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Fuzzy Relational Equations of k-Regular Block Fuzzy Matrices

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Abstract. In this paper, consistency of fuzzy relational equations involving k-regular block fuzzy matrices is discussed. Solutions of xM=b, where M is a block fuzzy matrix whose diagonal blocks are k-regular are determined in terms of k-generalized inverses of the diagonal block matrices.

Keywords: Fuzzy matrices, k-regular block fuzzy matrices, fuzzy relational equations, k-g-inverses.

AMS Mathematics Subject Classifications (2012): 15B

1.Introduction

Let F_n be the set of all n×n fuzzy matrices over the fuzzy algebra F=[0,1] under the operations (+,·) defined as a+b=max{a,b} and a·b=min{a,b} for all a, b∈F. A study of the theory of fuzzy matrices were made by Kim and Roush [2] analogous to that of Boolean matrices. A∈F_n is regular if there exists X such that AXA=A; X is called a generalized (g^-) inverse of A and is denoted as A^- [2]. A{1} denotes the set of all g-inverses of a regular matrix A. A fuzzy relational equation of the form $xA = b$ is consistent when A is a regular matrix and x=bA^- is a solution. Hence the set of all solutions Ω(A, b) of $xA = b$ is non-empty. It is shown that Ω(A, b) has a unique maximal element [4, p. 98], however the minimal element is not unique. Recently, a development of k-regular fuzzy matrices is made by Meenakshi and Jenita [5] analogous to that of generalized inverse of a complex matrix [1] and as a generalization of a regular fuzzy matrix [2, 3]. A matrix A∈F_n, is said to be right k-regular (left k-regular) if there exists a matrix X∈F_n such that $A^kXA = A^k$ ($AXA^k=A^k$) for some positive integer k. X is called a right k-g-inverse (left k-g-inverse) of A. In particular, for k=1 it reduces to g-inverses of a fuzzy matrix. However a right k-g-inverse and a left k-g-inverse of a fuzzy matrix are distinct (refer example (2.22) of [5]). A k-regular matrix as one that has a generalized inverse lays the foundation in the study on fuzzy relational equations. Here, solutions of xM=b, where M is a block fuzzy matrix whose diagonal blocks are k-regular are determined in terms of k-generalized inverses of the diagonal block matrices as a generalization of results available in the literature [6] on consistency of fuzzy relational equations involving block fuzzy matrices whose diagonal blocks are regular(refer p.193-200 [3]).

2. Notations and Preliminaries

For a matrix $A \in F_n$, let R(A), C(A), A^T and A^- be row space, column space, transpose and g-inverse of A respectively.

Definition 2.1[5] A matrix $A \in F_n$, is said to be right k-regular if there exists a matrix $X \in F_n$ such that $A^k XA = A^k$, for some positive integer k. X is called a right k-g-inverse of A.

Let $A_r\{1^k\} = \{X / A^k XA = A^k\}$.

Definition 2.2[5] A matrix $A \in F_n$, is said to be left k-regular if there exists a matrix $Y \in F_n$ such that $AYA^k = A^k$, for some positive integer k. Y is called a left k-g-inverse of A.

Let $A_\ell\{1^k\} = \{Y / AYA^k = A^k\}$.

In the sequel, we shall make use of the following results found in [5].

Lemma 2.1. For A, $B \in F_n$ and a positive integer k, the following hold.

(i) If A is right k-regular and $R(B) \subseteq R(A^k)$ then $B = BXA$ for each right k-g-inverse X of A.

(ii) If A is left k-regular and $C(B) \subseteq C(A^k)$ then $B = AYB$ for each left k-g-inverse Y of A.

Lemma 2.2. Let $A \in F_n$ and k be a positive integer, then $X \in A\{1_r^k\} \Leftrightarrow X^T \in A^T\{1_\ell^k\}$.

3. Fuzzy Relational Equations of k-Regular Block Fuzzy Matrices

In this section, we are concerned with fuzzy relational equations of the form xM=b, where $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a block fuzzy matrix whose diagonal blocks A and D are k-regular. We have determined conditions under which the consistency of xM=b implies those of $yA = c$ and $zD = d$ where $b = \begin{pmatrix} c & d \end{pmatrix}$.

Theorem 3.1.

Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with A and D are right k-regular, $R(C) \subseteq R(A^k)$ and $R(B) \subseteq R(D^k)$. If $xM = b$ is solvable then $yA = c$ and $zD = d$ are solvable, where $b = \begin{pmatrix} c & d \end{pmatrix}$.

Proof: Since $xM = b$ is solvable, let $x = \begin{pmatrix} \beta & \gamma \end{pmatrix}$ is a solution.

Then, $\begin{pmatrix} \beta & \gamma \end{pmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{pmatrix} c & d \end{pmatrix} \Rightarrow \begin{pmatrix} \beta A + \gamma C & \beta B + \gamma D \end{pmatrix} = \begin{pmatrix} c & d \end{pmatrix}$

Hence we get the equations: $\beta A+\gamma C=c$ and $\beta B+\gamma D=d$. (1)

By Lemma (2.1), A is right k-regular, $R(C)\subseteq R(A^k)\Rightarrow C=CA^-A$ for each right k-g-inverse A^- of A and D is right k-regular, $R(B)\subseteq R(D^k)\Rightarrow B=BD^-D$ for each right k-g-inverse D^- of D.

Substituting C and B in Equation (1), we get the equations:

$(\beta+\gamma CA^-)A=c$ and $(\beta BD^-+\gamma)D=d$.

Thus $yA=c$ and $zD=d$ are solvable and $(\beta+\gamma CA^-)=y$, $(\beta BD^-+\gamma)=z$ are the solutions.

Theorem 3.2.

Let $M=\begin{bmatrix} A & B \\ C & D\end{bmatrix}$ with A and D are right k-regular. If $yA^k=c$ and $zD^k=d$ are solvable, $c\geq dD^-C^k$ and $d\geq cA^-B^k$ then $xM^k=b$ is solvable when $M^k=\begin{bmatrix} A^k & B^k \\ C^k & D^k\end{bmatrix}$ and $b=\begin{pmatrix} c & d\end{pmatrix}$

Proof: Since $yA^k=c$ and $zD^k=d$ are solvable, let $y=cA^-$ and $z=dD^-$ are the Solutions $\Rightarrow cA^-A^k=c$ and $dD^-D^k=d$.

From the given conditions, $c\geq dD^-C^k$ and $d\geq cA^-B^k$ we get, $c=c+dD^-C^k$ and $d=d+cA^-B^k$.

$$\begin{aligned}\text{Now, } \begin{pmatrix} cA^- & dD^-\end{pmatrix}\begin{bmatrix} A^k & B^k \\ C^k & D^k\end{bmatrix} &= \begin{pmatrix} cA^-A^k+dD^-C^k & CA^-B^k+dD^-D^k\end{pmatrix}\\ &= \begin{pmatrix} c+dD^-C^k & CA^-B^k+d\end{pmatrix}\\ &= \begin{pmatrix} c & d\end{pmatrix}\\ &= b\end{aligned}$$

Thus $xM^k=b$ is solvable.

Hence the Theorem.

Remark 3.1. For *k=1*, the Theorem (3.1) and Theorem (3.2) reduces to the following:

Theorem 3.3[3].

Let $M=\begin{bmatrix} A & B \\ C & D\end{bmatrix}$ with A and D are regular with $R(C)\subseteq R(A)$ and $R(B)\subseteq R(D)$. The Schur compliment M/A=D-CA$^-$B and M/D =A-BD$^-$C are fuzzy matrix. Then $xM=b$ is solvable iff $yA=c$ and $zD=d$ are solvable, $c\geq dD^-C$ and $d\geq cA^-B$.

Theorem 3.4.

Let M = $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with A and D are left k-regular, $C(B) \subseteq C(A^k)$ and $C(C) \subseteq C(D^k)$. If $Mx = d$ is solvable then $Ay = b$ and $Dz = c$ are solvable, where $d = \begin{pmatrix} b \\ c \end{pmatrix}$.

Proof: This can be proved along the same lines as that Theorem (3.1) and by using Lemma (2.2) and hence omitted.

Theorem 3.5.

Let M = $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with A and D are left k-regular. If $A^k y = b$ and $D^k z = c$ are solvable, $c \geq C^k A^- b$ and $b \geq B^k D^- c$ then $M^k x = d$ is solvable where $M^k = \begin{bmatrix} A^k & B^k \\ C^k & D^k \end{bmatrix}$ and $d = \begin{pmatrix} b \\ c \end{pmatrix}$.

Proof: Since $A^k y = b$ and $D^k z = c$ are solvable, let $y = A^- b$ and $z = D^- c$ are the solutions $\Rightarrow$ $A^k A^- b = b$; $D^k D^- c = c$.

From the given conditions, $c \geq C^k A^- b$ and $b \geq B^k D^- c$ we get, $c = c + C^k A^- b$ and $b = b + B^k D^- c$.

Now,
$$\begin{bmatrix} A^k & B^k \\ C^k & D^k \end{bmatrix}\begin{pmatrix} A^- b \\ D^- c \end{pmatrix} = \begin{pmatrix} A^k A^- b + B^k D^- c \\ C^k A^- b + D^k D^- c \end{pmatrix}$$
$$= \begin{pmatrix} b + B^k D^- c \\ C^k A^- b + c \end{pmatrix}$$
$$= \begin{pmatrix} b \\ c \end{pmatrix}$$
$$= d$$

Thus $M^k x = d$ is solvable.
Hence the Theorem.

Remark 3.2. For k=1, the Theorem (3.4) and Theorem (3.5) reduces to the following.

Theorem 3.6[3]

Let M = $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with A and D are regular, M/A and M/D exists. $C(C) \subseteq C(D)$ and $C(B) \subseteq C(A)$. Then $Mx = d$ is solvable iff $Ay = b$ and $Dz = c$ are solvable, $c \geq CA^- b$ and $b \geq BD^- c$.

Remark 3.3. In particular, for B=0, the Theorem (3.1) and Theorem (3.4) reduces to the following.

Corollary 3.1.

For the matrix M=$\begin{bmatrix} A & O \\ C & D \end{bmatrix}$ with A and D are k-regular such that

(i) $R(C) \subseteq R(A^k)$. If $xM = b$ is solvable then $yA = c$ and $zD = d$ are solvable.

(ii) $C(C) \subseteq C(D^k)$. If $Mx = d$ is solvable then $Ay = b$ and $Dz = c$ are solvable.

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