_{\alpha k}(\omega)\}_{k\in N}$

The mappings $f_{\alpha k}, \alpha \in Q_1, k \in N$ are integrable and by the quality

$m_{\alpha k}(A)= \int_A f_{\alpha k}\, d\bar{\mu}, \alpha \in Q_1, k \in N, A \in \bar{\mathcal{A}}$

The sequence $\{m_{\alpha k}\}_{\substack{\alpha\in Q1\\ k\in N}}$ of $\bar{\mu}$- continuous measure of bounded variation are defined.

It only remains to shoe that $\bar{\mu}_\alpha(\Omega) = cl\{m_{\alpha k}(\Omega)\}_{k\in N}$ for every $\alpha \in Q_1$.

Since $\bar{\mu}_\alpha(\Omega)$ is closed and $m_{\alpha k}(\Omega) \in \bar{\mu}(\Omega)$ for every $k \in N$, it follows that

$cl\{m_{\alpha k}(\Omega)\}_{k\in N} \subseteq \bar{\mu}(\Omega)$.

To prove that $\bar{\mu}_\alpha(\Omega) \subseteq cl\{m_{\alpha k}(\Omega)\}_{k\in N}$ we show that the assumption

$x \notin cl\{m_{\alpha k}(\Omega)\}_{k\in N}$ implies that $x \notin \mu_\alpha(\Omega)$. Really, $x \notin cl\{m_{\alpha k}(\Omega)\}_{k\in N}$ from closeness

of the set $cl\{m_{\alpha k}(\Omega)\}_{k\in N}$ it follows that there exist an $\varepsilon > 0$ such that for all $k \in N$.

$$\| x - m_{\alpha k}(\Omega) \|_\lambda > \varepsilon.$$

If the mapping h : $\Omega \to x$ is such that x = $\int_\Omega h\, d\bar{\mu}$ the $\varepsilon < \| x - m_{\alpha k}(\Omega) \|_\lambda$

$$= \| \int_\Omega h\, d\bar{\mu} - \int_\Omega f_{\alpha k}\, d\bar{\mu} \| \le \int \| h - f_{\alpha k} \|_\lambda\, d\bar{\mu}$$

$$= \| h - f_{\alpha k} \|_{L\lambda} \text{ For all, } k \in N,$$

Which means that h $\notin T_{X_\lambda}$ where T_{X_λ} is the set measurable integrable selection of X_{λ_α} finally we can conclude

$X_\lambda = \int_\Omega h\, d\bar{\mu} \notin \int_\Omega X_\lambda\, d\bar{\mu} = \bar{\mu}_\alpha(\Omega)$

Having proved the inclusion in both directions, we get $\bar{\mu}_\alpha(\Omega) = cl\{m_{\alpha k}(\Omega)\}_{k\in N}$,

The preceding arguments are now repeated for the case when instead of Ω we have any $A \in \bar{\mathcal{A}}$.

To conclude the proof, conditions (1)-(4) can be proved.

Note: The next theorem discusses the problem how to construct the fuzzy valued mapping using the sequence of single valued measures. In this theorem we suppose that χ is finite dimensional.

Theorem 5.2. Let $\{m_{\alpha k}\}_{\substack{\alpha\in q1\\ k\in N}}$, $m_{\alpha k} : \bar{\mathcal{A}} \to x$, be a sequence of $\bar{\mu}$- continuous fuzzy measures of bounded variation such that

1. For all $\alpha, \beta \in Q_1, \alpha < \beta$ and every A$\in \bar{\mathcal{A}}$
   $\{m_{\beta k}(\text{A})\}_{k\in N} \subseteq \{m_{\alpha k}\ (A)\}_{k\in N}$,
2. If $\{\alpha_i\}_{i\in N}$ is a non-decreasing sequence converging to $\alpha \in Q_1$ then
   $\overline{co}\{m_{\alpha k}(\text{A})\}= \bigcap_{i=1}^{\infty} \overline{co}\,\{m_{\alpha i k}(\text{A})\}_{k\in N}$, for all A $\in \bar{\mathcal{A}}$
3. For every A $\in \bar{\mathcal{A}}$
   $\overline{co}\{m_{\alpha k}(\text{A})\}_{\alpha\in Q1,}$k $\in N, \in p_{ck}(x)$,
4. There exists an integrable mapping h: $\Omega \rightarrow R$ such that $\|(dm_{\alpha k}\ /\ d_{\bar{\mu}}(\omega)\ \|_{\lambda} < h(\omega)$ for all $\alpha \in Q_1$ and k $\in$ N.

Then the sequence $\{m_{\alpha k}\}_{\substack{\alpha\in Q1\\ k\in N}}$ generates one and only one $\bar{\mu}$- continuous fuzzy – valued measure of bounded vartiation defined by

$$\bar{\mu}(A)(x) = sup\,\{\alpha: x \in \bar{\mu}_{\alpha}(A)\}, x \in x, A \in \bar{\mathcal{A}}\ .$$

Proof: Let us take first a family of multi-valued mappings $\bar{\mu}_{\alpha}: \bar{\mathcal{A}} \rightarrow p_{ck}(x), \alpha \in (0.1]$, by

If $\alpha \in Q_1$, then $\bar{\mu}_{\alpha}$(A)= $\overline{co}\{m_{\alpha k}(A)\}_{k\in N}$ for all A $\in \bar{\mathcal{A}}$,

If $\alpha \in (0.1] \setminus Q_1$, then $\bar{\mu}_{\alpha}(A) = \bigcap_{i=1}^{\infty} \overline{co}\{m_{\alpha i K}(A)\}_{K\in N}$ for all A $\in \bar{\mathcal{A}}$, where $\{\alpha_i\}_{K\in N} \subset Q_1$ is a

Non decreasing sequence converging to α.

Similarly we can be proved that that for every fixed A $\in \bar{\mathcal{A}}$ the family $\{\bar{\mu}_{\alpha}(A)\}_{\alpha\in(0.1]}$ defines one and only one fuzzy set which we denote by $\bar{\mu}(A)$.

We show that the mapping $\bar{\mu}$: $\bar{\mathcal{A}} \rightarrow \boldsymbol{\mathcal{F}}_{ck}\,(x)$ is a fuzzy valued measure.

Since $\bar{\mu}_{\alpha}(A)$ are closed subset of the compact set from$(3')$, it follows that $\bar{\mu}(A) \in p_{cl}\ (x)$. In order prove that $\bar{\mu}_{\alpha}, \alpha \in (0.1], \bar{\mu}$- continuous multi- valued measure of bounded variation we consider two cases when $\alpha \in Q_1$ and when $\alpha \in (0.1] \setminus Q_1$ in both cases we shall show that there exists an integrably bounded multi-valued mapping $X_{\alpha}: \Omega \rightarrow \rho(x)$ such that $\bar{\mu}_{\alpha}(A) = \int_A X_{\lambda_{\alpha}} d\bar{\mu}$.

By $f_{\alpha k}: \Omega \rightarrow x, \alpha \in Q_1$, k $\in N$ we denote the Radon- Nikodym derivative of $m_{\alpha k}$ i.e. $m_{\alpha k}(A) = \int_A f_{\alpha k} \text{d}\bar{\mu}$ for all A $\in \bar{\mathcal{A}}$.

And by $X_{\lambda_{\alpha}}(\omega), \alpha \in Q_1$ we denote the set $\overline{co}\{f_{\alpha k}(\omega)\}_{k\in N}$ from the set of integrable selections $T_{X_{\lambda}}$ of $X_{\lambda_{\alpha}}$ is equal to the set $\overline{co}\ \{f_{\alpha k}\}_{k\in N}$ since, for every f $\in \overline{co}$ $\{f_{\alpha k}\}_{k\in N}$ and every $\varepsilon > 0$ therexist $a_1, a_2, \dots, a_n \in R, \sum_{j=1}^{n} a_j = 1$ such that

$$\|\ f(\omega) - \textstyle\sum_{j=1}^{n} a_j\, f_{\alpha k}(\omega)\ \| < \varepsilon, \bar{\mu}\text{-}a.e.,$$

we get $\|\ f(\omega)\ \| = \left\|f(\omega) - \sum_{j=1}^{n} a_j f_{\alpha k j}\,(\omega)\right\|_{\lambda}$

$$\leq \left\|f(\omega) - \textstyle\sum_{j=1}^{n} a_j f_{\alpha k j}\,(\omega)\right\|_{\lambda} + \left\|\textstyle\sum_{j=1}^{n} a_j f_{\alpha k j}\,(\omega)\right\|_{\lambda} \leq \varepsilon + h(\omega),$$

This means that the mapping X_α is integrably bounded

To prove that $\bar{\mu}_\alpha(A) = f_A X_{\lambda_\alpha} d\bar{\mu}$, it is sufficient to show that

$$\overline{co}\left\{\int_\Omega f_{\alpha k}\, d\bar{\mu}\right\}_{k\in N} = \int_\Omega \overline{co}\, \{f_{\alpha k}(\omega)\}_{k\in N} d\bar{\mu}$$

Since x is finite dimensional, $\int_\Omega \overline{co}\ \{f_{\alpha k}(\omega)\}_{k\in N} d\bar{\mu}$ is compact and convex.

It is obvious that $f_{\alpha k} \in \overline{co}\, \{f_{\alpha k}\}_{k\in N}$ and

$$\{\int_\Omega f_{\alpha k}\, d\bar{\mu}\}_{k\in N} \subseteq \int_\Omega \overline{co}\ \{f_{\alpha k}(\omega)\}_{k\in N}\, d\bar{\mu}$$

which implies that $\overline{co}\ \{\int f_{\alpha k} d\bar{\mu}\}_{k\in N} \subseteq \int_\Omega \overline{co}\, \{f_{\alpha k}(\omega)\}_{k\in N}\, d\bar{\mu}$, on the other hand, if we suppose that $\in \int_\Omega \overline{co}\ \{f_{\alpha k}(\omega)\}_{k\in N} d\bar{\mu}$ then there exists

$$f \in \overline{co}\, \{f_{\alpha k}\}_{k\in N} \text{ such that}$$

$\chi = \int_\Omega \mathrm{f}\, d\bar{\mu}$. For every $\varepsilon > 0$ there exists $a_1, a_2, \ldots\ldots a_n$, $\sum_{j=1}^{n} a_j = 1$,for which $\varepsilon > \left\|f - \sum_{j=1}^{n}(a_j f_{\alpha kj})\right\|_\lambda = \int_\Omega \left\|f - \sum_{j=1}^{n}(a_j f_{\alpha kj})\right\|_\lambda \geq d\bar{\mu}\ \left\|\int_\Omega f - \sum_{j=1}^{n}(a_j f_{\alpha kj})\, d\bar{\mu}\right\|_\lambda = \left\|\int_\Omega f d\bar{\mu} - \sum_{j=1}^{n} a_j \int_\Omega f_{\alpha kj} d\bar{\mu}\right\|$

The last relation shows that $\chi = \int_\Omega \mathrm{f}\, d\bar{\mu}$ belongs to $\overline{co}\ \{\int_\Omega f_{\alpha k} d\bar{\mu}\}_{k\in N}$ that is

$$\int_\Omega \overline{co}\, \{f_{\alpha k}(\omega)\}_{k\in N}\, d\bar{\mu} \subseteq \overline{co}\left\{\int_\Omega f_{\alpha k}\, d\bar{\mu}\right\}_{k\in N}$$

This together with the opposite inclusion proved earlier, means that

$$\bar{\mu}(\Omega) = \overline{co}\ \{\int_\Omega f_{\alpha k}\, d\bar{\mu}\}_{k\in N}$$
$$= \int_\Omega \overline{co}\ \{f_{\alpha k}(\omega)\}_{k\in N}\, d\bar{\mu} = \int_\Omega X_{\lambda_\alpha}\, d\bar{\mu}$$

Now, we consider the case when $\alpha \in (0.1] \setminus Q_1$ then for any non-decreasing sequence $\{a_i\}_{k\in N} \subset Q$ converging to α

$$\bar{\mu}_\alpha(\Omega) = \overline{co}\ \bar{\mu}_{\alpha i}(\Omega) = \cap_{i=1}^{\infty} \overline{co}\ \{m_{\alpha k}(\Omega)\}_{k\in N}$$
$$= \cap_{i=1}^{\infty} \overline{co}\ \{\int_\Omega f_{\alpha ik}\, d\bar{\mu}\}_{k\in N}$$
$$= \cap_{i=1}^{\infty} \int_\Omega \overline{co}\, \{f_{\alpha iK}(\omega)\}_{k\in N}$$

From non-increasingness and compactness of the sets from the sequence

$$\{\int_\Omega\ \overline{co}\, \{f_{\alpha iK}(\omega)\}_{k\in N}\, d\bar{\mu}\}_{k\in N} \text{ one gets } \cap_{i=1}^{\infty} \int_\Omega \overline{co}\ \{f_{\alpha iK}(\omega)\}_{k\in N}\, d\bar{\mu}$$
$$= H - \lim_{i\to\infty} f_\Omega \overline{co}\ \{f_{\alpha ik}(\omega)\}_{k\in N}\, d\bar{\mu}$$

The multi-valued mappings $X_{\lambda\alpha i}, X_{\lambda\alpha i},(\omega) = \overline{co}\{f_{\alpha ik}(\omega)\}_{k\in N}$ are uniformly integrably bounded so we can apply Lebesgue dominated convergence theorem (see [4]).

$$\bar{\mu}_\alpha(\Omega) = H - \lim_{i-\infty} f_\Omega \overline{co}\ \{f_{\alpha ik}(\omega)\}\, d\bar{\mu}$$
$$= \int_\Omega H - \lim_{i\to\infty} \overline{co}\ \{f_{\alpha ik}(\omega)\}_{k\in N}\, d\bar{\mu}$$

Since $\{\{\overline{co}\ f_{\alpha ik}(\omega)\ \}_{k\in N}\}_{i\in N}$ is a sequence of non increasing closed set with nonempty intersection one has

$$H - \lim_{i\to\infty} \overline{co}\{f_{\alpha ik}(\omega)\}_{k\in N} = \cap_{i=1}^{\infty} \overline{co}\ \{f_{\alpha ik}(\omega)\}_{k\in N}$$

This is denoted by $X_{\lambda_\alpha}(\omega)$,$\alpha \in (0.1] \setminus Q_1$Using same arguments we could repeat the proof when instead of Ω we have any $A \in \bar{\mathcal{A}}$.

So we have proved that for every$\alpha \in (0.1]$, $\bar{\mu}_\alpha(A) = \int_A X_{\lambda_\alpha} d\bar{\mu}$, A ∈ $\bar{\mathcal{A}}$. Which means that $\bar{\mu}_\alpha$ is a $\bar{\mu}$- continuous multi valued measure of bounded variation. Using above one concludes that the fuzzy valued mapping $\bar{\mu}: \bar{\mathcal{A}} \rightarrow \boldsymbol{\mathcal{F}}_{ck}(x)$ defind by $\bar{\mu}(A)(x) = \sup\{\alpha: x \in \bar{\mu}_\alpha(A)\}$ is $\bar{\mu}$ −continuous fuzzy valued measure of bounded variation.

The case when x = R

In this special case when $x = R$ is considered. In the set of reals the convex and compact subset is a closed interval, which means that to every subset A from σ- algebra $\bar{\mathcal{A}}$ a fuzzy number $\bar{\mu}(A)$ is related.

For every $\alpha \in (0.1]$, α-level of $\bar{\mu}(A)$ (i.e. $\mu_\alpha(A)$ is a closed interval

$$\bar{\mu}_\alpha(A) = [l_{\lambda\alpha}(A), s_{\lambda\alpha}(A)]$$

It is obvious that there is a one-to-one correspondence $\bar{\mu}$ and a pair or single-valued measures ι and s.

The fuzzy-valued measure $\bar{\mu} : \bar{\mathcal{A}} \rightarrow \boldsymbol{\mathcal{F}}_{ck}(\boldsymbol{R})$ can be defined by two families of single- valued measures

$l_{\lambda\alpha} : \bar{\mathcal{A}} \rightarrow \boldsymbol{R}$ and $s_{\lambda\alpha}: \bar{\mathcal{A}} \rightarrow \boldsymbol{R}$, $\alpha \in (0.1]$.

If $\alpha \in (0.1]$ is fixed, then], $\bar{\mu}_\alpha : \bar{\mathcal{A}} \rightarrow p_{ck}(\boldsymbol{R})$ is a multivalued measure and it can be generated by a sequence of single-valued measures $\{m_{\alpha i}\}_{i \in N}$.

It is easy to see that every $\alpha \in (0.1]$ and every A ∈ $\bar{\mathcal{A}}(A)$

$l_{\lambda\alpha}(A) = \inf_{i \in N} \{m_{\alpha i}(\mathrm{A})\}$, $s_{\lambda\alpha}(A) = \sup_{i \in N} \{m_{\alpha i}(\mathrm{A})\}$

Hence the proof.

Example 5.1. In this example let the set Ω be the set of real's, σ- algebra A the Boral field $B(R)$ of real's and the fuzzy measure $\bar{\mu}$ the lebsegue fuzzy measure on R further, let the Banach space $(x, \|\ \|)$ be the of reals with the usual norm. for every $\alpha \in [0.1]$ and every A ∈ $B(R)$ let

$$l_{\lambda\alpha}(A) = \mu(A \cap [0, \alpha]),$$

$$T_{\lambda\alpha}(\mathrm{A}) = \mu(A \cap [\alpha - 1,1]),$$

Then the functions l:[0.1]× B(R) and s: [0.1] × B (R)→ *R* satisfy conditions 1-3(above theorem) which means that they generate a fuzzy - valued measure.

For instance, the fuzzy – valued measure of intervals [-$\frac{1}{2}$,$\frac{1}{2}$] and [0.1] are the fuzzy sets given below.

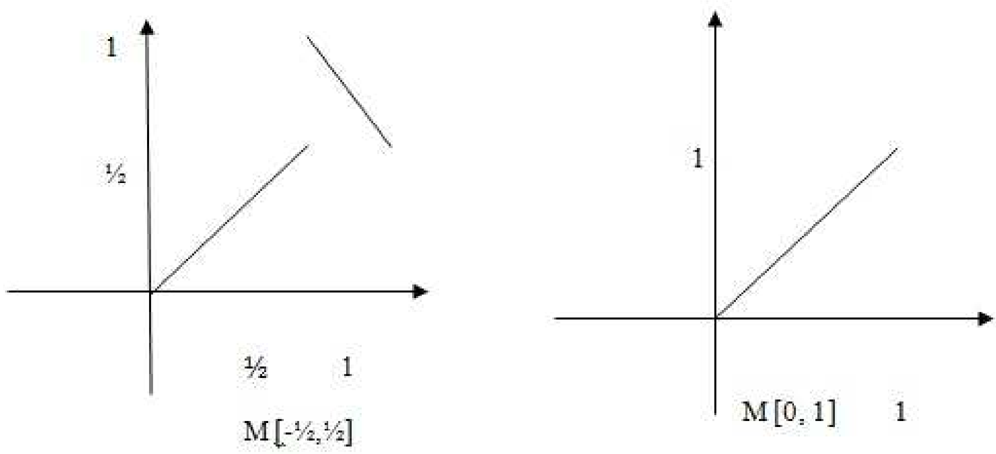

Figure 1:

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