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Fuzzy Conjunctive Grammar

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Abstract. Inspired by, Conjunctive Grammar (CG) in crisp case, we study the simplication of Fuzzy Conjunctive Grammars (FCG).

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1. Introduction

Conjunctive grammars, introduced in [8], are context-free grammars augmented with an explicit set theoretic intersection operation. Every rule in a conjunctive grammar is of the form

$$\mathbf{A \rightarrow \alpha_1 \& \ldots\ldots \& \alpha_n} \tag{1}$$

where $\boldsymbol{n \geq 1}$ and $\boldsymbol{\alpha_i}$ are strings consisting of terminal and non-terminal symbols. Each rule of the form (1) indicates that any string that can be generated from each α_i can therefore be generated by **A.**

The generative capacity of conjunctive grammars covers some important non context-free language constructs, such as $\mathbf{\{a^n b^n c^n \,/\, n \geq 0\}}$and $\mathbf{\{a^m b^n c^m / \, m,n \geq 1\}}$ where the latter is known to be not in the intersection closure of context-free languages [15]. The language of all computations of any given Turing machine is also known to be conjunctive, which has certain implications on undecidability and descriptional complexity [8].

In particular, in [8] Okhotin defined a sub-family of conjunctive grammars called linear conjunctive grammars **(LCG)**, analogously to the definition of linear grammars as a subfamily of context-free grammars. **LCG** are an interesting sub-family of **CG** as they have especially efficient parsing algorithms, see [10], making them very appealing from a computational standpoint. In addition, many of the interesting languages generated by conjunctive grammars can in fact be generated by linear conjunctive grammars. In [9] Okhotin proved that **LCG** are equivalent to a type of trellis automata [14].

In the classical computation theory, an important issue is the characterization of formal languages. For instance, regular languages can be characterized by some finite automata, regular expressions and regular grammars. However, the problems, such as vagueness and imprecision are frequently encountered in the study of natural languages.

To reduce the gap between the precision of formal languages and the imprecision of natural languages, various fuzzy languages have been proposed.

By introducing the concept of fuzziness into the structure of formal grammars, Lee and Zadeh (1969) [16] established fuzzy grammars for such languages, which is an extension of the corresponding notion in the theory of formal languages. As a further extension, Kim et al. (1975) [4] proposed one type of L-fuzzy grammar based on the distributive lattice and Boolean lattice and another type of L-fuzzy grammar based on the lattice-ordered group and lattice-ordered monoid by assigning the element of lattice to the rewriting rules of a formal grammar. Gerla (1992) [5] also studied fuzzy grammars and recursively enumerable fuzzy languages, and proved that a fuzzy language can be generated by a fuzzy grammar if an only if it is recursively enumerable.

As one of the generators of fuzzy language, fuzzy grammars have been used to solve the issues such as intelligent interface design (Senay 1992) [6], lexical analysis, clinical monitoring (Steimann and Adlassning 1994) [3], neural networks (Giles et al. 1999) [7], and pattern recognition (Depalma and Yau 1975) [4].

Usually, fuzzy grammars with maxmin compositions take values in the unit interval **[0, 1]**. It is well-known that in the fuzzy automata theory (Mordeson and Malik 2002) [17] the following four approaches to represent a fuzzy language **μ (*L*)** are equivalent:

(i) μ (*L*) is accepted by a certain fuzzy deterministic finite automaton;

(ii) μ (*L*) is accepted by a certain fuzzy nondeterministic finite automaton;

(iii) μ (*L*) is described by a certain fuzzy regular expression;

(iv) μ (*L*) is generated by a certain fuzzy regular grammar.

Recently, all the above approaches have been extended to represent fuzzy languages with membership values in some special domains. It was proved by Li and Pedrycz (2005) [11] that **(i)** and **(ii)** are not equivalent for some truth-valued lattice-ordered monoids. The nondeterministic lattice-valued finite automata are more powerful than deterministic lattice-valued finite automata in the sense of maintaining the same ability of recognizing fuzzy languages. Li and Pedrycz (2005) [11] also introduced fuzzy regular expressions and investigated the relationships among **(i)**, **(ii)** and **(iii)**. Subsequently, Sheng (2006) [12] and Guo (2009) [13] introduced regular grammars with truth values in lattice-ordered monoid and discussed the relationship among **(i)**, **(ii)** and **(iv)**, respectively.

The paper is organized as follows,in **section 2** we introduced some notations and definition about **FCG**.It also presents some example for the explaining the concepts.In **section 3** we study simplification of **FCG** (It means removing usless variables, ε -conjunct,unit conjuct) and also we find out the fuzzy conjunctive grammar in Comskey normal form ,which is equivalent to **FCG**.

2. Basic concepts

Definition 2.1. (Fuzzy context-free grammar) A fuzzy context-free grammar (FCFG) is a quadruple G = (V, ∑, P, S), where

- V is a finite set of non-terminal symbol (Variables)
- ∑ is a finite set of terminal symbols disjoint from V
- S ∈ V is the designated start symbols
- P is a set of fuzzy production of the form $A \overset{r}{\to} \alpha$,where A ∈ V, $\alpha \in (V \cup \Sigma)^*$ and r ∈ [0,1].

Note 2.2.

- Here we using all notations of $r_{\{any\ symbols\}}$ –mentioned to a membership values it is lies between [0, 1].
- For example {r, $r_{1,\ldots\ldots,}$ r_n, r_ε, r_w, $r_{\alpha\beta}$, $r_{\alpha_{1_{\ell_1}}}$, $r_{\alpha_{2_{\ell_2}}}, \ldots\ldots\ldots, r_{\alpha_{n_{\ell_n}}}$} are some membership values it is lies between [0,1].
- The string of the word derived immediately (which means one steply derived) we denoted simply by $\Longrightarrow$, suppose the word derived by taken more steps, we using $\Longrightarrow^*$

Definition 2.3. (Fuzzy conjunctive grammar) A fuzzy conjunctive grammar (FCG) is a quadruple G = (V, ∑, P, S), where

- V is a finite set of non-terminal symbol (Variables)
- ∑ is a finite set of terminal symbols disjoint from V
- S ∈ V is the designated start symbols
- P is a finite set of fuzzy production of the form $B_i \overset{r_i}{\rightarrow}_G \alpha_i$ and $A \overset{r}{\rightarrow}_G (\alpha_1 \& \ldots\ldots \& \alpha_n)$, where A, $B_i \in V$, $\alpha_i \in (V \cup \sum)^*$ and **r = min {r_1,r_2,.......,r_n}** ,for i = 1,2,….,n. If n =1 , then $A \overset{r}{\rightarrow} \alpha_1$ and call it ordinary fuzzy production. Otherwise, the rule is called proper fuzzy conjunctive.

Definition 2.4. (Conjunctive formula) Any element $\mathcal{B}, \mathcal{C}, \mathcal{D}$, ε is a fuzzy conjunctive formulas over V ⋃∑⋃{ (,) , &} are defined by the following recursion.

- The empty string ε is a conjunctive formula.
- Every symbol in V⋃∑ is a conjunctive formula.
- If $\mathcal{B}$ *and* $\mathcal{C}$ are conjunctive formulas, then $\mathcal{BC}$ is a conjunctive formula.
- If $\mathcal{B}_1, \mathcal{B}_2, \ldots\ldots, \mathcal{B}_n$ are conjunctive formulas, then $\mathcal{B}_1 \& \ldots\ldots \& \mathcal{B}_n$ is a conjunctive formula.

Definition 2.5. Let G = (V, ∑, P, S) fuzzy conjunctive grammar and let $\Longrightarrow_G$ be denoted by immediate derivation. For any $s_1, s_2, \ldots.., s_n \in (V \cup \sum \cup \{ (\ ,)\ , \& \})^*$, then the set of conjunctive formulas is defined as follows.

- $A \overset{r}{\rightarrow}_G \alpha_1 \in P$, then $s_1 A\, s_1 \overset{r}{\Rightarrow}_G s_1 \alpha_1 s_2$.
- If $A \overset{r}{\rightarrow}_G (\alpha_1 \& \ldots\ldots \& \alpha_n) \in P$, then $s_1 A\, s_1 \overset{r}{\Rightarrow}_G s_1 (\alpha_1 \& \ldots\ldots \& \alpha_n)\, s_2$.
- If $\alpha_i \overset{r_i}{\rightarrow} w$, then $s_1 (w \& \ldots\ldots \& w)\, s_2 \overset{r}{\Rightarrow}_G s_1 w\, s_1$, where **r = min {$r_1$,$r_2$,.......,$r_n$}.**

Definition 2.6. The fuzzy languages μ ($\mathcal{B}$) of a conjunctive formula $\mathcal{B}$ is defined as

$$\mu(\mathcal{B}) = \{ (w, r) / w \in \textstyle\sum^*, \quad \mathcal{B} \overset{r\ *}{\Rightarrow}_G w \}$$

and the fuzzy language μ(L) of a fuzzy conjunctive grammar G is defined as

$$\mu(L, G) = \mu(S) = \{ (w, r) / w \in \textstyle\sum^* : \ S \overset{r\ *}{\Rightarrow}_G w \}.$$

In particular if $\mathcal{B}_i \overset{r_i}{\rightarrow} w$, for i = 1,2,….,n ,then the conjunctive formula $\mathcal{B}_1 \& \ldots\ldots \& \mathcal{B}_n \overset{r}{\rightarrow} w$,

where **r = min {r_1,r_2,.......,r_n}.**

Hence ($\mathcal{B}_1 \& \ldots\ldots \& \mathcal{B}_n$) generates intersection of fuzzy languages μ ($\mathcal{B}_i$)

i.e., μ ($\mathcal{B}_1$ &……&$\mathcal{B}_n$) =$\wedge_{i=1}^{n}$ μ ($\mathcal{B}_i$).

Definition 2.7. A fuzzy language μ(L) of $\sum^*$ is a fuzzy conjunctive languages, if there is a fuzzy conjunctive grammar G such that μ(L) = μ (L, G).

Example 2.8. The following fuzzy conjunctive grammar G = (V,$\sum$,P,S) generates the fuzzy non context-free languages {$a^n b^n c^n$: n = 0,1,…..}.

- V = { S, A, B ,C ,D }
- $\sum$ = { a, b, c},and
- P consists of the following derivation rules

$S \xrightarrow{0.4} (A/0.7 \ \& \ C/0.4)$, $A \xrightarrow{0.1} aA$, $A \xrightarrow{0.2} B$, $B \xrightarrow{0.3} bBc$, $B \xrightarrow{0.4} \varepsilon$, $C \xrightarrow{0.5} Cc$, $C \xrightarrow{0.6} D$, $D \xrightarrow{0.7} aDb$, $D \xrightarrow{0.8} \varepsilon$

Now the word **aabbcc** can be derived as follows

$$
\begin{aligned}
S &\overset{0.4\,*}{\Rightarrow}_G (A/0.7 \ \& \ C/0.4)\\
&\overset{0.1\,*}{\Rightarrow}_G (a\,A/0.1 \& \ Cc/0.5)\\
&\overset{0.1\,*}{\Rightarrow}_G (aaA/0.1 \ \& \ Ccc/0.5)\\
&\overset{0.2\,*}{\Rightarrow}_G (aaB/0.2 \ \& \ Dcc/0.6)\\
&\overset{0.3\,*}{\Rightarrow}_G (aabBc/0.3 \ \& \ a\,Dbcc/0.7)\\
&\overset{0.3\,*}{\Rightarrow}_G (aabbBcc/0.3 \ \& \ aa\,Dbbcc/0.7)\\
&\overset{0.4\,*}{\Rightarrow}_G (aabbcc/0.4 \ \& \ aabbcc/0.8)\\
S &\overset{0.1\,*}{\Rightarrow}_G (\mathrm{aabbcc})
\end{aligned}
$$

where, **r = min {0.4, 0.1, 0.1, 0.2, 0.3, 0.3, 0.4} =0.1**

3. Simplification of fuzzy conjunctive grammar

Definition 3.1. (Fuzzy epsilon conjunct) Let G = (V, $\sum$, P, S) be a fuzzy conjunctive grammar. A conjunct of the form $A \xrightarrow{r_\varepsilon} \varepsilon \,\&\, \varepsilon \,\&\, \ldots\ldots \,\&\, \varepsilon$, where A ∈ V, is called an fuzzy epsilon conjunct.

Definition 3.2. (Nullable variable) Let G = (V, $\sum$, P, S) be a fuzzy conjunctive grammar. Define nullable variable as, nullable (G) = { A ∈ V$/A \overset{r_\varepsilon\,*}{\Rightarrow}_G$ ε & ε &……..& ε}.

Definition 3.3. (Fuzzy unit conjunct) Let G = (V, $\sum$, P, S) be a fuzzy conjunctive grammar. A conjunct of the form $A \xrightarrow{r_u}$ B_1 & B_2&……..& B_n, where A, B_i ∈ V, i = 1, 2… n, is called a fuzzy unit conjunct.

Definition 3.4. (Variable dependency graph) For any grammar G , a variable dependency graph has its vertices labeled with variables, with an edge between vertices C and D if and only if there is a production form

$$C \overset{r}{\rightarrow} uDv, \text{ where } u, v \in \sum^{*} \text{ and } C, D \in V.$$

Definition 3.5. (Fuzzy useless and useful variable) Let G = (V, ∑, P, S) be a fuzzy conjunctive grammar. A variable A_i , i = 1,2,….,n is said to be useful if and only if there is at least one (w , r) ∈ μ (L, G), such that

$$\begin{aligned} S &\overset{r_{\alpha\beta}\,*}{\Longrightarrow}_G \alpha_1 A_1 \beta_1 \& \alpha_2 A_2 \beta_2 \& \ldots\ldots \& \alpha_n A_n \beta_n \\ &\overset{r_w\,*}{\Rightarrow}_G w \& w \& \ldots\ldots \& w \\ &\overset{r\,*}{\Rightarrow}_G w \end{aligned}$$

where, **r = min {** $\boldsymbol{r_{\alpha\beta}}$, $\boldsymbol{r_w}$**}**.

A variable is said to be not useful if it is not derives any terminal from the start symbol.

Theorem 3.6. (Elimination of fuzzy useless variable (or) production) Let G = (V, ∑, P, S) be a fuzzy conjunctive grammar. Then there exist an equivalent fuzzy conjunctive grammar $\hat{G}$ = ($\hat{V}$, ∑, S, $\hat{P}$) that does not contains any useless variable (or) productions.

Proof: The fuzzy conjunctive grammar $\hat{G}$ can be generated from G by an algorithm consisting of two parts. In the first part we construct an intermediate fuzzy conjunctive grammar G_1 = (V_1, ∑, S, P_1) such that V_1 contains only variable A for which $A \overset{r_*}{\Rightarrow} w/r_1 \;\&\; w/r_2 \;\&\ldots\ldots\&\; w/r_m$,w ∈ $\sum^*$, where **r = min {$\mathbf{r_1, r_2, \ldots\ldots, r_m}$}** is possible.

The steps in the algorithm are

1. Set V_1 to φ

2. Repeat, the following steps until no more variable are added to V_1. For every A ∈ V for which P has a production of the form

$$A \overset{r_\alpha}{\rightarrow} \alpha_{11}\,\alpha_{12}\ldots.\,\alpha_{1_{\ell_1}}/r_{\alpha_{1_{\ell_1}}} \;\&\; \alpha_{21}\,\alpha_{22}\ldots.\alpha_{2_{\ell_2}}/r_{\alpha_{2_{\ell_2}}} \&\ldots\ldots\ldots\&\alpha_{n1}\,\alpha_{n2}\ldots.\,\alpha_{n_{\ell_n}}/r_{\alpha_{n_{\ell_n}}}$$

where $\boldsymbol{r_\alpha}$**= min {**$\boldsymbol{r_{\alpha_{1_{\ell_1}}}}$**,** $\boldsymbol{r_{\alpha_{2_{\ell_2}}}}$**, ………,** $\boldsymbol{r_{\alpha_{n_{\ell_n}}}}$**}** and each $\alpha_{i_{\ell_i}}$ ∈ (V ∪ ∑) , for $\ell_i \geq 1$, i=1,2,…,n.

3. Take P_1 as all the production in P whose symbols are all in (V_1 ∪ ∑). Clearly this procedure terminates. For every A ∈ V_1 , for which $A \overset{r_*}{\Rightarrow} w/r_1 \;\&\; w/r_2 \;\&\ldots\ldots\&\; w/r_m$, w ∈ $\sum^*$.

In the second part of the construction, we get the final answer $\hat{G}$ from G_1.We draw the variable dependency graph for G_1 and from it find all variables that cannot be reached from S. Those variables are removed from the variable set and also remove productions involving them. The result is the fuzzy conjunctive grammar $\hat{G}$ = ($\hat{V}$, ∑, S, $\hat{P}$). Let (w, r) ∈ μ (L, G), then

$$S \overset{r_{\alpha\beta}\,*}{\Longrightarrow}_G \alpha_1 A_1 \beta_1 \& \alpha_2 A_2 \beta_2 \& \ldots\ldots \& \alpha_n A_n \beta_n$$

$$\overset{r_w\,*}{\Rightarrow}_G \; w\&\, w\& \ldots\ldots \&\, w$$
$$\overset{r\,*}{\Rightarrow}_G \; w$$

By the construction of $\hat{G}$, for all A_i's and associate productions are should not removed from $\hat{G}$ hence.

$$S \overset{r_{\alpha\beta}\,*}{\Longrightarrow}_{\hat{G}} \alpha_1 A_1 \beta_1 \& \alpha_2 A_2 \beta_2 \& \ldots\ldots \& \alpha_n A_n \beta_n$$
$$\overset{r_w\,*}{\Rightarrow}_{\hat{G}} \; w\&\, w\& \ldots\ldots \&\, w$$
$$\overset{r\,*}{\Rightarrow}_{\hat{G}} w$$

$(w, r) \in \mu\,(L, \hat{G})$

$$\boldsymbol{\mu\,(L, G) \le \mu\,(L, \hat{G}\,)} \tag{2}$$

Conversely,

Let $(w, r) \in \mu\,(L, \hat{G}\,)$

Since , $\hat{P} \subseteq P$, $(w, r) \in \mu\,(L, G)$

$$\boldsymbol{\mu(L, \hat{G}\,) \le \mu(L, G)} \tag{3}$$

From **(2)** and **(3)** we get

$$\boldsymbol{\mu(L,G) = \mu(L, \hat{G}).}$$

Theorem 3.7. (Elimination of fuzzy ε-production) Let $G = (V, \sum, P, S)$ be a fuzzy conjunctive grammar with (ε,r) not in $\mu(L,G)$. Then there exist an equivalent fuzzy conjunctive grammar $\hat{G} = (\hat{V}, \sum, S, \hat{P})$ having no ε-productions.

Proof: We first find the set V_N of all nullable variable of G, using the following steps

1. For all productions $A \overset{r_\varepsilon}{\to} \varepsilon$, put A into V_N.

(Or) $A \overset{r_{\varepsilon_1}}{\longrightarrow} \varepsilon \,\&\, \varepsilon \,\&\, \ldots\ldots\ldots \&\, \varepsilon$, put A into V_N.

2. Repeat the following steps until no further variable are added to V_N. For all productions

$$B \overset{r_\ell}{\to} A_{11} A_{12} \ldots A_{1\,\ell_1} / r_{\ell_1} \;\&\; A_{21} A_{22} \ldots A_{2\,\ell_2} / r_{\ell_2} \;\&\ldots\ldots\&\; A_{n1} A_{n2} \ldots A_{n\,\ell_n} / r_{\ell_n}$$

where, $\mathbf{r}_{\boldsymbol{\ell}} = \mathbf{min}\,\{\mathbf{r}_{\boldsymbol{\ell}_1}, \mathbf{r}_{\boldsymbol{\ell}_2}, \ldots\ldots, \mathbf{r}_{\boldsymbol{\ell}_n}\}$

And $A_{11}, A_{12}, \ldots, A_{1_{\ell_1}}$, $A_{21}, A_{12}, \ldots, A_{2_{\ell_2}}, \ldots\ldots, A_{n1}, A_{n2}, \ldots, A_{n_{\ell_n}}$ are in V_N put B in to V_N. Once the set V_N has been found, we are ready to construct $\hat{P}$. To do so, we look at all productions in P of the form

$$A \overset{r_\alpha}{\to} \alpha_{11} \alpha_{12} \ldots \alpha_{1_{\ell_1}} / r_{\alpha_{1_{\ell_1}}} \,\&\, \alpha_{21} \alpha_{22} \ldots \alpha_{2_{\ell_2}} / r_{\alpha_{2_{\ell_2}}} \&\ldots\ldots\ldots\& \alpha_{n1} \alpha_{n2} \ldots \alpha_{n_{\ell_n}} / r_{\alpha_{n_{\ell_n}}}$$

For $\ell_i \ge 1$, $i = 1, 2\ldots m$, where $\boldsymbol{r_\alpha} = \mathbf{min}\,\{\boldsymbol{r_{\alpha_{1_{\ell_1}}}}, \boldsymbol{r_{\alpha_{2_{\ell_2}}}}, \ldots, \boldsymbol{r_{\alpha_{n_{\ell_n}}}}\}$ and each $\alpha_{i_{\ell_i}} \in V \cup \sum$

For each production of P ,we put into $\hat{P}$ that productions as well as all those generated by replacing nullable variable with ε in all possible combinations and its membership function is calculated as per the rule. For example, if α_{ij} and $\alpha_{\ell\,k}$ are both nullable, there will be one production in $\hat{P}$ with α_{ij} replaced with ε, one in which $\alpha_{\ell\,k}$ is replaced with ε, and one in which both α_{ij} and $\alpha_{\ell\,k}$ are replaced with ε, and the membership is of each productions is found as usual.

There is one exception: If all α_{ij} are nullable, the production

$$A \xrightarrow{r_\varepsilon} \varepsilon \text{ (or) } A \xrightarrow{r_{\varepsilon_1}} \varepsilon \,\&\, \varepsilon \,\&\, \ldots\ldots \&\, \varepsilon$$

is not put into $\hat{P}$. Let $(w, r) \in \mu(L, G)$, then

$$\begin{aligned}
S &\overset{r_{\alpha\beta}\,*}{\Longrightarrow}_G \alpha_1 A_1 \beta_1 \& \alpha_2 A_2 \beta_2 \& \ldots\ldots \& \alpha_n A_n \beta_n \\
&\overset{r_{\alpha\beta_\varepsilon}\,*}{\Longrightarrow}_G \alpha_1 A_1\, \varepsilon \,\&\, \varepsilon\, A_2 \beta_2 \& \ldots\ldots \& \,\varepsilon\, A_n \beta_n \\
&\overset{r_w\,*}{\Rightarrow}_G\; w \& w \& \ldots\ldots \& w \\
&\overset{r\,*}{\Rightarrow}_G\; w
\end{aligned}$$

Let $S \Longrightarrow_{\hat{G}} \alpha_1 A_1 \beta_1 \& \alpha_2 A_2 \beta_2 \& \ldots\ldots \& \alpha_n A_n \beta_n$. Since there is no ε-production in $\hat{G}$ the production involves variable directly derives sentential form without ε

$$\begin{aligned}
S &\overset{r_{\alpha\beta}\,*}{\Longrightarrow}_{\hat{G}} \alpha_1 A_1 \beta_1 \& \alpha_2 A_2 \beta_2 \& \ldots\ldots \& \alpha_n A_n \beta_n \\
&\overset{r_w\,*}{\Rightarrow}_{\hat{G}}\; w \& w \& \ldots\ldots \& w \\
&\overset{r\,*}{\Rightarrow}_{\hat{G}}\; w
\end{aligned}$$

Since $(w, r) \in \mu(L, \hat{G})$

$$\boldsymbol{\mu(L, G) \leq \mu(L, \hat{G})} \tag{4}$$

Conversely,

Let $(w, r) \in \mu(L, \hat{G})$

$$\begin{aligned}
S &\overset{r_{\alpha\beta}\,*}{\Longrightarrow}_{\hat{G}} \alpha_1 A_1 \beta_1 \& \alpha_2 A_2 \beta_2 \& \ldots\ldots \& \alpha_n A_n \beta_n \\
&\overset{r_{\alpha\beta i}\,*}{\Longrightarrow}_G \alpha_1 A_1 \beta_1 \& \alpha_2 A_2 \beta_2 \& \ldots\ldots \& \alpha_n A_n \beta_n \\
&\overset{r_w\,*}{\Rightarrow}_G\; w \& w \& \ldots\ldots \& w \\
&\overset{r\,*}{\Rightarrow}_G\; w
\end{aligned}$$

$$\boldsymbol{\mu(L, \hat{G}) \leq \mu(L, G)} \tag{5}$$

From **(4)** and **(5)** we get $\boldsymbol{\mu(L, \hat{G}) = \mu(L, G)}$.

Lemma 3.8. (Substitution of fuzzy unit conjunct) Let $G = (V, \sum, P, S)$ be a fuzzy conjunctive grammar.

Let $A \xrightarrow{r_j} \alpha_1/r_{\alpha_1} \,\&\ldots\ldots\&\, \alpha_{k-1}/r_{\alpha_{k-1}} \,\&\, B/r_{B_j} \,\&\, \alpha_{k+1}/r_{\alpha_{k+1}} \,\&\ldots\ldots\&\, \alpha_m/r_{\alpha_m}$ (6)

$(m \geq 1,\ 1 \leq k \leq m,\ \alpha_i \in \sum^*, 1 \leq j \leq n)$ be some rule for A that contains a unit conjunct, where $\mathbf{r = \min\{ r_{\alpha_1}, \ldots\ldots, r_{\alpha_{k+1}}, r_{B_j}, r_{\alpha_{k+1}}, \ldots\ldots\ldots, r_{\alpha_m} \}}$. Let $B \xrightarrow{r_{B_1}} \beta_{11}/r_{\beta_{11}} \,\&\ldots\&\, \beta_{1_{\ell_1}}/r_{\beta_{1_{\ell_1}}}$, $B \xrightarrow{r_{B_2}} \beta_{21}/r_{\beta_{21}} \,\&\ldots\&\, \beta_{2_{\ell_2}}/r_{\beta_{2_{\ell_2}}}$, ……..,$B \xrightarrow{r_{B_n}} \beta_{n_1}/r_{\beta_{n_1}} \,\&\ldots\&\, \beta_{n_{\ell_n}}/r_{\beta_{n_{\ell_n}}}$, be all rules for B that do not contain the unit conjunct $B \xrightarrow{1} B$ $(\beta_{ij} \neq B,\ \beta_{ij} \in (V \cup \sum)^*)$.

If $A \neq B$, then the rule **(6)** can be replaced with the collection of rules

$A \xrightarrow{r_1} \alpha_1/r_{\alpha_1} \,\&\ldots\ldots\&\, \alpha_{k-1}/r_{\alpha_{k-1}} \,\&\, \beta_{11}/r_{\beta_{11}} \&\ldots\&$
$\beta_{1_{\ell_1}}/r_{\beta_{1_{\ell_1}}} \,\&\, \alpha_{k+1}/r_{\alpha_{k+1}} \,\&\ldots\ldots\&\, \alpha_m/r_{\alpha_m}$,

$A \xrightarrow{r_2} \alpha_1/r_{\alpha_1}$ & ... ... & $\alpha_{k-1}/r_{\alpha_{k-1}}$ & $\beta_{21}/r_{\beta_{21}}$&...&$\beta_{2_{\ell_2}}$ $/r_{\beta_{2_{\ell_2}}}$
& $\alpha_{k+1}/r_{\alpha_{k+1}}$ & ... ... & α_m/r_{α_m} ,
.
.
$A \xrightarrow{r_m} \alpha_1/r_{\alpha_1}$ & ... ... & $\alpha_{k-1}/r_{\alpha_{k-1}}$ & β_{i_1} $/r_{\beta_{i_1}}$&
$\beta_{i_{\ell_i}}$ $/r_{\beta_{i_{\ell_i}}}$ & $\alpha_{k+1}/r_{\alpha_{k+1}}$ & ... ... & α_m/r_{α_m} .

If A=B, then the rule **(6)** may be just removed.

Theorem 3.9. (Elimination of fuzzy unit productions) Let G = (V, ∑, P, S) be any fuzzy conjunctive grammar without ε- productions and it has unit production of the form $A \xrightarrow{r} B_1$ &B_2& ... ... &B_n , A, $B_i \in V$. Then there exist fuzzy conjunctive grammar $\hat{G}$= ($\hat{V}$, ∑,$\hat{P}$,S) that don't have any unit-production and that is equivalent to G.

Proof: Let G = (V, ∑, P, S) be any fuzzy conjunctive grammar with unit production of the form

$$A \xrightarrow{r_u} B_1 \,\&B_2\& \ldots \ldots \&B_n \quad (7)$$

For every A, $B_i \in V$ and i=1,2,…..,n.

If all (or) some B_i's itself has unit production, then by the rule of fuzzy conjunctive grammar

$$A \xrightarrow{r_c} C_1\&C_2\& \ldots \ldots \&C_m \quad (8)$$

By Lemma 3.8 each C_i is replaced by production of the form

$$C_i \xrightarrow{r_\alpha} \alpha_{i_1}\& \,\alpha_{i_2}\&\ldots\ldots\& \,\alpha_{i_{\ell_i}}$$

Finally we remove the unit of production cyclically [8]. Then A becomes

$$A \xrightarrow{r_\beta} \beta_1\& \;\beta_2\& \ldots \ldots \& \;\beta_\ell \text{ , where } \beta_i \in (V \cup \textstyle\sum)^* .$$

Then include the above production in $\hat{P}$. Continue the process until there is no unit production in P. Now $\hat{G}$ = ($\hat{V}$, ∑,S, $\hat{P}$) is a grammar with no unit production.

Let (w , r) ∈ μ (L, G), then

$$S \overset{r_A\,*}{\Rightarrow}_G A_1 \,\&\, A_2 \,\&\ldots\ldots\&\, A_n$$
$$\overset{r_{\alpha\beta}\,*}{\Longrightarrow}_G \alpha_1 B_1 \,\beta_1 \& \,\alpha_2 B_2 \,\beta_2 \&\ldots\ldots\& \;\alpha_n B_n \,\beta_n$$
$$\overset{r_w\,*}{\Rightarrow}_G \;w\&\, w\&\ldots\ldots\&\, w$$
$$S \overset{r\,*}{\Rightarrow}_G w$$

But in grammar $\hat{G}$, we can get,

$$S \overset{r_{\alpha\beta}\,*}{\Longrightarrow}_{\hat{G}} \alpha_1 B_1 \,\beta_1 \& \,\alpha_2 B_2 \,\beta_2 \&\ldots\ldots\& \;\alpha_n B_n \,\beta_n$$
$$\overset{r_w\,*}{\Rightarrow}_{\hat{G}} \;w\&\, w\&\ldots\ldots\&\, w$$
$$S \overset{r\,*}{\Rightarrow}_{\hat{G}} w$$

(w, r) ∈ μ (L, $\hat{G}$)

$$\boldsymbol{\mu(L,G) \leq \mu(L, \hat{G})} \quad (9)$$

Conversely,

Let $(w, r) \in \mu(L, \widehat{G})$

$$S \overset{r_{\alpha\beta}\ *}{\Longrightarrow}_{\widehat{G}} \alpha_1 B_1 \beta_1 \& \alpha_2 B_2 \beta_2 \& \ldots\ldots \& \alpha_n B_n \beta_n$$
$$\overset{r_w\ *}{\Rightarrow}_{\widehat{G}}\ w \& w \& \ldots\ldots \& w$$
$$S \overset{r\ *}{\Rightarrow}_{\widehat{G}}\ w$$

But in grammar G we can get

$$S \overset{r_A\ *}{\Longrightarrow}_{G} A_1 \& A_2 \& \ldots\ldots \& A_n$$
$$\overset{r_{\alpha\beta}\ *}{\Longrightarrow}_{G} \alpha_1 B_1 \beta_1 \& \alpha_2 B_2 \beta_2 \& \ldots\ldots \& \alpha_n B_n \beta_n$$
$$\overset{r_w\ *}{\Rightarrow}_{G}\ w \& w \& \ldots\ldots \& w$$
$$S \overset{r\ *}{\Rightarrow}_{G}\ w$$

$$\boldsymbol{\mu (L, \widehat{G}) \leq \mu (L, G)} \qquad (10)$$

From **(9)** and **(10)** we get

$$\boldsymbol{\mu (L, G) = \mu (L, \widehat{G})}$$

Definition 3.10. A fuzzy conjunctive grammar is in Chomsky normal form if all productions are of the form

$$A \overset{r}{\rightarrow} B_1C_1/r_1 \& B_2C_2/r_2 \& \ldots \ldots \& B_mC_m/r_m, \text{ where } \mathbf{r = min\ \{r_1, r_2, \ldots\ldots, r_m\}}$$
$$A \overset{r_a}{\rightarrow}\ a,\ A \in V,\ a \in \textstyle\sum.$$

Theorem 3.11. Any fuzzy conjunctive grammar G = (V,$\sum$,P,S) with $(\varepsilon, r) \notin \mu(L,G)$ has equivalent fuzzy conjunctive grammar $\widehat{G} = (\widehat{V}, \sum, S, \widehat{P})$ in Chomsky normal form.

Proof: First, we can assume without loss of generality that G has no ε-productions and unit productions. The construction of $\widehat{G}$ will be done in two steps.

Step 1: Construct a fuzzy conjunctive grammar $G_1 = (V_1, \sum, S, P_1)$ form G by considering all productions in P in the form

$$A \overset{r_\alpha}{\rightarrow} \alpha_{11}\, \alpha_{12} \ldots .\, \alpha_{1_{\ell_1}} / r_{\alpha_{1_{\ell_1}}} \& \alpha_{21}\, \alpha_{22} \ldots . \alpha_{2_{\ell_2}} / r_{\alpha_{2_{\ell_2}}} \& \ldots\ldots\ldots \& \alpha_{n1}\, \alpha_{n2} \ldots .\, \alpha_{n_{\ell_n}} / r_{\alpha_{n_{\ell_n}}} \qquad (11)$$

where $\boldsymbol{r_\alpha = \min \{ r_{\alpha_{1_{\ell_1}}}, r_{\alpha_{2_{\ell_2}}}, \ldots \ldots, r_{\alpha_{n_{\ell_n}}} \}}$ and each α_{ij} is a symbol either in V (or) in $\sum$.

If **n = 1**, then result follows from [17].

If **n ≥ 2**, for each $a \in \sum$, introduced the variable B_a . Then each production of P in the form **(11)** can be written as

$$A \overset{r_\alpha}{\rightarrow} C_{11}C_{12} \ldots\ldots C_{1_{\ell_1}}/r_{\alpha_{1_{\ell_1}}} \& \ldots\ldots \& C_{n1}C_{n2} \ldots\ldots C_{n_{\ell_n}}/r_{\alpha_{n_{\ell_n}}}$$

where $C_{ij} = \alpha_{ij}$, if $\alpha_{ij} \in V$ and

$C_{ij} = B_a$, if $\alpha_{ij} = a$.

Then we include this production in P_1. Also, for every B_a, $B_a \overset{1}{\rightarrow} a$, we include the above production in P_1. This part of the algorithm removes all terminals from productions. At the end of this step we have fuzzy conjunctive grammar G_1 all of whose production have the form.

$$A \xrightarrow{r_a} a \,\&\, a \,\&\, \ldots\ldots \&\, a \tag{12}$$

(Or)

$$A \xrightarrow{r_\alpha} C_{11}C_{12}\ldots\ldots C_{1_{\ell_1}}/r_{\alpha_{1_{\ell_1}}} \,\&\ldots\ldots\&\, C_{n1}C_{n2}\ldots\ldots C_{n_{\ell_n}}/r_{\alpha_{n_{\ell_n}}} \tag{13}$$

where, $C_{ij} \in V_1$, we can easy to verify that

$$\boldsymbol{\mu (L,G_1) = \mu (L, G).}$$

Step 2: In this step, we make the right hand side of all the productions in P_1 as combination of two variables by introducing new variable set and put into $\hat{P}$. First we put all productions of the form **(12)** and **(13)** in $\hat{P}$. If any fuzzy conjunctive formula have already in Chomsky normal form, we put in to $\hat{P}$, then as follows:

$$C_{11} \xrightarrow{r_{\alpha_{1_{\ell_1}}}} C_{11}C_{11}$$

$$D_{11} \xrightarrow{1} C_{12}D_{12}$$

$$D_{12} \xrightarrow{1} C_{13}D_{13}$$

.

.

.

$$D_{1_{\ell_1-2}} \xrightarrow{1} C_{1_{\ell_1-1}} \; C_{1_{\ell_1}}$$

$$C_{21} \xrightarrow{r_{\alpha_{2_{\ell_2}}}} C_{21}D_{21}$$

$$D_{21} \xrightarrow{1} C_{22}D_{22}$$

$$D_{22} \xrightarrow{1} C_{23}D_{23}$$

.

.

.

$$D_{1_{\ell_2-2}} \xrightarrow{1} C_{1_{\ell_2-2}} \; C_{2_{\ell_2}}$$

.

.

.

$$C_{n1} \xrightarrow{r_{\alpha_{n_{\ell_n}}}} C_{n1}D_{n1}$$

$$D_{n1} \xrightarrow{1} C_{n2}D_{n2}$$

$$D_{n2} \xrightarrow{1} C_{n3}D_{n3}$$

.

.

$$\mathrm{D}_{\mathrm{n}_{\ell_{n-2}}} \xrightarrow{1} \mathrm{C}_{\mathrm{n}_{\ell_{n-1}}} \ \mathrm{C}_{\mathrm{n}_{\ell_{\mathrm{n}}}}$$

$$\boldsymbol{A \xrightarrow{r_\alpha} C_{11}D_{11} / r_{\alpha_{1_{\ell_1}}} \ \& C_{21}D_{22} / r_{\alpha_{2_{\ell_2}}} \ \& \ldots\ldots \& C_{nn}D_{nn} / r_{\alpha_{n_{\ell_n}}}}$$

Finally, the resulting fuzzy conjunctive grammar $\widehat{G}$ is in fuzzy Chomsky normal form. So we can easily verify that **μ(L,G) = μ(L, $\widehat{G}$).**

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