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Vertex Domination Critical in Circulant Graphs

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Abstract. A graph G is vertex domination critical if for any vertex v of G, the domination number of G – v is less than the domination number of G. We call these graphs γ-critical if domination number of G is γ. In this paper, we determine the domination and the total domination number of Cir(n,A) for two particular generating sets A of Z_n, and then study γ-critical in these graphs.

Keywords: Domination, total domination, circulant graph.

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1. Introduction

A vertex in a graph G dominates itself and its neighbors. A set of vertices S in a graph G is a dominating set, if each vertex of G is dominated by some vertex of S. The domination number γ(G) of G is the minimum cardinality of a dominating set of G. A dominating set S is called a total dominating set if each vertex v of G is dominated by some vertex u ≠ v of S. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G.

We denote the open neighborhood of a vertex v of G by $N_G(v)$, or just N(v), and its closed neighborhood by N[v]. For a vertex set $S \subseteq V(G)$, $N(S) = \cup_{v\in S}N(v)$ and $N[S] = \cup_{v\in S}N[v]$. So, a set of vertices S in G is a dominating set, if N[S] = V(G). Also, S is a total dominating set, if N(S) = V(G). For notation and graph theory terminology in general we follow [3]. Rashmanlou and Pal et al. [5-17] studied different kinds of fuzzy graphs. We call a dominating set of cardinality γ(G), a γ(G) – set and a total dominating set of cardinality $\gamma_t(G)$, a $\gamma_t(G)$ – set. A graph G is called vertex domination critical if γ(G – v) < γ(G), for every vertex v in G. For references on the vertex domination critical graphs see [1,2,3].

Jafari Rad [4], determines the domination number and the total domination number of graph *Cir*(*n*,{1, 3}), for any integer *n*, and then study γ – criticality in *Cir*(*n*, {1, 3}).

Let $n \geq 7$ be a positive integer. The circulant graph *Cir*(*n* , *A*) where *A* ={1, *n* – *1*, 3, *n* – 3, 5, *n* – 5, …, *2k –1*, *n* – (2*k* – 1), *2k +1*, *n* – (2*k +1*)} is the graph with vertex set{v_0,

$v_1, \ldots, v_{n-1}\}$, and edge set$\{\{v_i, v_{i+j}\} : i\in\{0, 1, \ldots, n-1\}, j\in \{1, n-1, 3, n-3, \ldots, 2k+1, n-(2k+1)\}\}$, k is an integer such that$0 \le k < \lceil \frac{n-3}{4} \rceil$.

Let $n \ge 9$ be a positive integer. The circulant graph $Cir(n, B)$ where $B = \{1, n-1, 2, n-2, 4, n-4, \ldots, 2k, n-2k, 2k+2, n-(2k+2)\}$ is the graph with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$, and edge set$\{\{v_i, v_{i+j}\} : i\in\{0, 1, \ldots, n-1\}, j \in\{1, n-1, 2, n-2, \ldots, 2k+2, n-(2k+2)\}\}$, k is an integer such that$0 \le k < \lceil \frac{n-5}{4} \rceil$.

All arithmetic on the indices is assumed to be modulo n.

In this paper, we first determine the domination number and the total domination number in the circulant graphs $Cir(n, A)$ and $Cir(n, B)$ for any integer n, and then study γ – criticality and γ_t (G) – criticality in these class of graphs.

For two vertices x and y in a graph G we denote the distance between x and y by $d_G(x, y)$, or just $d(x, y)$.

2. Domination and total domination

Let G be a circulant graph with n vertices. Let cycle $C = C(G)$ be the subgraph of G with vertex set$\{v_0, v_1, \ldots, v_{n-1}\}$and edge set $\{\{v_i, v_{i+1}\} : i\in\{0, 1, \ldots, n-1\}\}$. For a subset $S\subseteq V(G)$ with at least three vertices, we say that $x, y\in S$ are *consecutive* if there is no vertex $z \in S$ such that z lies between x and y in C. For two consecutive vertices x, y in a subset of vertices S, we define $|x - y| = d_C(x, y)$. So, $|x - y|$ equals to the number of edges in a shortest path between x and y in the cycle C.

Theorem 2.1. For any integer $n \ge 7$,

$$\gamma(Cir(n, A)) = \begin{cases} \lceil \frac{n}{2k+3} \rceil + 1 & n \equiv 4, 6, 8, \ldots, 2k+2 \pmod{2k+3} \\ \lceil \frac{n}{2k+3} \rceil & \text{otherwise} \end{cases}$$

Proof: Let S be a $\gamma(G)$-set of $G = Cir(n, A)$. Any vertex of G dominates $2k+3$ vertices of G including itself, so $|S| \ge \lceil \frac{n}{2k+3} \rceil$.

We claim that if $n\equiv 2t \pmod{2k+3}$, for an integer t such that $2 \le t \le k+1$, then $|S| \ge \lceil \frac{n}{2k+3} \rceil + 1$.

To see this, assume to the contrary that $n\equiv 2t \pmod{2k+3}$, and $|S| = \lceil \frac{n}{2k+3} \rceil$. There are two consecutive vertices $v_l, v_l'\in S$ such that $|l - l'| < 2k+3$. Let $v_l'' \ne v_l$ be a consecutive vertex of v_l'. Without loss of generality assume that $|v_l'' - v_l| = 2k+3+2t$. Then there are $2k+2+2t$ possibilities for v_l' to lies between v_l and v_l''. In each possibly there exists a vertex between v_l and v_l'' which is not dominated by $\{v_l, v_l', v_l''\}$, a contradiction. Hence, for $n\equiv 2t \pmod{2k+3}$, $|S| \ge \lceil \frac{n}{2k+3} \rceil + 1$.

Now it is sufficient to get a dominating set S of required cardinality. We consider the following cases:

1. For $n \equiv 4 \pmod{2k+3}$, $S = \{v_{(2k+3)i} : 0 \le i < \lceil \frac{n}{2k+3} \rceil\} \cup \{v_{n-2}\}$.

2. For $n \equiv 6,8,10, 12,14, \ldots, 2k+2 \pmod{2k+3}$, $S = \{v_{(2k+3)i}: 0 \le i < \lceil \frac{n}{2k+3} \rceil\} \cup \{v_{n-1}\}$.

3. For $n \equiv 4, 6, 8,10, ..., 2k+2 \pmod{2k+3}$, $S = \{v_{(2k+3)i}: 0 \le i < \lceil \frac{n}{2k+3} \rceil\}$.

In each of the above cases, S is a dominating set for $Cir(n, A)$ of cardinality $\lceil \frac{n}{2k+3} \rceil + 1$ when $n \equiv 4,6,8, \ldots, 2k+2 \pmod{2k+3}$, and of cardinality $\lceil \frac{n}{2k+3} \rceil$ when $n \equiv 4, 6, 8,10, ..., 2k+2 \pmod{2k+3}$. Hence, the result follows. ■

Theorem 2.2. For any integer $n \ge 9$,

$$\gamma(Cir(n, B)) = \begin{cases} \lceil \frac{n}{2k+5} \rceil + 1, & n \equiv 6,8,10, \ldots, 2k+4 \pmod{2k+5} \\ \lceil \frac{n}{2k+5} \rceil & otherwise \end{cases}$$

Proof: Let S be a $\gamma(G)$-set of $G = Cir(n, B)$. Any vertex of G dominates $2k+5$ vertices of G including itself, so $|S| \ge \lceil \frac{n}{2k+5} \rceil$.

We claim that if $n \equiv 2t \pmod{2k+5}$, t is an integer such that $3 \le t \le k+1$, then $|S| \ge \lceil \frac{n}{2k+5} \rceil + 1$. To see this, assume to the contrary that $n \equiv 2t \pmod{2k+5}$, and $|S| = \lceil \frac{n}{2k+5} \rceil$. There are two consecutive vertices $v_l, v_l' \in S$ such that $|l - l'| < 2k+5$. Let $v_l'' \neq v_l$ is a consecutive vertex of v_l'. Without loss of generality we assume that $|v_l'' - v_l| = 2k+5+2t$. Then there are $2k+4+2t$ possibilities for v_l' to lies between v_l and v_l''. In each possibly there exists a vertex between v_l and v_l'' which is not dominated by$\{v_l, v_l', v_l''\}$, a contradiction. Hence, for $n \equiv 2t \pmod{2k+5}$, $|S| \ge \lceil \frac{n}{2k+5} \rceil + 1$.

Now it is sufficient to get a dominating set S of required cardinality. We consider the following cases:

1. For $n \equiv 6,8,10, 12,14, \ldots, 2k+4 \pmod{2k+5}$, $S = \{v_{(2k+5)i}: 0 \le i < \lceil \frac{n}{2k+5} \rceil\} \cup \{v_{n-3}\}$.

2. For $n \equiv 6, 8,10, ..., 2k+4 \pmod{2k+5}$, $S = \{v_{(2k+5)i}: 0 \le i < \lceil \frac{n}{2k+5} \rceil\}$.

In each of the above cases S is a dominating set for $Cir(n, B)$ of cardinality $\lceil \frac{n}{2k+5} \rceil + 1$ when $n \equiv 6,8, \ldots, 2k+4 \pmod{2k+5}$, and of cardinality $\lceil \frac{n}{2k+5} \rceil$ when $n \equiv 6, 8,10, ..., 2k+4 \pmod{2k+5}$.

Hence, the result follows. ■

Theorem 2.3. For any integer $n \ge 7$,

$$\gamma_t(Cir(n, A)) = \begin{cases} \lceil \frac{2n}{4k+4} \rceil + 1 & n \equiv 2,4,6, \ldots, 2k+2 \pmod{4k+4} \\ \lceil \frac{2n}{4k+4} \rceil & otherwise \end{cases}$$

Proof. Let S be a γ_t-set of $G = Cir(n, A)$. Note that $|A| = 2k+2$ and G is $2k+2$-regular. From the definition of the total domination number, it follows that $\lceil \frac{n}{2k+2} \rceil \le \gamma_t(G)$, $\gamma_t(G) = |S|$.

For $n \equiv 2j \pmod{4k+4}$, j is an integer such that $0 \le j < 2k+2$, we have

$\lceil\frac{n}{2k+2}\rceil = \lceil\frac{2n}{4k+4}\rceil$, so $\lceil\frac{2n}{4k+4}\rceil \leq \gamma_t(G)$.

For $n \equiv 2j \pmod{4k+4}$, j is an integer such that $0 \leq j < 2k+1$, n can be written as $n=(4k+4)l+2j = 2((2k+2)l+j)$, which l is integer and n is an even number. We partite $V(G)$ into two disjoint sets $I_1 = \{v_1, v_3, v_5, v_7, \ldots, v_{n-3}, v_{n-1}\}$ and $I_2 = \{v_0, v_2, v_4, v_6, \ldots, v_{n-4}, v_{n-2}\}$. Note that $|I_1| = |I_2| = (2k+2)l + j$. For any $x \in I_1$, $N(x) \subseteq I_2$, for any $y \in I_2$, $N(y) \subseteq I_1$. It follows that G is a balanced bipartite graph with bipartition sets I_1 and I_2. We can write $S = S_1 \cup S_2$, such that $S_1 \subseteq I_2$, $S_2 \subseteq I_1$, I_i is dominated by S_i, $1 \leq i \leq 2$ and $|S_1| = |S_2|$.

If $0<j \leq k+1$, then $|S_1| = |S_2| \geq \lceil\frac{(2k+2)l+j}{2k+2}\rceil = l+1$ and $\gamma_t(G) = |S| = |S_1| + |S_2| \geq 2\lceil\frac{(2k+2)l+j}{2k+2}\rceil = 2l+2$. On the other hand $2l+2 = \lceil\frac{(4k+4)(2l)+4j}{4k+4}\rceil + 1$, and so $\gamma_t(G) \geq \lceil\frac{(4k+4)(2l)+4j}{4k+4}\rceil + 1$.

If $k+2 \leq j \leq 2k+1$, then $|S_1| = |S_2| \geq \lceil\frac{(2k+2)l+j}{2k+2}\rceil = l+1$ and $\gamma_t(G) = |S| = |S_1| + |S_2| \geq 2\lceil\frac{(2k+2)l+j}{2k+2}\rceil = 2l+2$. On the other hand $2l+2 = \lceil\frac{(4k+4)(2l)+4j}{4k+4}\rceil$, and so $\gamma_t(G) \geq \lceil\frac{(4k+4)(2l)+2j}{2k+2}\rceil$.

If $j=0$, then $|S_1| = |S_2| \geq \lceil\frac{(2k+2)l}{2k+2}\rceil = l$ and $\gamma_t(G) = |S| = |S_1| + |S_2| \geq 2\lceil\frac{(2k+2)l}{2k+2}\rceil = 2l$. On the other hand $2l = \lceil\frac{(4k+4)2l}{4k+4}\rceil$, and so $\gamma_t(G) \geq \lceil\frac{(4k+4)2l}{4k+4}\rceil$.

Now it is sufficient to define a total dominating set S of required cardinality. We consider the following cases:

1. For $n \equiv 0 \pmod{4k+4}$, $S = \{v_{(4k+4)i+2k+1}, v_{(4k+4)i+4k+2} : 0 \leq i < [\frac{n}{4k+4}]\}$.

2. For $n \equiv 1,3,5,7, \ldots, 2k+1 \pmod{4k+4}$, $S = \{v_{(4k+4)i+2k+1}, v_{(4k+4)i+4k+2} : 0 \leq i < [\frac{n}{4k+4}]\} \cup \{v_0\}$.

3. For the cases $n \equiv 2,4,6, \ldots, 2k+2 \pmod{4k+4}$ and $n \equiv 2k+3, 2k+4, 2k+5, 2k+6, \ldots, 4k+2, 4k+3 \pmod{4k+4}$, $S = \{v_{(4k+4)i+2k+1,\, v(4k+4)i+2k+2} : 0 \leq I < [\frac{n}{4k+4}]\} \cup \{v_{n-2k}, v_{n-(2k+1)}\}$.

In each of the above cases S is a total dominating set of $Cir(n, A)$, cardinality of S is $\lceil\frac{n}{2k+2}\rceil + 1$ when $n \equiv 2,4, \ldots, 2k+2 \pmod{4k+4}$, and cardinality of S is $\lceil\frac{n}{2k+2}\rceil$ when $n \equiv 2, 4,6, \ldots, 2k+2 \pmod{4k+4}$. Hence, the result follows.■

Lemma 2.1. Let S be a subset of vertices of $G=Cir(n,B)$ with $k \geq 3$ and $G[S]$ has no isolated vertices. If $|S|$ is even, then S dominates at most $(2k+3)|S|$ vertices of G.

Proof: Let S be a subset of vertices of G with $|S| = t$, where t is even. Any two adjacent vertices of S dominate $4k+6$ vertices of G including themselves. S dominates at most $(4k+6)(\frac{|S|}{2}) = (2k+3)\,|S|$ vertices of G.■

Lemma 2.2. Let S be a subset of vertices of $G=Cir(n,B)$ with $k \geq 3$ and $G[S]$ has no isolated vertices. If $|S|$ is odd, then S dominates at most $(2k+3)\,|S| - (k+1)$ vertices of G.

Proof: Let S a be subset of vertices of G with $|S| = t$, where t is odd. Without loss of generality we may assume that $G[S]$ has $d = (\frac{|S|-3}{2}) + 1$ components $G_1, G_2, \ldots, G_d$ where $|V(G_1)| = 3$ and $|V(G_i)| = 2$ for $i = 2, 3, 4, \ldots, d$. Let $V(G_1) = \{x, y, z\}$, then $\{x, y, z\}$

dominates at most $5k+8$ vertices of G. S dominates at most $(4k+6)\left(\frac{|S|-3}{2}\right)+5k+8=$ $(2k+3)t-(k+1)$ vertices of G.■

Theorem 2.4. For any integer $n \geq 21$ and $k \geq 3$,

$$\gamma_t(Cir(n, B)) = \begin{cases} \lceil \frac{n}{2k+3} \rceil + 1, & n \equiv 3,4,5,6, \ldots, 2k+3 \pmod{4k+6} \\ \lceil \frac{n}{2k+3} \rceil & otherwise \end{cases}$$

Proof: Let S be a γ_t-set of $G = Cir(n,B)$. It follows from Lemma 2.1 and Lemma 2.2 that $|S| \geq \lceil \frac{n}{2k+3} \rceil$.

We claim that if $n \equiv 3,4,5, \ldots, 2k+3 \pmod{4k+6}$ and S is a total dominating for G**,** then $|S| \geq \lceil \frac{n}{2k+3} \rceil+1$.

To see this, assume to the contrary that $|S|=\lceil \frac{n}{2k+3} \rceil$. We have $n=(4k+6)l+j$, where l is a positive integer, $j \in \{3,4,5, \ldots, 2k+3\}$ then $|S|=\lceil \frac{(4k+6)l+j}{2k+3} \rceil = 2l+1$ is an odd number. So, the induced subgraph $G[S]$ has an odd component H with at least three vertices. We proceed to prove the following facts.

(i) Any component of $G[S]$ has at most three vertices.

Assume to the contrary that G_1 is a component of $G[S]$ and G_1 has at least 4 vertices. Without loss of generality assume that G_1 has 4 vertices. Then S dominates at most $6k+10+(4k+6)\left(\frac{|S|-3-4}{2}\right)+5k+8=(4k+6)l-k$ vertices of G, a contradiction.

(ii) H is the only odd component of $G[S]$**.**

Assume to the contrary that $H' \neq H$ is a component of $G[S]$ with $|V(H')|$ odd. It follows from fact **(i)** that $|V(H')|=3$. Since $|S|$ is odd, there is another component H'' with three vertices. Now S dominates at most $(4k+6)\left(\frac{|S|-3-3-3}{2}\right)+3(5k+8)=(4k+6)l-k$ vertices of G, a contradiction.

For $n \equiv k+3,k+4,k+5, \ldots, 2k+3 \pmod{4k+6}$ and we have $V(H)=\{v_f, v_q, v_p\}$ and S dominates $(4k+6)\left(\frac{|S|-3}{2}\right)+5k+8=(4k+6)l+k+2$ vertices of G, a contradiction.

For $n \equiv 3, 4, 5, \ldots, k+2 \pmod{4k+6}$. We have $n=(4k+6)l+j$, where l is a positive integer, $j \in \{3,4,5, \ldots, k+2\}$. It follows from facts that $G[S]$ has $l=\left(\frac{|S|-3}{2}\right)+1$ components $G_1, G_2, \ldots, G_l$ where $|V(G_i)|=2$ for $i= 2, 3, 4, \ldots, l$ and $|V(G_1)|=3$. Any two adjacent vertices of S dominates at most $4k+6$ consecutive vertices of $V(G)$.

$V(G)$ can be partitioned into l subset $I_1=\{v_0, v_1, v_2, \ldots, v_{4k+5}\}$, $I_2=\{v_{4k+6}, v_{4k+7}, v_{4k+8}, \ldots, v_{8k+11}\}$, $I_3=\{v_{8k+12}, v_{8k+13}, \ldots, v_{12k+17}\}$, …, $I_{l-1}=\{v_{(4k+6)(l-2)}, v_{(4k+6)(l-2)+1}, \ldots, v_{(4k+6)(l-1)-1}\}$, $I_l=\{v_{(4k+6)(l-1)+1}, v_{(4k+6)(l-1)+2}, \ldots, v_{n-1}\}$.

Note that, $|I_i|=4k+6$ for $i= 1, 2, 3, \ldots, l-1$ and $4k+9 \leq |I_l| \leq 5k+8$.

Without loss of generality we may assume that I_1 is dominated by $\{v_{2k+2}, v_{2k+3}\}$ of S and I_2 is dominated by $\{v_{6k+8}, v_{6k+9}\}$, then each of I_i is by two adjacent vertices of S. Then, vertices I_l $(4k+9 \leq |I_l| \leq 5k+8)$ is dominated by three consecutive vertices of S. In

each possibility there exists at least one vertex in I_l which is not dominated by this three vertices, a contradiction. This completes the claim.
Now it is sufficient to define a total dominating set S of required cardinality.
We consider the following case:

1. For $n \equiv 0 \pmod{4k+6}$, $S = \{v_{(4k+6)m+2k+2}, v_{(4k+6)m+2k+3}: 0 \le m < \lceil \frac{n}{4k+6} \rceil \}$.

2. For $n \equiv 1,2 \pmod{4k+6}$, $S = \{v_{(4k+6)m+2k+2}, v_{(4k+6)m+2k+3}: 0 \le m < \lceil \frac{n}{4k+6} \rceil \} \cup \{v_{n-3}\}$.

3. For $n \equiv 3,4,5,\ldots, 4k+5 \pmod{4k+6}$, $S = \{v_{(4k+6)m+2k+1}, v_{(4k+6)m+2k+2}: 0 \le m < \lceil \frac{n}{4k+6} \rceil \} \cup \{v_{n-(2k+2)}, v_{n-(2k+1)}\}$.■

Lemma 2.3. Let S be a subset of vertices of $G=Cir(n,B)$ with $k=2$ and $G[S]$ has no isolated vertices. If $|S|$ is even, then S dominates at most $7|S|$ vertices of G.
Proof: Let S be subset of vertices of G with $|S|= t$, where t is even. Any two adjacent vertices of S dominate *14* vertices of G including them selves. S dominates at most $14(\frac{|S|}{2}) = 7|S|$ vertices of G.■

Lemma 2.4. Let S be a subset of vertices of $G=Cir(n ,B)$, $k = 2$ and $G[S]$ has no isolated vertices. If $|S|$ is odd, then S dominates at most $7|S|-2$ vertices of G.
Proof: Let S be subset of vertices of G with $|S|= t$, where t is odd. Without loss of generality we may assume that $G[S]$ has $d = (\frac{|S|-3}{2}) + 1$ components $G_1, G_2, \ldots, G_d$, where $|V(G_1)|=3$ and $|V(G_i)|=2$ for $i= 2, 3, 4, \ldots, d$. let $V(G_1)=\{x, y, z\}$, then $\{x, y, z\}$ dominates at most *19* vertices of G. S dominates at most $14(\frac{|S|-3}{2}) + 19 = 7(|S|-3)+19 = 7|S|-2$ vertices of G.■

Theorem 2.5. For any integer $n \ge 17$ and $k = 2$,

$$\gamma_t(Cir(n , B)) = \begin{cases} \lceil \frac{n}{7} \rceil + 1 & n \equiv 3,4,5,6,7 \pmod{14} \\ \lceil \frac{n}{7} \rceil & otherwise \end{cases}$$

Proof: Let S be a γ_t-set of $G = Cir(n ,B)$. It follows from Lemma 2.3 and Lemma 2.4 that $|S| \ge \lceil \frac{n}{7} \rceil$.

We claim that if $n \equiv 3,4,5,6,7 \pmod{14}$ and S is a total dominating for G, then $|S| \ge \lceil \frac{n}{7} \rceil + 1$.

To see this, assume to the contrary that $|S|=\lceil \frac{n}{7} \rceil$. We have $n=14l+j$, where l is a positive integer, $j \in \{3,4,5, 6,7\}$. Then $|S|=\lceil \frac{14l+j}{7} \rceil = 2l+1$ is an odd number. So, the induced subgraph $G[S]$ has a component H with at least three vertices. We proceed to prove following facts.

i. Any component of $G[S]$ has at most three vertices.

Assume to the contrary that G_1 is a component of G[S] and G_1 has at least *4*verticies. Without loss of generality assume that G_1 has *4* verticies. Then S dominates at most $19+(14)(\frac{|S|-3-4}{2})+24=14l+1$ verticies of G, a contradiction.

ii. H is the only odd component of G[S].

Assume to the contrary that $H'\neq H$ is a component of G[S] with $|V(H')|$ odd. It follows from fact ***i*** that $|V(H')|=3$. Since $|S|$ is odd, there is another component H'' with three vertices. Now S dominates at most

$14(\frac{|S|-3-3-3}{2})+3(19)=14l+1$ verticies of G, a contradiction.

For $n\equiv 6,7 \pmod{14}$ and we have $V(H)=\{v_f, v_q, v_p\}$ and S dominates $14(\frac{|S|-3}{2})+19=14l+5$ vertices of G, a contradiction.

For $n\equiv 3, 4, 5 \pmod{14}$. We have $n=14l+j$, where l is a positive integer, $j\in\{3,4,5\}$. It follows from facts that $G[S]$ has $l=(\frac{|S|-3}{2})+1$ components $G_1, G_2, \ldots, G_l$ where $|V(G_i)|=2$ for $i=2, 3, 4, \ldots, l$ and $|V(G_1)|=3$. Any two adjacent of S at most 14 consecutive vertices of $V(G)$.

$V(G)$ can be partition into l subsets $I_1=\{v_1, v_2,\ldots,v_{14}\}$, $I_2=\{v_{15},v_{16},\ldots, v_{28}\}$, $I_3=\{v_{29}, v_{30},\ldots,v_{42}\}$, …, $I_{l-1}=\{v_{(14)(l-2)+1},v_{(14)(l-2)+2},\ldots,v_{(14)(l-1)}\}$, $I_l=\{v_{(14)(l-1)+1}, v_{(14)(l-1)+2}, \ldots, v_n\}$.

Note that, $|I_i|=14$ for $i=1, 2, 3, \ldots, l-1$ and $17\leq |I_l|\leq 19$.

Without loss of generality we may assume that I_1 is dominated by $\{v_7, v_8\}$ of S and I_2 is dominated by$\{v_{21},v_{22}\}$, then each of I_i is by two adjacent vertices of S. Vertices I_i ($17\leq |I_i|\leq 19$) is dominated by three consecutive vertices. In each possibility there exists at least one vertex in I_l which is not dominated by this three vertices, a contradiction.

This completes the claim.

Now it is sufficient to define a total dominating set S of required cardinality.

We consider the following case:

1. For $n\equiv 0 \pmod{14}$, $S=\{v_{(14)m+7}, v_{(14)m+8}: 0\leq m<[\frac{n}{14}]\}$.

2. For $n\equiv 1,2 \pmod{14}$, $S=\{v_{(14)m+7}, v_{(14)m+8}: 0\leq m<[\frac{n}{14}]\}\cup\{v_{n-2}\}$.

3. For $n\equiv 3,4,5,\ldots,13 \pmod{14}$, $S=\{v_{(14)m+7}, v_{(14)m+8}: 0\leq m<[\frac{n}{14}]\}\cup\{v_{n-6}, v_{n-5}\}$.■

Note 1. For any two adjacent vertices v_a and v_b of $G=Cir(n,B)$, k=1 and $n\geq 13$. We have the following:

i. If $|v_a- v_b|=1$, then v_a and v_b dominate 10 verticies of G including themselves.

ii. If $|v_a- v_b|=2$, then v_a and v_b dominate 9 verticies of G including themselves.

iii. If $|v_a- v_b|=4$, then v_a and v_b dominate11verticies of G including themselves.

Therefore, any two adjacent vertices of G dominate at most 11 vertices of G including themselves.

Note 2. Let G_1 be a component of γ_t-set such that G_1 has three vertices v_a, v_b, v_c, we have the following:

i. If $|v_a- v_b|=|v_b- v_c|=1$ then G_1 dominates11verticies of G including themselves.

ii. If $|v_a- v_b|=|v_b- v_c|=2$ then G_1 dominates11 vertices of G including themselves.

iii. If $|v_a - v_b|=1$, $|v_b - v_c|=2$ then G_1 dominates14 vertices of G including themselves.

iv. If $|v_a - v_b|=4$, $|v_b - v_c|=2$ then G_1 dominates13 vertices of G including themselves.

v. If $|v_a - v_b|=|v_b - v_c|=4$ then G_1 dominates 15 verticies of G including themselves**.**

Therefore, any three vertices of G belong to a component dominate at most 15 vertices of G including themselves.

Lemma 2.5. Let S be a subset of vertices of $G=Cir(n,B)$ with $k = 1$ and $G[S]$ has no isolated vertices. If $|S|$ is even, then S dominates at most $11(\frac{|S|}{2})$ vertices of G.

Proof: Let S be a subset of vertices of G with $|S|= t$, where t is even. Any two adjacent vertices of S dominate *11* vertices of G including themselves. So S dominates at most $11(\frac{|S|}{2})$ vertices of G.■

Lemma 2.6. Let S be a subset of vertices of $G=Cir(n, B)$, $k = 1$ and $G[S]$ has no isolated vertices. If $|S|$ is odd, then S dominates at most $(\frac{|S|-3}{2})11+15$ vertices of G.

Proof: Let S be a subset of vertices of G with $|S|= t$, where t is odd. Without loss of generality we may assume that $G[S]$ has $d = (\frac{|S|-3}{2})+1$ components $G_1, G_2, \ldots, G_d$, where $|V(G_1)|=3$ and $|V(G_i)| = 2$ for $i= 2, 3, 4, \ldots, d$. Let $V(G_1)=\{x, y, z\}$, then $\{x, y, z\}$ dominates at most *15* vertices of G. So S dominates at most $(\frac{|S|-3}{2})11+15$ vertices of G.■

Theorem 2.6. For any integer $n \geq 13$ and $k = 1$**,**

$$\gamma_t(Cir(n, B)) = \begin{cases} \lceil\frac{2n}{11}\rceil+1 & n \equiv 3, 5, 10 \pmod{11} \\ \lceil\frac{2n}{11}\rceil & \textit{otherwise} \end{cases}$$

Proof: Let S be a γ_t-set of $G = Cir(n,B)$. It follows from Lemma 2.5 and Lemma 2.6 that $|S| \geq \lceil\frac{2n}{11}\rceil$. In the next we prove two claims as following.

Claim 1. If $n \equiv 3,5 \pmod{11}$ and S is a total dominating set for G**,** then $|S| \geq \lceil\frac{2n}{11}\rceil+1$.

Let $n \equiv 3,5 \pmod{11}$ and let S be a total dominating set for G. Assume to the contrary that $|S|=\lceil\frac{2n}{11}\rceil$. We have $n=11l+j$, where l is a positive integer, $j \in \{3,5\}$. Then $|S|=\lceil\frac{22l+2j}{11}\rceil= 2l+1$ is an odd number. So, the induced subgraph $G[S]$ has an odd component H with at least three vertices. We proceed to following facts.

(i) Any component of $G[S]$ has at most three vertices.

Assume to the contrary that G_1 is a component of $G[S]$ and G_1 has at least *4* vertices. Without loss of generality assume that G_1 has *4* vertices. Then S dominates at most $15+(11)(\frac{|S|-3-4}{2})+20=11l+2$ vertices of G, a contradiction.

(ii) H is the only odd component of $G[S]$**.**

Assume to the contrary that $H'\neq H$ is a component of G[S] with $|V(H')|$ odd. Then $|V(H')|$=3. Since $|S|$ is odd, there is another component H'' with three vertices. Now S dominates at most $11(\frac{|S|-3-3-3}{2})+3(15)=11l+1$ verticies of G, a contradiction.

For $n\equiv 5$ (mod 11) and we have $V(H)=\{v_f, v_q, v_p\}$ and S dominates$11(\frac{|S|-3}{2})+15=11l+4$ vertices of G, a contradiction.

When $n\equiv 3$ (mod 11). We have $n=11l+3$, where l is a positive integer.

According to note 1, $\{v_5,v_9\}$ dominate 11 vertices. $N(\{v_5, v_9\})=\{v_1,v_2,\ldots, v_{13}\}-\{v_2, v_{12}\}$ and v_{12} is dominated by v_{16}, $N(\{v_{16}, v_{20}\})=\{v_{12},v_{13},\ldots, v_{24}\}-\{v_{13}, v_{23}\}$ and v_{23} is dominated by v_{27}, $N(\{v_{27}, v_{31}\})=\{v_{23},v_{24},\ldots, v_{35}\}-\{v_{24}, v_{34}\}$, we continue this process and $N(\{v_{(l-2)11+5}, v_{(l-2)11+9}\})=\{v_{11l-21+1},v_{11l-20},\ldots, v_{11l-9}\}-\{v_{11l-20},v_{11l-10}\}$, So $\{v_{11l-8},v_{11l-7},v_{11l-6},\ldots,v_{n-1},v_n\}\cup\{v_2, v_{11l-10}\}$is dominated by three vertices. In each possibility there exits at least one vertex in Last subset which is not dominated by this 3 vertices, a contradiction.

This completes the Claim 1.

Claim 2. If $n\equiv 10$ (mod 11) and let S be a total dominating set for G, then $|S|\geq\lceil\frac{2n}{11}\rceil+1$.

Assume to the contrary that $|S|=\lceil\frac{2n}{11}\rceil$. We have $n=11l+10$ where l is a positive integer. Then $|S|=\lceil\frac{22l+20}{11}\rceil=2l+2$ is an even number. We have any component of G has at least two vertices. Now we are proving any component of G has exactly two vertices.

Assume to the contrary that G_1 is a component of G and it has at least 3 vertices. Let G_1 has 3 vertices. So $|S|$ is an even number, there exist $G_1'\neq G_1$ is a component of G[S] with $|V(G_1')|$ is odd, then at least $|V(G_1')|$ is 3. If $|V(G_1')|$=3, then S dominates at most $\boldsymbol{11}(\frac{|S|-3-3}{2})+2(15)=11l+8$ vertices of G, a contradiction.

So the induced subgraph $G[S]$ has components with two vertices. It follows from Note 1 and process of case 2, $S=\{v_5, v_9, v_{16}, v_{20}, v_{27}, v_{31}, \ldots, v_{(l-1)11+5}, v_{(l-1)l+9}, v_{11l+5}, v_{n-1}\}$. We have $N(\{v_5, v_9\})=\{v_1,v_2,\ldots, v_{13}\}-\{v_2, v_{12}\}$,$N(\{v_{16}, v_{20}\})=\{v_{12},v_{13},\ldots, v_{24}\}-\{v_{13}, v_{23}\}$, $N(\{v_{27}, v_{31}\})=\{v_{23},\ldots, v_{35}\}-\{v_{24}, v_{34}\}$, …,$N(\{v_{(l-1)11+5}, v_{(l-1)11+9}\})=\{v_{(l-1)11+1},\ldots, v_{11l-17}\}-\{v_{(l-2)11+2},v_{11l-16}\}$, $N(\{v_{11l+5},v_{11l+9}\})=\{v_{11l+1},\ldots, v_3\}-\{v_{11l+2},v_2\}$ and v_2 is not dominated by S, a contradiction.

This completes the Claim 2.

Now it is sufficient to define a total dominating set S of required cardinality. We consider the following case:

1. For $n\equiv 0$ (mod 11), $S=\{v_{(11)m+5}, v_{(11)m+9}: 0\leq m<\lceil\frac{n}{11}\rceil\}$.

2. For $n\equiv 1,2,4$ (mod 11), $S=\{v_{(11)m+5}, v_{(11)m+9}: 0\leq m<\lceil\frac{n}{11}\rceil\}\cup\{v_{n-2}\}$.

3. For $n\equiv 3,5,6,7,8$ (mod 11), $S=\{v_{(11)m+5}, v_{(11)m+9}: 0\leq m<\lceil\frac{n}{11}\rceil\}\cup\{v_{n-2},v_{n-3}\}$.

4. For $n\equiv 9$ (mod 11), $S=\{v_{(11)m+5}, v_{(11)m+9}: 0\leq m<\lceil\frac{n}{11}\rceil\}\cup\{v_{n-2},v_{n-4}\}$.

5. For $n\equiv 10$ (mod 11), $S=\{v_{(11)m+5}, v_{(11)m+9}: 0\leq m<\lceil\frac{n}{11}\rceil\}\cup\{v_{n-2},v_{n-3},v_{n-5}\}$.■

Theorem 2.7. For $n \geq 7$, *Cir(n , A)* is γ-*critical* if and only if $n \equiv 4$ (mod *2k+3*).
Proof. First we show that if $n \equiv 4$ (mod *2k+3*) that *G* is γ-*critical*. Let *x* be a vertex of *G* = *Cir((2k+3)l+4 , A)*, for some positive integer *l*. Since *G* is transitive, we assume that $x = v_{n-2}$. It is easy to see that $S = \{v_{(2k+3)i}: 0 \leq i \leq \lceil \frac{n}{2k+3} \rceil\}$ is a dominating set for $G - x$. It follows that $\gamma(G - x) \leq \lceil \frac{n}{2k+3} \rceil < \lceil \frac{n}{2k+3} \rceil + 1 = \gamma(G)$. Hence, *G* is γ-*critical.*
Suppose now that $n \equiv 4$ (mod *2k+3*), we show that *G* is not γ-*critical*. Let *T* be a subset of vertices with $|T| < \gamma(G)$. Without loss of generality we let $|T| = \gamma(G)-1$. We show that any $|T|$ vertices of *G* dominate at most $n - 2$ vertices of *G*.

We consider the following cases:

1. For $n \equiv 4,6,8, \ldots, 2k+2$ (mod *2k+3*), by Theorem 2.1 $\gamma(G) = \lceil \frac{n}{2k+3} \rceil$.
If $n \equiv 0$ (mod *2k+3*), then $n = (2k+3)l$ for some intrger *l*. It follows that $\gamma(G) = l$. Now *T* dominates at most$(2k+3)(l-1) \leq n - 2$ vertices of *G*. Similarly, for $n \equiv 2,3,5,7,9,11, \ldots, 2k+1$(mod $2k+3$), *T* dominates at most $(2k+3)(l-1) \leq n - 2$ vertices of *G*.
We assume that $n \equiv 1$ (mod $2k+3$). There is an integer *l* such that
$n = (2k+3)l+1$, $|T| = \lceil \frac{n}{2k+3} \rceil - 1 = l$.

If there are two consecutive vertices *x*, *y* in *T* such that $|x - y| < 2k+3$, then $N_G(x) \cap N_G(y) \neq \emptyset$. Hence, $\{x, y\}$ dominates at most $4k+5$ vertices of *G* and $T \setminus \{x,y\}$ dominates at most $(2k+3)(l-2)$ vertices of *G*. So, *T* dominates at most $n - 2$ vertices of *G*.

It remains to assume that for any two consecutive vertices *a,b* in *T*, $|a - b| \geq 2k+3$. In this case, there are two consecutive vertices *x,y* in *T* such that $|x - y| > 2k+3$. Then there exit two vertices *u,v* lie between *x* and *y* in *G*, and *T* does not dominate $\{u,v\}$. So, *T* dominates at most $n - 2$ vertices of *G*, which is a contradiction.

2. For $n \equiv 2t$ (mod *2k+3*), *t* is an integer with $3 \leq t \leq k+1$ by Theorem 2.1, $\gamma(G) = \lceil \frac{n}{2k+3} \rceil + 1$. There are two consecutive vertices $v_l, v_l' \in S$ such that $|l - l'| < 2k+3$. Let $v_l'' \neq v_l$ be a consecutive vertex of v_l'. Without loss of generality we assume that $|v_l'' - v_l| = 2k+3+2t$. Then there are $2k+2+2t$ possibilities for v_l' to lies between v_l and v_l''. In each possibly there exists at least two vertex between v_l and v_l'' which is not dominated by $\{v_l, v_l', v_l''\}$.

So, *T* dominates at most $n - 2$ vertices of *G,* which is a contradiction.■

Theorem 2.8. For $n \geq 9$, *Cir(n, B)* is γ-*critical* if and only if $n \equiv 6$ *(mod 2k+5).*
Proof: First we show that if $n \equiv 6$ (mod *2k+5*) that *G* is γ-*critical*. Let *x* be a vertex of *G* = *Cir((2k+5)l+6 , A)*for some positive integer *l*. Since *G* is transitive, we assume that $x = v_{n-3}$. It is easy to see that $S = \{v_{(2k+5)i}: 0 \leq i \leq \lceil \frac{n}{2k+5} \rceil\}$ is a dominating set for $G - x$. It follows that $\gamma(G - x) \leq \lceil \frac{n}{2k+5} \rceil < \lceil \frac{n}{2k+5} \rceil + 1 = \gamma(G)$. Hence, *G* is γ-*critical.*

Suppose now, that $n \equiv 6$ (mod *2k+5*). We show that *G* is not γ-*critical*. Let *T* be a subset of vertices with $|T| < \gamma(G)$. Without loss of generality we let $|T| = \gamma(G)-1$. We show that any $|T|$ vertices of *G* dominate at most $n - 2$ vertices of *G*.

We consider the following cases:

1. For $n \equiv 8, \ldots, 2k+2, 2k+4$ (mod *2k+5*), by Theorem 2.2, $\gamma(G) = \lceil \frac{n}{2k+5} \rceil$.

If $n \equiv 0$ (mod $2k+5$), then $n = (2k+5)l$ for some integer l. It follows that $\gamma(G) = l$. Now, T dominates at most $(2k+5)(i-1) \leq n-2$ vertices of G. Similarly for $n \equiv 2,3,4,5,7,9,11,\ldots,2k+3$ (mod $2k+5$), T dominates at most $(2k+5)(i-1) \leq n-2$ vertices of G. We assume that $n \equiv 1$ (mod $2k+5$). There is an integer l such that $n = (2k+5)l+1$. Without loss of generality we let $|T| = \lceil \frac{n}{2k+5} \rceil - 1 = l$.

If there are two consecutive vertices x, y in T such that $|x-y| < 2k+5$, then $N_G(x) \cap N_G(y) \neq \emptyset$. Hence, $\{x,y\}$ dominates at most $4k+9$ vertices of G and $T \setminus \{x,y\}$ dominates at most $(2k+5)(l-2)$ vertices of G. So, T dominates at most $n-2$ vertices of G.

It remains to assume that for any two consecutive vertices a,b in T, $|a-b| \geq 2k+5$. In this case there are two consecutive vertices x,y in T such that $|x-y| > 2k+5$. Then there exit two vertices u, v lie between x and y in C, and T does not dominate $\{u,v\}$. So, T dominates at most $n-2$ vertices of G, which is a contradiction.

2. For $n \equiv 2t$ (mod $2k+5$), t is an integer with $4 \leq t \leq k+2$, by Theorem 2.2, $\gamma(G) = \lceil \frac{n}{2k+5} \rceil + 1$. There are two consecutive vertices v_l, $v_l' \in S$ such that $|l - l'| < 2k+5$. Let $v_l'' \neq v_l$ be a consecutive vertex of v_l'. Without loss of generality we assume that $|v_l'' - v_l| = 2k+5+2t$. Then there are $2k+4+2t$ possibilities for v_l' to lies between v_l and v_l''. In each possibly there exists at least two vertex between v_l and v_l'' which is not dominated by $\{v_l, v_l', v_l''\}$.

So, T dominates at most $n-2$ vertices of G, which is a contradiction. ■

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