Intern. J. Fuzzy Mathematical Archive
Vol. 1, 2013, 56-61
ISSN: 2320 –3242 (P), 2320 –3250 (online)
Published on 12 February 2013
www.researchmathsci.org

International Journal of
Fuzzy Mathematical Archive

Inverses of *k*-Regular Interval-Valued Fuzzy Matrices

P. Poongodi
Department of Mathematics, Karpagam University
Coimbatore -641021, India.
Email: poongodi_happi@rediffmail.com

Received 11 December 2012; accepted 7January 2013

Abstract. In this paper, we discuss various k-g inverses associated with a k-regular Interval-Valued Fuzzy Matrix. We obtain some characterization of the set of all k-g inverses.

Keywords: Fuzzy Matrices, k-regular interval-valued fuzzy matrices, k-g inverses

AMS Mathematics Subject Classification (2010): 15B15, 15A09

1. Introduction

We deal with Interval-Valued Fuzzy Matrices (IVFM) that is, matrices whose entries are intervals and all the intervals are subintervals of the interval [0,1]. Recently the concept of IVFM a generalization of fuzzy matrix was introduced and developed by Shyamal and Pal [5], by extending the max.min operations on fuzzy algebra F= [0,1], for elements a,b $\in$ F, a+b = max{a,b} and a.b = min {a,b}. In [2], Meenakshi and Kaliraja have represented an IVFM as an interval matrix of its lower and upper limit fuzzy matrices. In [3], Meenakshi and Jenita have introduced the concept of k –regular fuzzy matrix and discussed about inverses associated with a k –regular fuzzy matrix as a generalization of results on regular fuzzy matrix developed in [1]. A matrix A$\in$Fn, the set of all nxn fuzzy matrices is said to be right (left) k-regular if there exists X(Y) $\in$ Fn, such that $A^kXA = A^k$ ($AYA^k=A^k$), X(Y) is called a right (left) k-g inverse of A, where k is a positive integer. Recently we have extended the concept of k-regularity of fuzzy matrices to IVFM and determined the structure of k-regular IVFM in [4].

In this paper, we discuss various k-g inverses of k-regular interval-valued fuzzy matrices. In section 2, some basic definitions and results needed are given. In section 3, characterization of various k-g inverses of k-regular IVFM are determined.

2. Preliminaries

In this section, some basic definitions and results needed are given. Let $(IVFM)_n$ denotes the set of all nxn Interval-Valued Fuzzy Matrices.

Definition 2.1. An Interval-valued Fuzzy Matrix (IVFM) of order mxn is defined as $A=(a_{ij})_{mxn}$, where $a_{ij} = [a_{ijL}, a_{ijU}]$, the ij^{th} element of A is an interval representing the membership value. All the elements of an IVFM are intervals and all the intervals are the subintervals of the interval [0,1].

For $A =(a_{ij}) = ([a_{ijL}, a_{ijU}])$ and $B = (b_{ij}) = ([b_{ijL}, b_{ijU}])$ of order mxn their sum denoted as A+B defined as ,

$$A + B = (a_{ij} + b_{ij}) = ([(a_{ijL} + b_{ijL}), (a_{ijU} + b_{ijU})]) \quad ...(2.1)$$
$$=([\max\{a_{ijL}, b_{ijL}\}, \max\{a_{ijU}, b_{ijU}\}])$$

For $A = (a_{ij})_{mxn}$ and $B =(b_{ij})_{nxp}$ their product denoted as AB is defined as,

$$AB = (c_{ij}) = \left\{ \sum_{k=1}^{n} a_{ik}\, b_{kj} \right\} \qquad i=1, 2, \ldots, m \text{ and } j=1, 2, \ldots, p$$

$$= \left[\sum_{k=1}^{n} (a_{ikL} \tilde{} b_{kjL}),\ \sum_{k=1}^{n} (a_{ikU} \tilde{} b_{kjU}) \right] \qquad i=1, 2, \ldots, m \text{ and } j=1, 2, \ldots, p \ \ldots(2.2)$$

In particular if $a_{ijL} = a_{ijU}$ and $b_{ijL} = b_{ijU}$ then (2.2) reduces to the standard max. min composition of Fuzzy Matrices [1].

$A \le B$ if and only if $a_{ijL} \le b_{ijL}$ and $a_{ijU} \le b_{ijU}$

In [2], representation of an IVFM as an interval matrix of its lower and upper limit fuzzy matrices is introduced and discussed the regularity of an IVFM in terms of its lower and upper limit fuzzy matrices.

Definition 2.2. For a pair of Fuzzy Matrices $E = (e_{ij})$ and $F=(f_{ij})$ in $F_{m,n}$ such that $E \le F$, it can be defined that the interval matrix is denoted as [E, F], whose ij^{th} entry is the interval with lower limit e_{ij} and upper limit f_{ij}, that is $([e_{ij},f_{ij}])$. In particular for E = F, IVFM [E,E] reduces to the fuzzy matrix $E \in F_{m,n}$.

For $A = (a_{ij}) = ([a_{ijL}, a_{ijU}]) \in (IVFM)_{mn}$, let us define $A_L = (a_{ijL})$ and $A_U = (a_{ijU})$. Clearly A_L and A_U belongs to $F_{m,n}$ such that $A_L \le A_U$ and from Definition (2.2) A can be written as $A = [A_L, A_U]$ …(2.3).

For $A \in (IVFM)_{mn}$, A^T, R (A), C (A) denotes the transpose of A, row space of A, column space of A respectively.

Definition 2.3. A matrix $A \in (IVFM)_n$ is said to be right k – regular if there exist a matrix $X \in (IVFM)_n$, such that $A^k X A = A^k$, for some positive integer k. X is called a right k – g inverse of A. Let $A\{1_r^k\} = \{ X / A^k X A = A^k \}$.

Definition 2.4. A matrix $A \in (IVFM)_n$ is said to be left k – regular if there exist a matrix $Y \in (IVFM)_n$, such that $A Y A^k = A^k$, for some positive integer k. Y is called a left k – g inverse of A. Let $A \{1_\ell^k\} = \{ Y / A Y A^k = A^k \}$.

In particular for a fuzzy matrix A, $a_{ijL} = a_{ijU}$ then Definition (2.3) and Definition (2.4) reduce to right left k-regular fuzzy matrices respectively found in [3].

In the sequel we shall make use of the following results on IVFM found in [2] and [4].

Lemma 2.5. For $A = [A_L, A_U] \in (IVFM)_{mn}$ and $B = [B_L, B_U] \in (IVFM)_{np}$, the following hold.

(i) $A^T = [A_L^T, A_U^T]$
(ii) $AB = [A_LB_L, A_UB_U]$.

Lemma 2.6. For $A, B \in (IVFM)_{mn}$
(i) $R(B) \subseteq R(A) \Leftrightarrow B = XA$ for some $X \in (IVFM)_m$
(ii) $C(B) \subseteq C(A) \Leftrightarrow B = AY$ for some $Y \in (IVFM)_n$

Theorem 2.7. For $A, B \in (IVFM)_n$, with $R(A) = R(B)$ and $R(A^k) = R(B^k)$ then A is right k – regular IVFM $\Leftrightarrow$ B is right k – regular IVFM.

Theorem 2.8. For $A, B \in (IVFM)_n$, with $C(A) = C(B)$ and $C(A^k) = C(B^k)$ then A is left k – regular IVFM $\Leftrightarrow$ B is left k – regular IVFM.

3. Inverses of k-Regular Interval-Valued Fuzzy Matrices

In this section, we shall introduce the concept of k-g inverses associated with a k-regular IVFM as an extension of k-g inverses of a k-regular fuzzy matrix [3] and as a generalization of generalized inverses of a regular IVFM [2].

Definition 3.1. A matrix $A \in (IVFM)_n$, is said to have a $\{3^k\}$ inverse if there exists a matrix $X \in (IVFM)_n$ such that $(A^kX)^T = A^kX$, for some positive integer k. X is called the $\{3^k\}$ inverse of A. Let $A\{3^k\} = \{ X / (A^kX)^T = A^kX\}$.

Definition 3.2. A matrix $A \in (IVFM)_n$, is said to have a $\{4^k\}$ inverse if there exists a matrix $X \in (IVFM)_n$ such that $(XA^k)^T = XA^k$, for some positive integer k. X is called the $\{4^k\}$ inverse of A. Let $A\{4^k\} = \{X / (XA^k)^T = XA^k\}$.

Remark 3.3. In particular for k=1, Definitions (3.1) and (3.2) reduces to set of {3} and {4} inverses respectively of a IVFM and in the case $A_L = A_U$, Definitions (3.1) and (3.2) reduces to set of $\{3^k\}$ and $\{4^k\}$ inverses respectively of fuzzy matrices.

Theorem 3.4. Let $A = [A_L, A_U] \in (IVFM)_n$. Then A has a $\{3^k\}$ inverse $\Leftrightarrow$ A_L and $A_U \in F_n$ have $\{3^k\}$ inverses.

Proof. Let $A=[A_L, A_U] \in (IVFM)_n$.
Since A has a $\{3^k\}$ inverse, then there exist $X \in (IVFM)_n$ such that, $(A^kX)^T = A^kX$
Let $X = [X_L, X_U]$, then by Lemma(2.5)(ii),

$$\begin{aligned}(A^kX)^T = A^kX &\Leftrightarrow [A_L^kX_L, A_U^kX_U]^T = [A_L^kX_L, A_U^kX_U] \\ &\Leftrightarrow [(A_L^kX_L)^T, (A_U^kX_U)^T] = [A_L^kX_L, A_U^kX_U] \\ &\Leftrightarrow (A_L^kX_L)^T = A_L^kX_L \text{ and } (A_U^kX_U)^T = A_U^kX_U.\end{aligned}$$

Hence $A \in (IVFM)_n$ has a $\{3^k\}$inverse $\Leftrightarrow$ A_L and $A_U \in F_n$ have$\{3^k\}$inverses.

Theorem 3.5. Let $A = [A_L, A_U] \in (IVFM)_n$. Then A has a $\{4^k\}$ inverse $\Leftrightarrow$ A_L and $A_U \in F_n$ have $\{4^k\}$ inverses.

Proof. Let $A = [A_L, A_U] \in (IVFM)_n$.
Since A has a $\{4^k\}$ inverse, then there exist $X \in (IVFM)_n$, such that $(XA^k)^T = XA^k$
Let $X = [X_L, X_U]$, then by Lemma(2.5)(ii),

$$(XA^k)^T = XA^k \Leftrightarrow [X_LA_L{}^k, X_UA_U{}^k]^T = [X_LA_L{}^k, X_UA_U{}^k]$$
$$\Leftrightarrow [(X_LA_L{}^k)^T, (X_UA_U{}^k)^T] = [X_LA_L{}^k, X_UA_U{}^k]$$
$$\Leftrightarrow (X_LA_L{}^k)^T = X_LA_L{}^k \text{ and } (X_UA_U{}^k)^T = X_UA_U{}^k.$$

Hence $A \in (IVFM)_n$ has a $\{4^k\}$inverse $\Leftrightarrow A_L$ and $A_U \in Fn$ have $\{4^k\}$inverse.

Theorem 3.6. Let $A \in (IVFM)_n$ and k be a positive integer,
(i) if $X \in A\{1_r{}^k\}$ with $R(X) = R(A^kX)$ then, $A \in X\{1_\ell{}^k\}$
(ii) if $X \in A\{1_\ell{}^k\}$with $C(X) = C(XA^k)$ then, $A \in X\{1_r{}^k\}$
Proof.
(i) Since $X \in A\{1_r{}^k\}$ by Definition (2.3), $A^kXA = A^k$.
Since $R(X) = R(A^kX)$, by Lemma (2.6), $X = YA^kX$, for some $Y \in (IVFM)_n$,

$$XAX^k = YA^kX\,AX^k$$
$$= YA^kX^k$$
$$= YA^kXX^{k-1} = XX^{k-1} = X^k$$

Hence $A \in X\{1_\ell{}^k\}$
(ii) Proof is similar to (i) and hence omitted.

Theorem 3.7. For $A \in (IVFM)_n$ and for any $G \in (IVFM)_n$, if $A^kX = A^kG$, where X is a $\{1_r{}^k, 3^k\}$ inverse of A then G is a $\{1_r{}^k, 3^k\}$ inverse of A.
Proof. Since X is a $\{1_r{}^k, 3^k\}$ inverse of A, by Definitions (2.3) and (3.1),

$$A^kXA = A^k \text{ and } (A^kX)^T = A^kX.$$

Post multiplying by A on both sides of $A^kX = A^kG$, we get $A^kGA = A^kXA = A^k$
Since $A^kX = A^kG \Rightarrow (A^kG)^T = (A^kX)^T. = A^kX = A^kG$.
Hence G is a $\{1_r{}^k, 3^k\}$ inverse of A.

Theorem 3.8. For $A \in (IVFM)_n$ and for any $G \in (IVFM)_n$, if $XA^k = GA^k$, where X is a $\{1_\ell{}^k, 4^k\}$ inverse of A then G is a $\{1_\ell{}^k, 4^k\}$ inverse of A.
Proof: Proof is similar to that of Theorem(3.7) and hence omitted.

Theorem 3.9. For $A \in (IVFM)_n$, X is a $\{1_r{}^k, 3^k\}$ inverse of A and G is a $\{1_\ell{}^k, 3\}$inverse of A then $A^kX = A^kG$.
Proof. Since X is a $\{1_r{}^k, 3^k\}$ inverse of A, by Definitions (2.3) and (3.1)

$$A^kXA = A^k \text{ and } (A^kX)^T = A^kX$$

Since G is a $\{1_\ell{}^k, 3\}$ inverse of A, by Definition (2.4) and Remark (3.3),

$$AGA^k = A^k \text{ and } (AG)^T = AG$$

$$A^kG = (A^kXA)G = (A^kX)(AG)$$
$$= (A^kX)^T(AG)^T$$
$$= X^T(A^T)^kG^TA^T$$
$$= X^T(AGA^k)^T$$
$$= X^T(A^k)^T$$
$$= (A^kX)^T = A^kX. \text{ Hence the theorem.}$$

Theorem 3.10. For $A \in (IVFM)_n$, X is a $\{1_\ell^{\,k}, 4^k\}$ inverse of A and G is a $\{1_r^{\,k}, 4\}$inverse of A then $XA^k = GA^k$.
Proof. This can be proved in the same manner as that of Theorem (3.9) and hence omitted.

In general, for an IVFM A, there is no relation between k-regularity of A, A^TA and AA^T. Here, the relation shall be discussed under certain conditions on their row spaces.

Theorem 3.11. For $A \in (IVFM)_n$, with $R(A) = R(A^TA)$ and $R(A^k) = R((A^TA)^k)$ then, A is right k-regular $\Leftrightarrow$ A^TA is right k-regular.
Proof. This follows from Theorem (2.7), by replacing B by A^TA.

Theorem 3.12. For $A \in (IVFM)_n$, with $C(A) = C(AA^T)$ and $C(A^k) = C((AA^T)^k)$ then, A is left k-regular $\Leftrightarrow$ AA^T is left k-regular.
Proof. This follows from Theorem(2.8), by replacing B by AA^T.

Theorem 3.13. For $A \in (IVFM)_n$, if A^TA is a right k-regular IVFM and $R(A^k) \subseteq R((A^TA)^k)$ then A has a $\{1_r^{\,k}, 3^k\}$ inverse. In particular for k=1, $Y = (A^TA)^-A^T$ is a {1, 3} inverse of A.
Proof. Since A^TA is right k-regular IVFM, By Definition (2.3),

$$(A^TA)^k(A^TA)^-(A^TA) = (A^TA)^k \text{ for some right k-g-inverse } (A^TA)^- \text{ of } A^TA.$$

Since $R(A^k) \subseteq R((A^TA)^k)$, by Lemma(2.6), $A^k = X(A^TA)^k$ for some $X \in (IVFM)_n$ and take $Y = (A^TA)^-A^T$.

$$\begin{aligned} A^kYA = A^k\ (YA) &= (X(A^TA)^k)((A^TA)^-A^TA) \\ &= X((A^TA)^k(A^TA)^-A^TA)) \\ &= X(A^TA)^k \\ &= A^k. \end{aligned}$$

Take $Z = (A^TA)^-(A^k)^T$.

$$\begin{aligned} A^kZ = (A^k)Z &= (X(A^TA)^k)((A^TA)^-(A^k)^T) \\ &= X(A^TA)^k(A^TA)^-(A^TA)^kX^T \\ &= X(A^TA)^k(A^TA)^-(A^TA)(A^TA)^{k-1}X^T \\ &= X(A^TA)^k(A^TA)^{k-1}X^T \\ &= X(A^TA)^{2k-1}X^T \\ &= (X(A^TA)^{2k-1}X^T)^T \\ &= (A^kZ)^T \end{aligned}$$

Hence A has a $\{1_r^{\,k}, 3^k\}$ inverse. In particular for k=1, $Y = (A^TA)^-A^T$ is a {1, 3} inverse of A.

Theorem 3.14. For $A \in (IVFM)_n$, if AA^T is a left k-regular IVFM and $C(A^k) \subseteq C((AA^T)^k)$ then A has a $\{1_\ell^{\,k}, 4^k\}$ inverse. In particular for k=1, $Z = A^T(A^TA)^-$ is a {1, 4} inverse of A.
Proof. Proof is similar to Theorem (3.13) and hence omitted.

Theorem 3.15. Let $A \in (IVFM)_n$, be a right k-regular IVFM and $R((A^TA)^k) \subseteq R(A^k)$ then, A^TA has a $\{3^k\}$ inverse.

Proof. Since A is right k-regular IVFM. By Definition (2.3),

$A^kXA = A^k$ for some right k-g-inverse $X \in (IVFM)_n$, of A.

Since $R((A^TA)^k) \subseteq R(A^k)$, by Lemma(2.6), $(A^TA)^k = ZA^k$ for some $Z \in (IVFM)_n$ and take $Y = XA$.

$$(A^TA)^k\, Y = (ZA^k)\,(XA)$$
$$= ZA^k\, XA = ZA^k = (A^TA)^k = ((A^TA)^k)^T = ((A^TA)^kY)^T$$

Hence A^TA has a (3^k) inverse.

Theorem 3.16. Let $A \in (IVFM)_n$, be a left k-regular IVFM and $C((AA^T)^k) \subseteq C(A^k)$ then, AA^T has a $\{4^k\}$ inverse.

Proof. This can be proved in the same manner as that of Theorem (3.15) and hence omitted.

REFERENCES

1. Kim, K.H. and Roush, F.N., Generalized fuzzy matrices, *Fuzzy Sets and Systems*, 4 (1980), 293 – 315.
2. Meenakshi, AR., and Kaliraja,M., Regular Interval-Valued Fuzzy Matrices, *Advances in Fuzzy Mathematics*, 5 (1) (2010), 7 -15.
3. Meenakshi, AR., and Jenita, P., Generalized Regular Fuzzy Matrices, *Iranian Journal of Fuzzy Systems*, 8 (2), (2011) 133 – 141.
4. Meenakshi, AR. and Poongodi,P., Generalized Regular Interval Valued Fuzzy Matrices, *International Journal of Fuzzy Mathematics and Systems*, 2 (1) (2012), 29 - 36.
5. Shyamal.A.K. and Pal. M., Interval-Valued Fuzzy Matrices, *Journal of Fuzzy Mathematics,* 14 (3) (2006) 582 – 592.