Intern. J. Fuzzy Mathematical Archive
Vol. 1, 2013, 41-48
ISSN: 2320-3242 (P), 2320-3250 (online)
Published on 9 February 2013
www.researchmathsci.org

International Journal of
Fuzzy Mathematical Archive

Second Order Fuzzy S-Hausdorff Spaces

A.Kalaichelvi
Department of Mathematics
Avinashilingam Institute for Home Science and Higher Education for Women
Coimbatore – 641 043
email : kalaichelviadu@yahoo.com

Received 6 December 2012; accepted 7January 2013

Abstract. In this paper, the definition of S-Hausdorff space introduced by Srivastava, R., Lal, S.N. and Srivastava, A.K. [6] is extended to second order fuzzy topological spaces in two different ways. It is shown that these two concepts are hereditary and productive. The relations between first order and second order concepts are studied. The behavior of these concepts with regard to $i(\hat{\delta})$, $i_{\varepsilon}(\hat{\delta})$, $i^{*}(\hat{\delta})$, $\omega_2(\tau)$, $(\omega_2(\tau))_{\varepsilon}$ and $\omega_2^{*}(\tau)$ are also analysed.

AMS Mathematics Subject Classification (2010): 94D05

Keywords: **second order fuzzy set, S-Hausdorff Spaces**

1. Introduction

A **fuzzy set** on a set X is a map defined on X with values in I, where I is the closed unit interval [0, 1]. Equivalently fuzzy sets which are named as first order fuzzy sets in this study deal with crispy defined membership functions or degrees of membership. It is doubtful whether, for instance, human beings have or can have a crisp image of membership functions in their minds. Zadeh [8] therefore suggested the notion of a fuzzy set whose membership function itself is a fuzzy set. This leads to the following definition of a second order fuzzy set or a fuzzy set of type 2. A **second order fuzzy set** on a nonempty set X is a map from X to I^I.

First order fuzzy sets are denoted by f, g, h, . . . and second order fuzzy sets are denoted by $\hat{f}$, $\hat{g}$, $\hat{h}$, ...

In this paper the terms 'fuzzy set' and 'first order fuzzy set' are used synonymously.

Whenever a fuzzy set is considered without mentioning the order, it always refers to a first order fuzzy set.

Similar terminology applies to all concepts related to first order fuzzy sets.

Fundamental definitions and properties of second order fuzzy sets and second order fuzzy topological spaces are introduced in [4]. Six important and interesting connections $¢_1$, $¢_2$, $¢_3$, $¢_4$, $¢_5$ and $¢_6$ between first order and second order fuzzy topological spaces are also discussed in [4].

The concept of fuzzy topological spaces was introduced and developed by Chang[1]and for other applications one may refer[2& 3].Connections between crisp topological spaces and second order fuzzy topological spaces are given in [5].

With every crisp topology τ on a nonempty sex X, three second order fuzzy topologies $\omega_2(\tau)$, $(\omega_2(\tau))_\varepsilon$ and $\omega_2{}^*(\tau)$ on X are associated. Also with every second order fuzzy topology $\hat{\delta}$ on a nonempty set X, three crisp topologies $i(\hat{\delta})$, $i_\varepsilon(\hat{\delta})$ and $i^*(\hat{\delta})$ on X are associated [5].

In this paper the definition of S-Hausdorff space introduced by Srivastava, R., Lal, S.N. and Srivastava, A.K. [6] is extended to second order fuzzy topological spaces in two different ways. It is shown that these two concepts are hereditary and productive. The relations between first order and second order concepts are studied. The behavior of these concepts with regard to $i(\hat{\delta})$, $i_\varepsilon(\hat{\delta})$, $i^*(\hat{\delta})$, $\omega_2(\tau)$, $(\omega_2(\tau))_\varepsilon$ and $\omega_2{}^*(\tau)$ are also analysed.

2. Fundamental Definitions And Notations

Definition 2.1. [4] A second order Chang fuzzy topology $\hat{\delta}$ on a nonempty set X is a collection of second order fuzzy sets on X satisfying the following conditions :

(i) $\hat{0}, \hat{1} \in \hat{\delta}$ where, for any $x \in X$

$\hat{0}(x)$ = the zero function **0** and I $\hat{1}(x)$ = the constant function **1** on I

(ii) $\hat{f}_\lambda \in \hat{\delta}$ for each $\lambda \in \Lambda$ implies $(\bigvee_{\lambda\in\Lambda} \hat{f}_\lambda) \in \hat{\delta}$

(iii) $\hat{f}_i \in \hat{\delta}$ for i = 1, 2, . . . , m implies $(\bigwedge_{i=1}^{m} \hat{f}_i) \in \hat{\delta}$

The pair (X, $\hat{\delta}$) is called a **second order Chang fuzzy topological space**.

A **second order Lowen fuzzy topology** $\hat{\delta}$ on X is defined by replacing axiom (i) in the above definition by axiom (i)′.

(i)′ All constant function $\hat{\alpha} \in \hat{\delta}$ where, for any $x \in X$, $\hat{\alpha}(x)$ = the constant function α on I. The pair (X, $\hat{\delta}$) is called a **second order Lowen fuzzy topological space**. The elements of $\hat{\delta}$ are called **second order fuzzy open sets.**

Definition 2.2. [4] Every first order fuzzy topology $\delta = \{f_\lambda / \lambda \in \Lambda\}$ on a non-empty set X defines a second order fuzzy topology $\hat{\delta} = \{\hat{f}_\lambda / f_\lambda \in \delta\}$ on X where $\hat{f}_\lambda(x)(\alpha)$

$= f_\lambda(x)$, for every $x \in X$ and for every $\alpha \in I$. The correspondence $\delta \to \hat{\delta}$ is denoted as $\not{c}_1$.

Let $\hat{\delta} = \{\hat{f}_\lambda / \lambda \in \Lambda\}$ be a second order fuzzy topology (Lowen) on X.

Fix $\alpha \in I$. The collection δ_α = distinct elements of the collection $\{(\hat{f}_\lambda)_\alpha / \hat{f}_\lambda \in \hat{\delta}\}$ defines a first order fuzzy topology on X where $(f_\lambda)_\alpha(x) = \hat{f}_\lambda(x)(\alpha)$, for every $x \in X$. $\not{c}_3$ denotes the correspondence $\hat{\delta} \to \delta_\alpha$.

The collection $S^\Diamond = \{(f_\lambda)^\Diamond / \hat{f}_\lambda \in \hat{\delta}\}$ is a subbase for a first order fuzzy topology $\delta^\Diamond$ on X where $(f_\lambda)^\Diamond(x) = \bigvee_{\alpha \in I} \hat{f}_\lambda(x)(\alpha)$, for every $x \in X$. $\not{c}_4$ denotes the correspondence $\hat{\delta} \to \delta^\Diamond$.

Given a second order fuzzy topology on a nonempty set X, the association $\not{c}_6$, gives a way of getting another second order fuzzy topology on the same set X. That is, given a second order fuzzy topology $\hat{\delta} = \{\hat{f}_\lambda / \lambda \in \Lambda\}$ on a nonempty set X, the collection $\hat{\delta}_c = \{(\hat{f}_\lambda)_c / \hat{f}_\lambda \in \hat{\delta}\}$ is also a second order fuzzy topology on X.

Definition 2.3. [5] Let $(X, \hat{\delta})$ be a second order fuzzy topological space. Define

(1) $i_\varepsilon(\hat{\delta})$ to be the topology generated by the collection $\{(\{(A_{\hat{f}})_\varepsilon / \hat{f} \in \hat{\delta}\}$ where for $\varepsilon \in (0, 1)$, $(A_{\hat{f}})_\varepsilon = \{x \in X / \hat{f}(x)^{-1}(\varepsilon, 1] = I\}$.

(2) $i^*(\hat{\delta})$ to be the topology generated by the collection $\{A_{\hat{f}} / \hat{f} \in \hat{\delta}\}$ where $(A_{\hat{f}}) = \{x \in X / \hat{f}(x)^{-1}(\varepsilon, 1] = I$ for some $\varepsilon \in (0, 1)\}$.

(3) $i(\hat{\delta})$ to be the topology having the collection $\{(A_{\hat{f}})_\varepsilon / \hat{f} \in \hat{\delta}, \varepsilon \in (0,1)\}$ as a sub basis.

Definition 2.4. [5] Let (X, τ) be a topological space. Define

(1) $(\omega_2(\tau))_\varepsilon$ to be the second order fuzzy topology generated by $\hat{K}_\varepsilon$ where for $\varepsilon \in (0, 1)$, $\hat{K}_\varepsilon = \{\hat{f} \in (I^I)^X / (A_{\hat{f}})_\varepsilon \in \tau\}$.

(2) $\omega_2(\tau)$ to be the second order fuzzy topology generated by $\hat{K}$ where $\hat{K} = \{\hat{f} \in (I^I)^X / (A_{\hat{f}})_\varepsilon \in \tau$, for every $\varepsilon \in (0, 1)\}$.

(3) $\omega_2^*(\tau)$ to be the second order fuzzy topology generated by $\hat{K}_*$ where $\hat{K}_* = \{\hat{f} \in (I^I)^X / A_{\hat{f}} \in \tau\}$.

Definition 2.5. [6] A fuzzy topological space (X, δ) is said to be **fuzzy S-Hausdorff** if for any pair of distinct fuzzy points x_t, y_s in X, there exist f, g ∈ δ such that $x_t \in f$, $y_s \in g$ and f Λ g = **0**.

3. Second Order Fuzzy S-Hausdorff Spaces

Definition 3.1. A second order fuzzy topological space (X, $\hat{\delta}$) is said to be second order fuzzy **S-Hausdorff of type 1, denoted by (S – H)₁,** iff for any pair of distinct second order fuzzy points $\hat{x}_r, \hat{y}_s$ in X, there exist $\hat{f}$, $\hat{g} \in \hat{\delta}$ such that $\hat{x}_r \in \hat{f}$, $\hat{y}_s \in \hat{g}$ and $\hat{f}\ \Lambda_1\ \hat{g} = \hat{0}$, where $\hat{f}\ \Lambda_1\ \hat{g} = \hat{0}$ means given x ∈ X, either $\hat{f}(x) = 0$ or $\hat{g}(x) = 0$.

Definition 3.2. A second order fuzzy topological space (X, $\hat{\delta}$) is said to be **second order fuzzy S-Hausdorff of type 2, denoted by $(S – H)_2$,** is defined by replacing the condition $\hat{f}\ \Lambda_1\ \hat{g} = \hat{0}$ in the above definition by $\hat{f}\ \Lambda_2\ \hat{g} = \hat{0}$, where $\hat{f}\ \Lambda_2\ \hat{g} = \hat{0}$ means given x ∈ X and α ∈ I either $\hat{f}(x)(\alpha) = 0$ or $\hat{g}(x)(\alpha) = 0$.

Note:

(i) (X, $\hat{\delta}$) is $(S – H)_1 \Rightarrow$ (X, $\hat{\delta}$) is $(S – H)_2$. But the converse need not be true.

(ii) Subspace of a $(S – H)_1$ space is $(S – H)_1$.

Definition 3.3. Let $(X, \hat{\delta}_1)$, $(Y, \hat{\delta}_2)$ be two second order fuzyz topological spaces. If $\hat{f}_1 \in \hat{\delta}_1$ and $\hat{f}_2 \in \hat{\delta}_2$ then the **star product**

$\hat{f}_1 * \hat{f}_2$ on X x Y is defined as follows :

$$(\hat{f}_1 * \hat{f}_2)(x, y)(\alpha) = \hat{f}_1(x)(\alpha)\ \Lambda\ \hat{f}_2(y)(\alpha),$$

for every (x, y) ∈ X x Y and for every α ∈ I.

The **product topology** $\hat{\delta}_1 \times \hat{\delta}_2$ on X x Y is the second order fuzzy topology having the collection $\{\hat{f}_1 * \hat{f}_2 / \hat{f}_1 \in \hat{\delta}_1, \hat{f}_2 \in \hat{\delta}_2\}$ as a basis.

Theorem 3.4. Product of two $(S – H)_1$ spaces is $(S – H)_1$.

Proof. Let (X, $\hat{\delta}_1$) and (Y, $\hat{\delta}_2$) be two $(S – H)_1$ spaces. Consider two distinct fuzzy points $\hat{z}_r$, $\hat{u}_s$ in X x Y where z = (x, y) and u = (p, q).

∴ Either x ≠ p or y ≠ q.

Assumex ≠ p.

Then $\hat{x}_r \neq \hat{p}_s$.

∴ There exist $\hat{f}$, $\hat{g} \in \hat{\delta}_1$ such that $\hat{x}_r \in \hat{f}$, $\hat{p}_s \in \hat{g}$ and $\hat{f}\ \Lambda_1\ \hat{g} = \hat{0}$.

$\hat{f}, \hat{g} \in \hat{\delta}_1 \Rightarrow \hat{f} * \hat{1}, \hat{g} * \hat{1} \in \hat{\delta}_1 \times \hat{\delta}_2$.

For $\alpha \in I$, consider

$(\hat{f} * \hat{1})(x, y)(\alpha) = \hat{f}(x)(\alpha) \wedge \hat{1}(y)(\alpha) \geq r, \forall \alpha \in I$ ($\because \hat{x}_r \in \hat{f}$)

$\therefore \quad \hat{z}_r \in \hat{f} * \hat{1}$

Similarly $\quad \hat{u}_s \in \hat{g} * \hat{1}$

Also $\hat{f} \wedge_1 \hat{g} = \hat{0} \Rightarrow (\hat{f} * \hat{1}) \wedge_1 (\hat{g} * \hat{1}) = \hat{0}$.

The proof is similar if $y \neq q$.

Hence $(X \times Y, \hat{\delta}_1 \times \hat{\delta}_2)$ is $(S-H)_1$.

Definition 3.5. Let $\{(X_\lambda, \hat{\delta}_\lambda) / \lambda \in \Lambda\}$ be a family of second order fuzzy topological spaces and $X = \Pi_{\lambda \in \Lambda} X_\lambda$. The product topology on X is the one with basic second order fuzzy open sets of the form $\Pi_{\lambda \in \Lambda} \hat{f}_\lambda$ where $\hat{f}_\lambda \in \hat{\delta}_\lambda$ and $\hat{f}_\lambda = \hat{1}$ except for finitely many λ's.

Here $(\Pi_{\lambda \in \Lambda} \hat{f}_\lambda)((x_\lambda)_{\lambda\in\Lambda})(\alpha) = \wedge_{\lambda\in\Lambda} \hat{f}_\lambda(x_\lambda)(\alpha)$, for every $(x_\lambda)_{\lambda\in\Lambda} \in \Pi_{\lambda \in \Lambda} X_\lambda$ and for every $\alpha \in I$.

Theorem 3.6. Arbitrary product of $(S-H)_1$ spaces is $(S-H)_1$.

Theorem 3.7. (X, δ) is fuzzy S-Hausdorff iff $(X, \hat{\delta})$ is $(S-H)_1$ where $(X, \hat{\delta})$ is got from (X, δ) through the association $¢_1$.

Proof. Given δ, $\hat{\delta} = \{\hat{f} / f \in \delta\}$ where $\hat{f}(x)(\alpha) = f(x)$, for every $x \in X$ and $\alpha \in I$.

Assume (X,δ) is fuzzy S-Hausdorff. Consider two distinct second order fuzzy points $\hat{x}_r, \hat{y}_s$ in X.

Then the first order fuzzy points x_r, y_s in X are distinct.

$\therefore$ there exists $f, g \in \delta$ such that $x_r \in f$, $y_s \in g$ and $f \wedge g = \mathbf{0}$. $\therefore \hat{f}, \hat{g} \in \hat{\delta}$ and

$x_r \in f \Rightarrow f(x) \geq r \Rightarrow \hat{f}(x)(\alpha) \geq r, \forall \alpha \in I \Rightarrow \hat{x}_r \in \hat{f}$

Similarly $\hat{y}_s \in \hat{g}$.

Also $f \wedge g = \mathbf{0} \Rightarrow \hat{f} \wedge_1 \hat{g} = \hat{0}$.

Hence $(X, \hat{\delta})$ is $(S-H)_1$.

Proof of the converse is similar.

Theorem 3.8. If $(X, \hat{\delta})$ is $(S-H)_1$ then for every $\alpha \in I$, (X, δ_α) is fuzzy S-Hausdorff where (X, δ_α) is got from $(X, \hat{\delta})$ through the association $¢_3$.

Proof. Given $\hat{\delta}$, $\delta_\alpha = \{f_\alpha \,/\, \hat{f} \in \hat{\delta}\}$ where $f_\alpha(x) = \hat{f}(x)(\alpha)$, $\forall\, x \in X$.

Consider two distinct first order fuzzy points x_r, y_s in X. Then $\hat{x}_r$, $\hat{y}_s$ are distinct.

$\therefore$ there exist $\hat{f}$, $\hat{g} \in \hat{\delta}$ such that

$\hat{x}_r \in \hat{f}$, $\hat{y}_s \in \hat{g}$ and $\hat{f}\ \Lambda_1\ \hat{g} = \hat{0}$.

$\hat{x}_r \in \hat{f} \Rightarrow \hat{f}(x)(\alpha) \geq r$, $\forall\, \alpha \in I \Rightarrow f_\alpha(x) \geq r \Rightarrow x_r \in f_\alpha$

Similarly $y_s \in g_\alpha$.

Also $\hat{f}\ \Lambda_1\ \hat{g} = \hat{0} \Rightarrow f_\alpha \wedge g_\alpha = \mathbf{0}$

$\therefore$ (X, δ_α) is fuzzy S-Hausdorff.

Theorem 3.9. If $(X, \hat{\delta})$ is $(S - H)_1$, then $(X, \delta^\lozenge)$ is fuzzy S-Hausdorff where $(X, \delta^\lozenge)$ is got from $(X, \hat{\delta})$ through the association $¢_4$.

Proof. Given $\hat{\delta}$, a subbase for $\delta^\lozenge$ is $S^\lozenge = \{f^\lozenge \,/\, \hat{f} \in \hat{\delta}\}$ where $f^\lozenge(x) = \bigvee_{\alpha \in I} \hat{f}_\lambda(x)(\alpha)$, $\forall\, x \in X$.

Consider two distinct first order fuzzy points, x_r, y_s in X. Then $\hat{x}_r$, $\hat{y}_s$ are distinct.

$\therefore$ There exist $\hat{f}$, $\hat{g} \in \hat{\delta}$ such that $\hat{x}_r \in \hat{f}$, $\hat{y}_s \in \hat{g}$ and $\hat{f}\ \Lambda_1\ \hat{g} = \hat{0}$.

$\hat{x}_r \in \hat{f} \Rightarrow \hat{f}(x)(\alpha) \geq r$, $\forall\, \alpha \in I \Rightarrow \bigvee_{\alpha \in I} \hat{f}(x)(\alpha) \geq r \Rightarrow f^\lozenge(x) \geq r \Rightarrow x_r \in f^\lozenge$.

Similarly $y_s \in g^\lozenge$

Also $\hat{f}\ \Lambda_1\ \hat{g} = \hat{0} \Rightarrow f^\lozenge \wedge g^\lozenge = \mathbf{0}$.

$\therefore$ $(X, \delta^\lozenge)$ is fuzzy S-Hausdorff.

Theorem 3.10. $(X, \hat{\delta})$ is $(S - H)_1$ iff $(X, (\hat{\delta})_c)$ is $(S - H)_1$ where $(X, (\hat{\delta})_c)$ is got from $(X, \hat{\delta})$ through the association $¢_6$.

Proof. The proof is immediate from the following two facts.

For $\hat{f}$, $\hat{g} \in \hat{\delta}$ and a second order fuzzy point $\hat{x}_r$ in X,

(1) $\hat{x}_r \in \hat{f} \Leftrightarrow \hat{x}_r \in (\hat{f})_c$

(2) $\hat{f}\ \Lambda_1\ \hat{g} = \hat{0} \Leftrightarrow (\hat{f})_c\ \Lambda_1\ (\hat{g})_c = \hat{0}$.

Theorem 3.11. If $(X, \hat{\delta})$ is $(S - H)_1$, then

(1) For $\varepsilon \in (0, 1)$, $(X, i_\varepsilon(\hat{\delta}))$ is Hausdorff.

(2) $(X, i^*(\hat{\delta}))$ is Hausdorff.

(3) $(X, i(\hat{\delta}))$ is Hausdorff.

Proof. Consider x, y $\in$ X such that $x \neq y$. Consider r, s $\in$ I such that $r > \varepsilon$, $s > \varepsilon$. $\hat{x}_r$, $\hat{y}_s$ in X are distinct.

$\therefore$ There exist $\hat{f}, \hat{g} \in \hat{\delta}$ such that $\hat{x}_r \in \hat{f}$, $\hat{y}_s \in \hat{g}$ and $\hat{f}\ \Lambda_1\ \hat{g} = \hat{0}$.

$$\hat{x}_r \in \hat{f} \Rightarrow \hat{f}(x)(\alpha) \geq r,\ \forall\ \alpha \in I \Rightarrow (\hat{f}(x))^{-1}(\varepsilon, 1] = I \Rightarrow x \in (A_{\hat{f}})_\varepsilon$$

Similarly $y \in (A_{\hat{g}})_\varepsilon$

$$\hat{f}, \hat{g} \in \hat{\delta} \Rightarrow (A_{\hat{f}})_\varepsilon, (A_{\hat{g}})_\varepsilon \in i_\varepsilon(\hat{\delta})$$

Also $\hat{f}\ \Lambda_1\ \hat{g} = \hat{0} \Rightarrow (A_{\hat{f}})_\varepsilon \cap (A_{\hat{g}})_\varepsilon = \varphi$.

$\therefore$ $(X, i_\varepsilon(\hat{\delta}))$ is Hausdorff.

Proofs of (2) and (3) are similar.

Theorem 3.12. If (X, τ) is Hausdorff, then

(1) For $\varepsilon \in (0, 1)$, $(X, (\omega_2(\tau))_\varepsilon)$ is $(S - H)_1$.

(2) $(X, \omega_2(\tau))$ is $(S - H)_1$.

(3) $(X, \omega_2^*(\tau))$ is $(S - H)_1$.

Proof.

(1) Consider two distinct second order fuzzy points $\hat{x}_r$, $\hat{y}_s$ in X. Then $x \neq y$.

$\therefore$ There exist U, V $\in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \varphi$.

$$x \in U \Rightarrow \hat{\chi}_U(x) = 1 \Rightarrow \hat{\chi}_U(x)(\alpha) \geq r,\ \forall\ \alpha \in I \Rightarrow \hat{x}_r \in \hat{\chi}_U$$

Similarly $\hat{y}_s \in \hat{\chi}_V$

Also $U \cap V = \varphi \Rightarrow \hat{\chi}_U\ \Lambda_1\ \hat{\chi}_V = \hat{0}$

$U, V \in \tau \Rightarrow \hat{\chi}_U, \hat{\chi}_V \in \hat{K}_\varepsilon$ where $\hat{K}_\varepsilon$ is a base of $(\omega_2(\tau))_\varepsilon$.

$(\because U = (A_{\hat{\chi}_U})_\varepsilon, V = (A_{\hat{\chi}_V})_\varepsilon)$

Hence $(X, (\omega_2(\tau))_\varepsilon)$ is $(S - H)_1$.

Proofs of (2) and (3) are similar.

Note 3.13. The theorems 2.4 to 2.12 proved for the separation axiom $(S - H)_1$ have exact parallels for the separation axiom $(S - H)_2$ also.

REFERENCES

1. Chang, C.L., Fuzzy Topological Spaces, *J. Math. Anal. Appl.,* 24 (1968) 182-190.

2. George J. Klir and Tina A. Folger, Fuzzy sets, Uncertainty and Informations, *Prentice-Hall of India Private Limited, New Delhi,* 1994.

3. George J. Klir and Bo Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, *Prentice Hall of India Private Limited, New Delhi*, 1997.

4. Kalaichelvi, A., Second order fuzzy topological spaces - I, *Acta Ciencia Indica*, XXXIII M, (3) (2007) 819-826.

5. Kalaichelvi, A., Second order fuzzy topological spaces - II, *International Journal of Mathematical Sciences and Applications*, 1(2) (2011) 551-560.

6. Srivastava, R., Lal, S.N. and Srivastava, A.K., Fuzzy Hausdorff Topological Spaces, *J. Math. Anal. Appl.,* 81 (1981) 497-506.

7. Srivastava, R., Lal, S.N. and Srivastava, A.K., On Fuzzy T_0 and R_0 Topological Spaces, *J. Math. Anal. Appl.,* 136 (1988) 66-73.

8. Zadeh, L.A., Fuzzy Sets, *Information and Control*, 8 (1965), 338-353.