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On the Number of Distinct Fuzzy Subgroups for Some Elementary Abelian Groups and Quaternion Groups

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Abstract. In this paper we compute the number of distinct fuzzy subgroups of elementary abelian groups of the form $\mathbb{Z}_p \times \mathbb{Z}_p$ and quaternion groups of the form Q_{4p} with respect to an equivalence relation $\approx$ existing in literature. The group algorithm programming (GAP) was applied in our computation.

Keywords: Fuzzy subgroups; quaternion groups; elementary abelian groups; equivalence relation; GAP

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1. Introduction

The theory of fuzzy set is very important and has many important applications, one of which is group theory [1]. Counting fuzzy subgroups of finite groups is a fundamental problem of fuzzy group theory. Research had focused on counting distinct fuzzy subgroups of finite groups with respect to an equivalence relation $\approx$[3], however the number of fuzzy subgroups of elementary abelian group (elementary abelian p-group) of the form $\mathbb{Z}_p \times \mathbb{Z}_p$ and quaternion groups of the form Q_{4p} had not been computed ($p = prime$) with respect to equivalence relation $\approx$. This paper was therefore designed to compute the number of distinct fuzzy subgroups of $\mathbb{Z}_p \times \mathbb{Z}_p$ and Q_{4p}.

The equivalence relation $\approx$ is preferred to other equivalence as its definition involves more group theoretical properties compared to other equivalence relation known in literature. For other equivalence relations see [4,5,6]. As explained in [3], the equivalence relation $\approx$ was used to obtain a formula for computing the number of distinct fuzzy subgroups of any finite groups, although this formula is not explicit as it contains computations of order of automorphism groups, automorphism groups structure, group actions, subgroup lattices and application of Burnside's lemma. It is also our aim to make this computation easier and less cumbersome, hence the introduction and use of

GAP, since it is efficient and reliable. The algorithm we have introduced reduces our computation to counting the chains of subgroups of the group say G that ends in G and applying our result in the corresponding formula derived in [3]. The paper is organized as follows. In section 2, we give a brief preliminary on fuzzy subgroups of a group and overview of the equivalence relation as defined in [3]. In section 3, we compute the number of fuzzy subgroups of $\mathbb{Z}_p \times \mathbb{Z}_p$ and Q_{4p}. In section 4, we give conclusion.

2. Preliminaries

Let G be a group and $F(G)$ be the collection of all fuzzy subset of G. An element θ of $F(G)$ is said to be fuzzy subgroup of G if satisfies the following conditions.

$(i)\theta(xy) \geq \min(\theta(x), \theta(y))$ for all $x, y \in G$

$(ii)\theta(x^{-1}) \geq \theta(x)$ for any $x \in G$. In this situation we have $\theta(x^{-1}) = \theta(x)$ for any $x \in G$ and $\theta(e) = \sup\theta(G)$ where e is the identity in G.

In mathematics, it is more convenient to study the relationship between objects rather than studying individual objects as members of a set. This leads to the formulation of the concept of relations sometimes called correspondences.

To classify the elements of a given set is basically to divide the set into disjoint subsets usually called classes. This concept leads to the concept of equivalence relation. This notion generalizes equality in the sense that, objects or elements that are related or equal in some sense are classified in the same class. This means that an equivalence relation on a set leads to the grouping of the set into literally disjoint subsets whose union gives back the set. Without any equivalence relation, the number of fuzzy subgroups of any finite group is infinite. The equivalence relation employed in [3] gave rise to the formula which has been used to calculate the number (N) of distinct fuzzy subgroups of some finite group and is given below;

$$N = \frac{1}{|Aut(G)|} \sum_{f \in Aut(G)} |Fix_{\overline{C}}(f)|$$

where $\overline{C}$ is the set consisting of all chains of subgroups of G terminated in G, $Fix_{\overline{C}}(f) = \{C \leq \overline{C} \mid f(C) = C\}$. For details see [3].

3. The number of distinct fuzzy subgroups of $\mathbb{Z}_p \times \mathbb{Z}_p$ and Q_{4p}

3.1. Distinct fuzzy subgroups of $\mathbb{Z}_p \times \mathbb{Z}_p$

An elementary abelian group is an abelian group in which every non-identity element is of order p. Our interest is to count the number of distinct fuzzy subgroups of $\mathbb{Z}_p \times \mathbb{Z}_p$ with respect to the equivalence relation $\approx$ using GAP [2] as a tool, we give the algorithms that were used in our computations.

Theorem 3.1. The number of distinct fuzzy subgroups of $\mathbb{Z}_p \times \mathbb{Z}_p$ with respect to $\approx$ is 4.

Proof: For case p=2; which has already been proved in [3], we give the algorithm that we have adopted;

```
GroupHomomorphismByImages( Group( [ (1,2), (3,4)] ),
Group( [ (1,2), (3,4) ] ), [ (1,2), (3,4) ],[ (1,2), (1,2)(3,4) ] )
[ Group( () ), Group( [ (1,2) ] ), Group( [ (3,4), (1,2) ] )]

gap> ℤ2:= CyclicGroup(IsPermGroup,2);
Group([ (1,2) ])
gap> ℤ2×ℤ2:= DirectProduct(ℤ2,ℤ2);
Group([ (1,2), (3,4) ])
gap> U:=AllSubgroups(ℤ2×ℤ2);
[ Group(()), Group([ (3,4) ]), Group([ (1,2) ]), Group([ (1,2)(3,4) ]),
Group([ (3,4), (1,2) ]) ]
gap> fi:=AutomorphismGroup(ℤ2×ℤ2);;Size(fi);
6
gap> for x in fi do Print(x, " ", Filtered(U,i->Images(x,i)=i), "\n"); od;
GroupHomomorphismByImages( Group( [ (1,2), (3,4) ] ),
Group( [ (1,2), (3,4) ] ),
[ (1,2), (3,4) ], [ (1,2), (3,4) ] )[ Group( () ),Group( [ (3,4) ] ),
Group( [ (1,2) ] ), Group( [ (1,2)(3,4) ] ),
Group( [ (3,4), (1,2) ] ) ]
GroupHomomorphismByImages( Group( [ (1,2), (3,4)] ),
Group( [ (1,2), (3,4) ] ), [ (1,2), (3,4) ],
[ (3,4), (1,2)(3,4) ] ) [ Group( () ), Group( [ (3,4), (1,2) ] ) ]
GroupHomomorphismByImages( Group( [ (1,2), (3,4) ] ),
Group( [ (1,2), (3,4) ] ), [ (1,2), (3,4) ],
[ (1,2)(3,4), (1,2) ] ) [ Group( () ), Group( [ (3,4), (1,2) ] ) ]
GroupHomomorphismByImages( Group( [ (1,2), (3,4) ] ),
Group( [ (1,2), (3,4) ] ), [ (1,2), (3,4) ],[ (1,2)(3,4), (3,4) ] )
[ Group( () ),Group( [ (3,4) ] ), Group([ (3,4), (1,2) ])]
```

GroupHomomorphismByImages(Group([(1,2), (3,4)]), Group([(1,2), (3,4)]), [(1,2), (3,4)], [(3,4), (1,2)])
[Group(()), Group([(1,2)(3,4)]), Group([(3,4),(1,2)])]

Notice that there are 6 elements of the automorphism group i.e. $Aut(D)=\{f_{1,}f_{2,}f_{3,}f_{4,}f_{5,}f_6\}$ and the subgroups that are fixed for each automorphism are displayed above. For example the automorphism $f_{1,}$ fixes all the subgroups of $\mathbb{Z}_2\times\mathbb{Z}_2$. Our computation is now reduced to counting the chains of subgroups of $\mathbb{Z}_2\times\mathbb{Z}_2$ that ends in $\mathbb{Z}_2\times\mathbb{Z}_2$.

$|fix_{\overline{C}}(f_1)|=8, |fix_{\overline{C}}(f_2)|=|fix_{\overline{C}}(f_3)|=2, |fix_{\overline{C}}(f_4)|=|fix_{\overline{C}}(f_5)|=|fix_{\overline{C}}(f_6)|=4$

Now the number(N) of distinct fuzzy subgroups of $Z_2\times Z_2$ with respect to $\approx$ is given by

$$N=\frac{8+(3\times4)+(2\times2)}{6}=4.$$

For case p=3, we have $Z_3\times Z_3$, the same algorithm was applied and we have the number(N) of distinct fuzzy subgroups of $Z_3\times Z_3$,with respect to $\approx$ is given by

$$N=\frac{(10\times2)+(18\times2)+(16\times4)+(12\times6)}{48}=4.$$

For case p=5, that is, $Z_5\times Z_5$, we also have that the number(N) of distinct fuzzy subgroups of $Z_5\times Z_5$ with respect to $\approx$ is 4. Continuing in this manner, observe that there exists an automorphism $f_k\in$ Aut($\mathbb{Z}_p\times\mathbb{Z}_p$**)** such that f_k fixes a similar pattern of subgroups for any p which in turn yields four(4) chains of subgroups of $\mathbb{Z}_p\times\mathbb{Z}_p$ that ends in $\mathbb{Z}_p\times\mathbb{Z}_p$**,** that is $|fix_{\overline{C}}(f_k)|=4$.

Showing that for any p there is a one-to-one correspondence between the number of distinct fuzzy subgroups of $\mathbb{Z}_p\times\mathbb{Z}_p$ and the number of chains of subgroups of $\mathbb{Z}_p\times\mathbb{Z}_p$ that ends in $\mathbb{Z}_p\times\mathbb{Z}_p$ fixed for a particular element $f_k\in$ Aut($\mathbb{Z}_p\times\mathbb{Z}_p$**).** Hence the result.

3.2. Distinct fuzzy subgroups of Q_{4p}

Quaternion groups (generalized quaternion) are well studied in literature. The group can be given by the presentation $Q_{4n}=\langle a,b\,|\,a^n=b^2,a^{2n}=1,b^{-1}ab=a^{-1}\rangle$ where a,b are its generators. In this case our $n\succ2=p=prime$. The subgroup structure are also well known. Every subgroup of Q_{4n} is either cyclic or quaternion. We study carefully the patterns of subgroups and chains that are fixed by each element of $Aut(Q_{4p})$ and then generalize from our observation.

Theorem 3.2. The number of distinct fuzzy subgroups of Q_{4p} for $p \succ 2$ with respect to $\approx$ is 16.

Proof: We use the same algorithm that we have adopted;

We begin with the case $p=3$, i.e. Q_{12} from GAP we have that Q_{12} have 8 subgroups namely

$S_1=\langle a^3\rangle, S_2=\langle a^2\rangle, S_3=\langle b\rangle, S_4=\langle a^5 b\rangle, S_5=\langle ab\rangle, S_6=\langle a\rangle, S_7=\langle e\rangle, S_8=\langle a,b\rangle$ and $|Aut(Q_{12})|=12$. Also for each $f_i \in \mathrm{Aut(Q_{12})}$ $i=1\ldots 12$ let

$fix(f_i)=\{S_i \le Q_{4p} \mid f_i(S_i)=S_i\}$ .From GAP we have

$fix(f_1)=fix(f_3)=\{S_8\},\ fix(f_2)=fix(f_4)=\{S_1,S_2,S_4,S_6,S_7,S_8\},$

$fix(f_7)=fix(f_9)=\{S_1,S_2,S_3,S_6,S_7,S_8\}$

$fix(f_8)=fix(f_{10})=fix(f_{11})=fix(f_{12})=\{S_1,S_2,S_6,S_7,S_8\},$

$fix(f_5)=fix(f_6)=\{S_1,S_2,S_5,S_6,S_7,S_8\}$

In what follows we have

$|fix_{\overline{C}}(f_1)|=|fix_{\overline{C}}(f_3)|=24, |fix_{\overline{C}}(f_2)|=|fix_{\overline{C}}(f_4)|=|fix_{\overline{C}}(f_5)|$

$=|fix_{\overline{C}}(f_6)|=|fix_{\overline{C}}(f_7)|=|fix_{\overline{C}}(f_9)|=16,$

$|fix_{\overline{C}}(f_8)|=|fix_{\overline{C}}(f_{10})|=|fix_{\overline{C}}(f_{11})|=|fix_{\overline{C}}(f_{12})|=12.$

The number of distinct fuzzy subgroups of Q_{12} with respect to $\approx$ is given by

$$N=\frac{(24\times 2)+3(16\times 2)+(12\times 4)}{12}=16.$$

In a similar manner we have

For $p=5$, i.e. Q_{20}, from GAP we have that Q_{20} have 10 subgroups namely

$K_1=\langle a\rangle, K_2=\langle a^2\rangle, K_3=\langle a^5\rangle, K_4=\langle b\rangle, K_5=\langle ab\rangle, K_6=\langle a^2 b\rangle, K_7=\langle a^3 b\rangle,$

$K_8=\langle a^4 b\rangle, K_9=\langle e\rangle, K_{10}=\langle a,b\rangle$

and $|Aut(Q_{20})|=40$. Also for each $f_i \in \mathrm{Aut(Q_{20})}$ $i=1\ldots 40$ let

$fix(f_i)=\{K_i \le Q_{20} \mid f_i(K_i)=K_i\}$ we have

$fix(f_1)=fix(f_4)=\{K_{10}\},\ fix(f_2)=fix(f_3)=fix(f_6)=fix(f_7)=fix(f_9)$

$=fix(f_{10})=\{K_1,K_2,K_3,K_6,K_9,K_{10}\},\ fix(f_9)=fix(f_{16})=fix(f_{20})=$

$fix(f_{23})=fix(f_{28})=fix(f_{32})=fix(f_{37})=\{K_1,K_2,K_3,K_9,K_{10}\},$

$fix(f_{11})=fix(f_{14})=fix(f_{21})=fix(f_{24})=$

$fix(f_{31})=fix(f_{34})=\{K_1,K_2,K_3,K_4,K_9,K_{10}\},$

In what follows we have

$|fix_{\overline{C}}(f_1)|=|fix_{\overline{C}}(f_4)|=32, |fix_{\overline{C}}(f_5)|=|fix_{\overline{C}}(f_9)|=|fix_{\overline{C}}(f_{16})|=|fix_{\overline{C}}(f_{20})|$
$=|fix_{\overline{C}}(f_{23})|=|fix_{\overline{C}}(f_{28})|=|fix_{\overline{C}}(f_{32})|=|fix_{\overline{C}}(f_{37})|=12,$
$|fix_{\overline{C}}(f_2)|=|fix_{\overline{C}}(f_3)|=|fix_{\overline{C}}(f_6)|=|fix_{\overline{C}}(f_7)|=|fix_{\overline{C}}(f_8)|$
$=|fix_{\overline{C}}(f_{10})|=|fix_{\overline{C}}(f_{11})|=|fix_{\overline{C}}(f_{14})|=|fix_{\overline{C}}(f_{21})|=|fix_{\overline{C}}(f_{24})|$
$=|fix_{\overline{C}}(f_{31})|=|fix_{\overline{C}}(f_{34})|=|fix_{\overline{C}}(f_{12})|=|fix_{\overline{C}}(f_{17})|=|fix_{\overline{C}}(f_{33})|$
$=|fix_{\overline{C}}(f_{38})|=|fix_{\overline{C}}(f_{35})|=|fix_{\overline{C}}(f_{39})|=|fix_{\overline{C}}(f_{13})|=|fix_{\overline{C}}(f_{18})|$
$=|fix_{\overline{C}}(f_{26})|=|fix_{\overline{C}}(f_{30})|=|fix_{\overline{C}}(f_{29})|=|fix_{\overline{C}}(f_{25})|=|fix_{\overline{C}}(f_{19})|$
$=|fix_{\overline{C}}(f_{22})|=|fix_{\overline{C}}(f_{27})|=|fix_{\overline{C}}(f_{15})|=|fix_{\overline{C}}(f_{36})|=|fix_{\overline{C}}(f_{40})|=16.$

The number of distinct fuzzy subgroups of Q_{20} with respect to $\approx$ is given by

$$N = \frac{(32\times 2)+5(16\times 6)+(12\times 8)}{40} = 16.$$

For $p=7$, i.e. Q_{28}, from GAP we have that Q_{28}, have 12 subgroups; $|Aut(Q_{28})|=84$ and the number of distinct fuzzy subgroups of Q_{28} is given by the equality; $N = \frac{(40\times 2)+7(16\times 10)+(12\times 12)}{84} = 16$.

Continuing in this manner, observe that there exists an automorphism $f_k \in$ Aut(Q_{4p}**)** such that f_k fixes a similar pattern of subgroups for any p which in turn yields Sixteen(16) chains of subgroups that is $|fix_{\overline{C}}(f_k)|=16$.

Showing that for any p there is a one-to-one correspondence between the number of distinct fuzzy subgroups of Q_{4p} and the number of chains of subgroups of Q_{4p} that ends in Q_{4p} fixed for a particular element $f_k \in$ Aut(Q_{4p}**).** Hence the result.

4. Conclusion

Our result has shown that the number of distinct fuzzy subgroups of $\mathbb{Z}_p \times \mathbb{Z}_p$ and Q_{4p} with respect to $\approx$ is independent of p. The next step is establishing an explicit formula for counting in general number of distinct fuzzy subgroups elementary abelian p-groups and generalized quaternion groups.

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