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Common Fixed Point Theorems in Fuzzy 2 and Fuzzy 3-Metric Spaces

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***Abstract*.** The purpose of this paper is to prove some common fixed point theorems in fuzzy-2, fuzzy-3 metric spaces by employing the notion of semi-compatible and reciprocal continuity. We extend and generalize the result of Chauhan et al. [1] and Som [19] from fuzzy metric spaces to fuzzy-2 metric and fuzzy-3 metric Spaces.

Keywords: Common fixed Point, Fuzzy metric space, fuzzy -2 metric spaces, fuzzy-3 metric spaces, semi-compatible, reciprocal continuity.

AMS Mathematics Subject Classification (2010): 47H10, 54H25

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [20]. Kramosil and Michalek [10] introduced the concept of fuzzy metric space. George and Veeramani [7] modified this concept of fuzzy metric space and defined Hausdroff topology on fuzzy metric space. Many authors have studied common fixed point theorems in fuzzy metric spaces. The most interesting results in this direction are due to Cho [2], George and Veeramani [7], Grabiec [8], Kaleva [9], Kramosil and Michalek [10], Mishra et al. [11] etc., Singh and Chauhan [15] introduced the concept of compatibility in fuzzy metric space and proved some common fixed point theorems in fuzzy metric space.

Gahler [5-6] investigated 2-metric spaces in a series of his papers. It is to be remarked that Sharma, Sharma and Iseki [14] studied for the first time contraction type mappings in 2-metric spaces. In 2002, Sharma [13] introduced fuzzy 2-metric space and fuzzy 3-metric space and obtained some common fixed point theorems for three mappings in this setting. Sharma [13] proved common fixed point theorems for commuting maps, thus modified and extended the results due to Fisher [4].

In 2007, Singh et al. [16] introduced the concept of semi-compatibility and weak compatibility in fuzzy -2, fuzzy-3 metric space.

For the sake of completeness, we recall some definitions and known results in fuzzy, fuzzy -2, fuzzy-3 metric spaces.

2. Preliminaries and definitions

Definition 2.1. [12] A binary operation $*: [0, 1] \times [0,1] \to [0,1]$ is called a continuous t-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a*1 = a$; for all $a \in [0,1]$
(iv) $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Example of t-norms are $a*b = ab$ and $a*b = \min\{a, b\}$.

Definition 2.2. [10] The 3-tuple (X, M,*) is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0,\infty)$ satisfying the following conditions for all x, y, z $\in$ X and s, t > 0:
(FM-1) $M(x, y, 0) = 0$;
(FM-2) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$;
(FM-3) $M(x, y, t) = M(y, x, t)$;
(FM-4) $M(x, y, t) * M(y, z, s) \geq M(x, z, t + s)$;
(FM-5) $M(x, y, \cdot): [0,\infty) \to [0,1]$ is left continuous.

Note that M(x, y, t) can be thought of as the degree of nearness between x and y with respect to t. We identify x = y with M(x, y, t) = 1, for all t > 0.
The following example shows that every metric space induces a fuzzy metric space.

Example 2.1. [10] Let (X, d) be a metric space. Define $a*b = \min\{a, b\}$ and for all x, y $\in$ X, $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all t > 0 and M(x, y, 0) = 0.Then (X, M, $*$) is a fuzzy metric space. It is called the fuzzy metric space induced by the metric d.

Definition 2.3. [8] Let (X, M, $*$) be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to convergent to a point x $\in$ X if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all t > 0. Further, the sequence $\{x_n\}$ is said to be Cauchy sequence in X, if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for all t > 0 and p > 0.The space X is said to be complete if every Cauchy sequence in it converges to a point of it.

Definition 2.4. [16] A function M is continuous in fuzzy metric space iff whenever $\{x_n\}\to x$ and $\{y_n\}\to y$ then $\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t)$ for each t > 0.

Definition 2.5. [3] A binary operation $*: [0,1] \times [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if ([0,1], $*$) is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 \leq a_2 * b_2 * c_2$ whenever $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2$ for all a_1, a_2, b_1, b_2, c_1 and c_2 are in [0, 1].

Definition 2.6. [13] The 3-tuple (X, M, $*$) is called a fuzzy-2 metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^3 \times [0,\infty)$ satisfying the following conditions for all x, y, z, u $\in$ X and $t_1, t_2, t_3 > 0$:
(FM'-1) $M(x, y, z, 0) = 0$;
(FM'-2) $M(x, y, z, t) = 1$, for all t > 0 and when at least two of the three points are equal,
(FM'-3) $M(x, y, z, t) = M(y, x, z, t) = M(z, x, y, t)$;
(Symmetry about three variables)

(FM'-4) $M(x, y, z, t_1 + t_2 + t_3) \geq M(x, y, u, t_1) * M(x, u, z, t_2) * M(u, y, z, t_3)$;
(This corresponds to tetrahedron inequality in 2-metric space).
(FM'-5) $M(x, y, z, \cdot): [0,\infty) \to [0,1]$ is left continuous.

The function value M(x, y, z, t) may be interpreted as the probability that the area of triangle is less than t.

Definition 2.7. [16] Let $(X, M, *)$ be a fuzzy-2 metric space. A sequence $\{x_n\}$ in X is said to convergent to a point $x \in X$ $\lim_{n\to\infty} M(x_n, x, a, t) = 1$, for all $a \in X, t > 0$. Further, the sequence $\{x_n\}$ is said to be Cauchy sequence in X, if $\lim_{n\to\infty} M(x_n, x_{n+p}, a, t) = 1$ for all $a \in X$, $t > 0$ and $p > 0$.The space X is said to be complete if every Cauchy sequence in it converges to a point of it.

Definition 2.8. [16] A function M is continuous in fuzzy-2 metric space iff whenever $\{x_n\}\to x$ and $\{y_n\}\to y$ then $\lim_{n\to\infty} M(x_n, y_n, a, t) = M(x, y, a, t)$ for all $a \in X$ and for each $t > 0$.

Definition 2.9. [16] A binary operation $*: [0,1] \times [0,1] \times [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 * d_1 \leq a_2 * b_2 * c_2 * d_2$ whenever $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2, d_1 \leq d_2$ for all $a_1, a_2, b_1, b_2, c_1, c_2, d_1$ and d_2 are in [0, 1].

Definition 2.10. [13] The 3-tuple $(X, M, *)$ is called a fuzzy-3 metric space if x is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^4 \times [0, \infty)$ satisfying the following conditions for all $x, y, z, w, u \in X$ and $t_1, t_2, t_3, t_4 > 0$:
(FM"-1) $M(x, y, z, w, 0) = 0$;
(FM"-2) $M(x, y, z, w, t) = 1$, for all $t > 0$, iff at-least two of the four points are equal
(FM"-3) $M(x, y, z, w, t) = M(y, x, z, w, t) = M(w, z, x, y, t) = \ldots$;
(Symmetry)
(FM"-4) $M(x, y, z, t_1+t_2+t_3+t_4) \geq M(x, y, z, u, t_1) * M(x, y, u, w, t_2) * M(x, u, z, w, t_3) * M(u, y, z, w, t_4)$;
(FM"-5) $M(x, y, z, w, \cdot): [0,\infty) \to [0,1]$ is left continuous.

Definition 2.11. [16] Let $(X, M, *)$ be a fuzzy-3 metric space. A sequence $\{x_n\}$ in X is said to convergent to a point $x \in X$ if

$$\lim_{n\to\infty} M(x_n, x, a, b, t) = 1,$$

for all $a, b \in X$, $t > 0$. Further, the sequence $\{x_n\}$ is said to be Cauchy sequence in X, if $\lim_{n\to\infty} M(x_n, x_{n+p}, a, b, t) = 1$ for all $a, b \in X$, $t > 0$ and $p > 0$. The space X is said to be complete if every Cauchy sequence in it converges to a point of it.

Definition 2.12. [16] A function M is continuous in fuzzy-3 metric space iff whenever $\{x_n\} \to x$ and $\{y_n\} \to y$ then $\lim_{n\to\infty} M(x_n, y_n, a, b, t) = M(x, y, a, b, t)$ for all $a, b \in X$ and for each $t > 0$.

Definition 2.13. [11] Let A and B mappings from a fuzzy metric space, $(X, M, *)$ into itself. The mappings are said to be compatible if

$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}Ax_n = \lim_{n\to\infty}Bx_n = x$ for some $x \in X$.
Singh et al.[16] extend this concept in fuzzy-2 metric spaces and in fuzzy-3 metric spaces as follows:

Definition 2.14. [16] Let A and B mappings from a fuzzy-2 metric space, $(X, M, *)$ into itself. The mappings are said to be compatible if $\lim_{n\to\infty} M(ABx_n, BAx_n, a, t) = 1$ for all $t > 0$ and $a \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}Ax_n = \lim_{n\to\infty}Bx_n = x$ for some $x \in X$.

Definition 2.15. [16] Let A and B mappings from a fuzzy-3 metric space, $(X, M, *)$ into itself. The mappings are said to be compatible if $\lim_{n\to\infty} M(ABx_n, BAx_n, a, b, t) = 1$ for all $t > 0$ and for all $a, b \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}Ax_n = \lim_{n\to\infty}Bx_n = x$ for some $x \in X$.

Definition 2.16. [17] Two self maps A and B of a fuzzy metric space $(X, M, *)$ are said to be weak compatible if they commute at their coincidence points, that is $Ax = Bx$ implies $ABx = BAx$.

Definition 2.17. [18] Let A and B mappings from a fuzzy metric space $(X, M, *)$ into itself. The mappings are said to be semi-compatible if $\lim_{n\to\infty}M(ABx_n, Bx, t) = 1$, for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}Ax_n = \lim_{n\to\infty}Bx_n = x$ for some $x \in X$.

Singh et al.[16] extend this concept in fuzzy-2 metric spaces and in fuzzy-3 metric spaces as follows:

Definition 2.18. [16] Let A and B mappings from a fuzzy-2 metric space $(X, M, *)$ into itself. The mappings are said to be semi-compatible if $\lim_{n\to\infty}M(ABx_n, Bx, a, t) = 1$ for all $t > 0$ and $a \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}Ax_n = \lim_{n\to\infty}Bx_n = x$ for some $x \in X$.

Definition 2.19. [16] Let A and B mappings from a fuzzy-3 metric space $(X, M, *)$ into itself. The mappings are said to be semi-compatible if $\lim_{n\to\infty}M(ABx_n, Bx, a, b, t) = 1$ for all $t > 0$ and $a, b \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}Ax_n = \lim_{n\to\infty}Bx_n = x$ for some $x \in X$.

Definition 2.20. [3] A Pair (A, S) of self maps of a fuzzy metric space $(X, M, *)$ is said to be reciprocal continuous if $\lim_{n\to\infty}ASx_n = Ax$ and $\lim_{n\to\infty}SAx_n = Sx$, whenever there exist a sequence $\{x_n\}$ such that $\lim_{n\to\infty}Ax_n = \lim_{n\to\infty}Sx_n = x$ for some $x \in X$.
In a similar manner, we extend the concept of reciprocal continuity in fuzzy-2 metric spaces and fuzzy-3 metric spaces as follows:

Definition 2.21. A Pair (A, S) of self maps of a fuzzy-2 metric space $(X, M, *)$ is said to be reciprocal continuous if $\lim_{n\to\infty}ASx_n = Ax$ and $\lim_{n\to\infty}SAx_n = Sx$, whenever there exist a sequence $\{x_n\}$ such that $\lim_{n\to\infty}Ax_n = \lim_{n\to\infty}Sx_n = x$ for some $x \in X$.

Definition 2.22. A Pair (A, S) of self maps of a fuzzy-3 metric space (X, M, $*$) is said to be reciprocal continuous if $\lim_{n\to\infty} ASx_n = Ax$ and $\lim_{n\to\infty} SAx_n = Sx$, whenever there exist a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some $x \in X$.

Example 2.2. Let(X, M,*) be a fuzzy metric space, where X = [0, 4]. Define self maps A and B on X as follows

$$Ax = \begin{cases} 5x-4, & \text{if } x \in [0,2] \\ x-3, & \text{if } x \in (2,4] \end{cases}, \quad Bx = \begin{cases} x^2, & \text{if } x \in [0,2] \\ 3x, & \text{otherwise} \end{cases}$$

$x \in [0,4]$.Define a sequence $x_n = \left(1-\frac{1}{n}\right)$;

Then $\lim_{n\to\infty} ABx_n = \lim_{n\to\infty} A\left(1-\frac{1}{n}\right)^2 = \lim_{n\to\infty} 5\left(1-\frac{1}{n}\right) - 4 = 1 = Ax$.

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} 5\left(1-\frac{1}{n}\right) - 4 = 1$

$$\lim_{n\to\infty} BAx_n = \lim_{n\to\infty} B\left(5\left(1-\frac{1}{n}\right)-4\right)$$
$$= \lim_{n\to\infty} B\left(1-\frac{5}{n}\right) = \lim_{n\to\infty}\left(1-\frac{5}{n}\right)^2 = 1 = Bx.$$

$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^2 = 1$

$\lim_{n\to\infty} ABx_n = Ax$ and $\lim_{n\to\infty} BAx_n = Bx$.

Thus pair (A, B) is reciprocally continuous.

3. Main results

Theorem 3.1. Let (X, M, $*$) be a complete fuzzy-2 metric space and $t*t \geq t$ for all $t \in [0,1]$ and let A, B, S and T be self maps of X such that

(3.1.1) $A(X) \cup B(X) \subseteq S(X) \cap T(X)$,

(3.1.2) Pairs (A, T) and (B, S) are semi compatible and are reciprocal continuous,

(3.1.3) $aM(Tx,Sy,v,t) + bM(Tx,Ax,v,t) + cM(Sy,By,v,t)$
$+ \max\{M(Ax,Sy,v,t), M(By,Tx,v,t)\} \leq qM(Ax,By,v,t)$

for all $x, y \in X$, where $a, b, c \geq 0$, $q > 0$ with $q < (a + b + c+1)$.

Then A, B, S and T have a unique common fixed point.

Proof: Let $x_o \in X$ be any arbitrary point.

Since $A(X) \subseteq S(X)$, there is a point $x_1 \in X$ such that $Ax_o = Sx_1$.

Again since $B(X) \subseteq T(X)$ for this x_1 there is a point $x_2 \in X$ such that $Bx_1 = Tx_2$.

Inductively, we construct a sequence $\{y_{2n}\}$ in X such that

$y_{2n} = Ax_{2n} = Sx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Tx_{2n+2}$, for all n = 0, 1, 2,…

Let $M_{2n} = M(y_{2n}, y_{2n+1}, t) < 1$ for all n.

Putting $x = x_{2n}$, $y = x_{2n+1}$ in equation (3.1.3),we get

$$aM(Tx_{2n},Sx_{2n+1},v,t) + bM(Tx_{2n},Ax_{2n},v,t) + cM(Sx_{2n+1},Bx_{2n+1},v,t)$$
$$+ \max\{M(Ax_{2n},Sx_{2n+1},v,t),M(Bx_{2n+1},Tx_{2n},v,t)\} \leq qM(Ax_{2n},Bx_{2n+1},v,t)$$

$$aM(y_{2n-1},y_{2n},v,t) + bM(y_{2n-1},y_{2n},v,t) + cM(y_{2n},y_{2n+1},v,t)$$
$$+ \max\{M(y_{2n},y_{2n},v,t),M(y_{2n+1},y_{2n-1},v,t)\} \leq qM(y_{2n},y_{2n+1},v,t)$$

Implies $(a + b)M_{2n-1} + cM_{2n} + 1 \leq qM_{2n}$

$(q- c) M_{2n} \geq (a + b)M_{2n-1} + 1$

$(q -c)M_{2n} > (a + b) M_{2n-1}$

i.e. $M_{2n} > pM_{2n-1} > M_{2n-1}$, where $p = \frac{(a+b)}{(q-c)} > 1$, (3.1.1.1)

Thus $\{M_{2n}, n \geq 0\}$ is an increasing sequence of positive real numbers in [0, 1] and therefore tends to a limit $L \leq 1$.
We claim that L = 1, suppose if not i.e. L<1.
If L < 1, taking limit as $n \to \infty$ in equation (3.1.1.1), we get L < L, which is a contradiction. Therefore L = 1.
Now consider, for any positive integer p,

$M(y_{2n}, y_{2n+p}, v, t) \geq M(y_{2n}, y_{2n+1}, v, \frac{t}{p}) * \ldots * M(y_{2n+p-1}, y_{2n+p}, v, \frac{t}{p})$

Implying $\lim_{n\to\infty} M(y_{2n}, y_{2n+p}, v, t) \geq 1 * \ldots * 1 = 1$.

Thus $\{y_{2n}\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $u \in X$ such that the sequence $\{y_{2n}\}$ converges to $u \in X$ and subsequently, the sequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n+1}\}$ and $\{Tx_{2n+2}\}$and its subsequences also converges to $u \in X$.
Now we will show that u is a common fixed point of A, B, S and T.

Step 1. Since the pair (A, T) is reciprocal continuous, we get

$\lim_{n\to\infty} TAx_n \to Tu$, also $\lim_{n\to\infty} TTx_n \to Tu$,

and since pair (A, T) is semi compatible, we have

$\lim_{n\to\infty} ATx_n \to Tu$.

Now we put $x = Tx_n$, $y = x_n$ in equation (3.1.3), we get

$$aM(TTx_n, Sx_n, v, t) + bM(TTx_n, ATx_n, v, t) + cM(Sx_n, Bx_n, v, t)$$
$$+ \max\{M(ATx_n, Sx_n, v, t), M(Bx_n, TTx_n, v, t)\} \leq qM(ATx_n, Bx_n, v, t)$$

Taking limit as $n \to \infty$, we get

$$aM(Tu, u, v, t) + bM(u, Tu, v, t) + cM(u, u, v, t)$$
$$+ \max\{M(Tu, u, v, t), M(u, Tu, v, t)\} \leq qM(Tu, u, v, t)$$
$$aM(Tu, u, v, t) + bM(u, Tu, v, t) + c + M(Tu, u, v, t) \leq qM(Tu,u,v,t)$$

this gives $c \leq (q - a - b - 1)M(Tu, u, v, t)$

i.e. $M(Tu, u, v, t) \geq (\frac{c}{q-a-b-1}) > 1$,

Thus Tu = u.

Step 2. Put x = u and $y = x_n$, in equation (3.1.3), we get

$$aM(Tu, Sx_n, v, t) + bM(Tu, Au, v, t) + cM(Sx_n, Bx_n, v, t)$$
$$+ \max\{M(Au, Sx_n, v, t), M(Bx_n, Tu, v, t)\} \leq qM(Au, Bx_n, v, t)$$

Taking limit as $n \to \infty$, we get

$$aM(u, u, v, t) + bM(u, Au, v, t) + cM(u, u, v, t)$$
$$+ \max\{M(Au, u, v, t), M(u, u, v, t)\} \leq qM(Au,u,v,t)$$
$$a + bM(u, Au, v, t) + c + \max\{M(Au, u, v, t), 1\} \leq qM(Au,u,v,t)$$

Implies $(a + c + 1) \leq (q - b)M(Au, u, v, t)$

i.e. $M(Au, u, v, t) \geq (\frac{a+c+1}{q-b}) > 1$,

This gives Au = u. Hence Au = u = Tu.

Step 3. Since the pair (B, S) is reciprocal continuous.
In this case, $\lim_{n\to\infty} SBx_n \to Su$, also $\lim_{n\to\infty} SSx_n \to Su$,
Since (B, S) is semi compatible, we have $\lim_{n\to\infty} BSx_n \to Su$.
Let $x = x_n$, $y = Sx_n$ in equation (3.1.3), we get

$$aM(Tx_n, SSx_n, v, t) + bM(Tx_n, Ax_n, v, t) + cM(SSx_n, BSx_n, v, t)$$

$+ \max\{M(Ax_n, SSx_n, v, t), M(BSx_n, Tx_n, v, t)\} \leq qM(Ax_n, BSx_n, v, t)$

Taking limit as $n \to \infty$, we get

$aM(u, Su, v, t) + bM(u, u, v, t) + cM(Su, Su, v, t)$

$+ \max\{M(u, Su, v, t), M(Su, u, v, t)\} \leq qM(u, Su, v, t)$

Implies $aM(u, Su, v, t) + b + c + M(u, Su, v, t) \leq qM(u,Su,v,t)$

$(b + c) \leq (q - a - 1)M(u, Su, v, t)$

i.e. $M(u, Su, v, t) \geq \left(\frac{b+c}{q-a-1}\right) > 1$

This gives $Su = u$. Hence $Au = Su = u = Tu$.

<u>Step 4.</u> Put $x = x_n$, $y = u$ in equation (3.1.3), we get

$aM(Tx_n,Su,v,t) + bM(Tx_n,Ax_n,v,t) + cM(Su,Bu,v,t)$

$+ \max\{M(Ax_n,Su,v,t), M(Bu,Tx_n,v,t)\} \leq qM(Ax_n,Bu,v,t)$

Taking limit as $n \to \infty$, we get

$aM(u,u,v,t) + bM(u,u,v,t) + cM(u,Bu,v,t)$

$+ \max\{M(u,u,v,t), M(Bu,u,v,t)\} \leq qM(u,Bu,v,t)$

$a + b + cM(u,Bu,v,t) + 1 \leq qM(u,Bu,v,t)$

Implies $(a+b+1) \leq (q - c)M(u, Bu, v, t)$

i.e. $M(u,Bu,v,t) \geq \left(\frac{a+b+1}{q-c}\right) > 1$, this gives $Bu = u$.

Hence $u = Au = Su = Bu = Tu$ is a common fixed point of A, B, S and T.

Uniqueness. Let $z \neq u$ be another common fixed point of A, B, S and T, then $Az = Bz = Sz = Tz = z$.

Put $x = u$ and $y = z$ in equation (3.1.3), we get

$aM(Tu, Sz, v, t) + bM(Tu, Au, v, t) + cM(Sz, Bz, v, t)$

$+ \max\{M(Au, Sz, v, t), M(Bz, Tu, v, t)\} \leq qM(Au,Bz,v,t)$

$aM(u, z, v, t) + bM(u, u, v, t) + cM(z, z, v, t)$

$+ \max\{M(u, z, v, t), M(z, u, v, t)\} \leq qM(u,z,v,t)$

$aM(u, z, v, t) + b + c + M(u, z, v, t) \leq qM(u,z,v,t)$

Implies $(b+c) \leq (q - a - 1)M(u,z,v,t)$

i.e. $M(u, z, v, t) \geq \left(\frac{b+c}{q-a-1}\right) > 1$

This gives $u = z$. Hence u is a unique common fixed point of A, B, S and T. This completes the proof of the theorem.

Theorem 3.2. Let $(X, M, *)$ be a complete fuzzy-3 metric space and $t*t \geq t$ for all $t \in [0,1]$ and let A, B, S and T be self maps of X such that

(3.2.1) $A(X) \cup B(X) \subseteq S(X) \cap T(X)$;

(3.2.2) Pairs (A, T) and (B, S) are semi compatible and reciprocal continuous;

(3.2.3) $aM(Tx, Sy, v, w, t) + bM(Tx, Ax, v, w, t) + cM(Sy, By, v, w, t)$
$+ \max\{M(Ax, Sy, v, w, t), M(By, Tx, v, w, t)\} \leq qM(Ax,By,v,w,t)$

for all $x, y \in X$, where $a,b,c \geq 0$, $q > 0$ with $q < a+b+c+1$.

Then A, B,S and T have a unique common fixed point.

Proof: Let $x_o \in X$ be any arbitrary point.

Since $A(X) \subseteq S(X)$, there is a point $x_1 \in X$ such that $Ax_o = Sx_1$.

Again since $B(X) \subseteq T(X)$ for this x_1 there is a point $x_2 \in X$ such that $Bx_1 = Tx_2$.

Inductively, we construct a sequence $\{y_{2n}\}$ in X such that

$y_{2n} = Ax_{2n} = Sx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Tx_{2n+2}$, for all $n = 0, 1, 2,\ldots$

Let $M_{2n} = M(y_{2n}, y_{2n+1}, v, w, t) < 1$ for all n.
Putting $x = x_{2n}$, $y = x_{2n+1}$ in equation (3.2.3), we get

$$aM(Tx_{2n},Sx_{2n+1},v,w,t) + bM(Tx_{2n},Ax_{2n},v,w,t) + cM(Sx_{2n+1},Bx_{2n+1},v,w,t) + \max\{M(Ax_{2n},Sx_{2n+1},v,w,t),M(Bx_{2n+1},Tx_{2n},v,w,t)\} \leq qM(Ax_{2n},Bx_{2n+1},v,w,t)$$

$$aM(y_{2n-1},y_{2n},v,w,t) + bM(y_{2n-1},y_{2n},v,w,t) + cM(y_{2n},y_{2n+1},v,w,t) + \max\{M(y_{2n},y_{2n},v,w,t),M(y_{2n+1},y_{2n-1},v,w,t)\} \leq q M(y_{2n},y_{2n+1},v,w,t)$$

Implies $(a + b) M_{2n-1} + cM_{2n} + 1 \leq qM_{2n}$

$$(q-c) M_{2n} \geq (a+b)M_{2n-1}+1$$

$$(q-c) M_{2n} > (a+b) M_{2n-1}$$

$$\text{i.e. } M_{2n} > pM_{2n-1} > M_{2n-1}, \text{ where } p = \frac{(a+b)}{(q-c)} > 1. \qquad (3.2.1.1)$$

Thus $\{M_{2n}, n \geq 0\}$ is an increasing sequence of positive real numbers in [0, 1] and therefore tends to a limit $L \leq 1$.
We claim that L = 1, suppose if not i.e. L < 1.
If L < 1, taking limit as $n \to \infty$ by equation (3.2.1.1), we get L < L, which is a contradiction. Therefore L = 1.
Now consider, for any positive integer p,

$$M(y_{2n},y_{2n+p},v,w,t) \geq M(y_{2n},y_{2n+1},v,w,\tfrac{t}{p}) * \ldots * M(y_{2n+p-1},y_{2n+p},v,w,\tfrac{t}{p})$$

Implying $\lim_{n\to\infty} M(y_{2n}, y_{2n+p}, v, w, t) \geq 1 * \ldots * 1 = 1$
Thus $\{y_{2n}\}$ is a Cauchy sequence in X. Since X is complete, there is a point $u \in X$ such that y_{2n} converges to $u \in X$ and subsequently, the sequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n+1}\}$ and $\{Tx_{2n+2}\}$ and its subsequences also converges to $u \in X$.
Now we show that u is a common fixed point of A, B, S and T.

Step 1. Since the pair (A, T) is reciprocal continuous, we get

$$\lim_{n\to\infty} TAx_n \to Tu, \text{ also } \lim_{n\to\infty} TTx_n \to Tu.$$

And since (A, T) is semi compatible, we have

$$\lim_{n\to\infty} ATx_n \to Tu.$$

Now we put $x = Tx_n$, $y = x_n$ in equation (3.2.3), we get

$$aM(TTx_n,Sx_n,v,w,t) + bM(TTx_n,ATx_n,v,w,t) + cM(Sx_n,Bx_n,v,w,t) + \max\{M(ATx_n,Sx_n,v,w,t),M(Bx_n,TTx_n,v,w,t)\} \leq qM(ATx_n,Bx_n,v,w,t)$$

Taking limit as $n \to \infty$, we get

$$aM(Tu,u,v,w,t) + bM(u,Tu,v,w,t) + cM(u,u,v,w,t) + \max\{M(Tu,u,v,w,t),M(u,Tu,v,w,t)\} \leq qM(Tu,u,v,w,t)$$

$$aM(Tu,u,v,w,t) + bM(u,Tu,v,w,t) + c + M(Tu,u,v,w,t) \leq qM(Tu,u,v,w,t)$$

$$c \leq (q - a - b - 1)M(Tu,u,v,w,t)$$

i.e $\quad M(Tu,u,v,w,t) \geq \left(\frac{c}{q-a-b-1}\right) > 1.$

This gives $Tu = u$.

Step 2. Put $x = u$ and $y = x_n$ in equation (3.2.3), we get

$$aM(Tu,Sx_n,v,w,t) + bM(Tu,Au,v,w,t) + cM(Sx_n,Bx_n,v,w,t) + \max\{M(Au,Sx_n,v,w,t),M(Bx_n,Tu,v,w,t)\} \leq qM(Au,Bx_n,v,w,t)$$

Taking limit as $n \to \infty$, we get

$$aM(u,u,v,w,t) + bM(u,Au,v,w,t) + cM(u,u,v,w,t)$$

$+ \max\{M(Au,u,v,w,t), M(u,u,v,w,t)\} \leq qM(Au,u,v,w,t)$

$a + bM(u,Au,v,w,t) + c + \max\{M(Au,u,v,w,t), 1\} \leq qM(Au,u,v,w,t)$

Implies $(a+c+1) \leq (q - b)M(Au, u, v, w, t)$

i.e. $M(Au,u,v,w,t) \geq (\frac{a+c+1}{q-b}) > 1.$

This gives $Au = u$. Hence $Au = u = Tu$.

Step 3. Since the pair (B, S) is reciprocal continuous, we have

$\lim_{n\to\infty} SBx_n \to Su$, also $\lim_{n\to\infty} SSx_n \to Su$.

And since (B, S) is semi compatible, we have

$\lim_{n\to\infty} BSx_n \to Su$.

Let $x = x_n$, $y = Sx_n$ in equation (3.2.3), we get

$$aM(Tx_n,SSx_n,v,w,t) + bM(Tx_n,Ax_n,v,w,t) + cM(SSx_n,BSx_n,v,w,t) + \max\{M(Ax_n,SSx_n,v,w,t), M(BSx_n,Tx_n,v,w,t)\} \leq qM(Ax_n,BSx_n,v,w,t)$$

Taking limit as $n\to\infty$, we get

$$aM(u,Su,v,w,t) + bM(u,u,v,w,t) + cM(Su,Su,v,w,t) + \max\{M(u,Su,v,w,t), M(Su,u,v,w,t)\} \leq qM(u,Su,v,w,t)$$

$aM(u, Su, v, w, t) + b + c + M(u,Su,v,w,t) \leq qM(u,Su,v,w,t)$

Implies $(b + c) \leq (q - a - 1)M(u, Su, v, w, t)$

Implies $M(u, Su, v, w, t) \geq (\frac{b+c}{q-a-1}) > 1$

This gives $Su = u$. Hence $Au = Su = u = Tu$.

Step 4. Put $x = x_n$, $y = u$ in equation (3.2.3), we get

$$aM(Tx_n, Su, v, w, t) + bM(Tx_n, Ax_n, v, w, t) + cM(Su, Bu, v, w, t) + \max\{M(Ax_n, Su, v, w, t), M(Bu, Tx_n, v, w, t)\} \leq qM(Ax_n, Bu, v, w, t)$$

Taking limit as $n\to\infty$, we get

$$a M(u, u, v, w, t) + b M(u, u, v, w, t) + c M(u, Bu, v, w, t) + \max\{M(u, u, v, w, t), M(Bu, u, v, w, t)\} \leq qM(u,Bu,v,w,t)$$

$a + b + c M (u, Bu, v, w, t) + 1 \leq q M(u,Bu,v,w,t)$

Implies $(a + b +1) \leq (q - c)M(u, Bu, v, w, t)$

i.e. $M (u, Bu, v, w, t) \geq (\frac{a+b+1}{q-c}) > 1.$

This gives $Bu = u$.

Hence $u = Au = Su = Bu = Tu$ is a common fixed point of A, B, S and T.

Uniqueness. Let $z \neq u$ be another common fixed point of A, B, S and T, then $Az = Bz = Sz = Tz = z$.

Put $x = u$ and $y = z$ in equation (3.2.3), we get

$$aM(Tu,Sz,v,w,t) + bM(Tu,Au,v,w,t) + cM(Sz,Bz,v,w,t) + \max\{M(Au,Sz,v,w,t), M(Bz,Tu,v,w,t)\} \leq qM(Au,Bz,v,w,t)$$

$$aM(u,z,v,w,t) + bM(u,u,v,w,t) + cM(z,z,v,w,t) + \max\{M(u,z,v,w,t), M(z,u,v,w,t)\} \leq qM(u,z,v,w,t)$$

$aM(u,z,v,w,t) + b + c + M(u,z,v,w,t) \leq qM(u,z,v,w,t)$

Implies $(b+c) \leq (q - a - 1)M(u, z, v, w, t)$

i.e. $M(u,z,v,w,t) \geq (\frac{b+c}{q-a-1}) > 1.$

This gives $u = z$.

Hence, u is a unique common fixed point of A,B, S and T. This completes the proof of the theorem.

4. Conclusion

Using the notion of semi-compatible and reciprocal continuity of mappings, theorem 3.1 and theorem 3.2 are generalization of some results of Chauhan et al. [1] and Som [19] results of from fuzzy metric spaces to fuzzy 2, fuzzy 3- metric spaces.

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