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Decomposition of Fuzzy η-Continuity and Fuzzy A – Continuity

M. Rajamani[1], V. Inthumathi[2] and G. Anusuya Devi[3]
[1]Department of Mathematics, N.G.M. College, Pollachi, Tamilnadu-642001
[2]Department of Mathematics, N.G.M. College, Pollachi, Tamilnadu-642001
Email: inthugops@yahoo.com
[3]Department of Mathematics, Karpagam University, Coimbatore-21
Email: anusuya.hari@gmail.com*

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Abstract. In this paper, we introduce and study the notions of fuzzy η-sets, fuzzy ηζ-sets and we obtain decompositions of fuzzy η-continuity and fuzzy A-continuity in fuzzy topological spaces.

Keywords: Fuzzy η-sets, fuzzy ηζ-sets, fuzzy η-continuity, fuzzy ηζ- continuity, A-continuity.

AMS Mathematics Subject Classification (2010): 54A40

1. Introduction

In 1986, Tong [12] introduced the notions of A-sets and A-continuity in topological spaces and established a decomposition of continuity. In 1989, he [13] introduced the notions of B-sets and B-continuity to obtain another new decomposition of continuity and Ganster and Reilly [4] have improved Tong's decomposition result and provided a decomposition of A-continuity. In 2004, Rajamani and Ambika [9], introduced the notions of fuzzy A-sets and fuzzy B-sets to obtain the decomposition of fuzzy continuity. In 2004, Santhi and Rajamani [11] introduced fuzzy AB-sets and fuzzy AB-continuity in fuzzy topological spaces. In 2006, Noiri and Sayed [6] introduced the notions of η-sets and obtained some decompositions. In 2008, Jafari et.al. [5] introduced the concept of fuzzy C-sets and obtained a decomposition of fuzzy A-continuity.

2. Preliminaries

Throughout this paper X and Y denote fuzzy topological spaces (X, τ) and (Y, σ) respectively. On which no separation axioms are assumed. If λ is a fuzzy set in a fuzzy topological space X, fuzzy closure, fuzzy interior, the fuzzy α-closure and fuzzy α-interior are denoted by $cl(\lambda)$, $int(\lambda)$, $cl_\alpha(\lambda)$ and $int_\alpha(\lambda)$ respectively.

Definition 2.1. A fuzzy set λ in a fuzzy topological space X is called

1. a fuzzy pre-open set [2] if $\lambda \leq$ int(cl (λ)) and a fuzzy pre-closed set if cl (int(λ)) $\leq \lambda$,
2. a fuzzy semi-open set [1] if $\lambda \leq$ cl(int(λ)) and a fuzzy semi-closed set if int (cl (λ)) $\leq \lambda$,
3. a fuzzy α-open set [2] if $\lambda \leq$ int(cl(int(λ))) and fuzzy α -closed set if cl(int(cl(λ))) $\leq \lambda$,
4. a fuzzy regular-open set [2] if λ = int(cl(λ)) and a fuzzy regular-closed set if λ = cl(int(λ)),
5. a fuzzy clopen [3] if and only if it is both fuzzy open and fuzzy closed.

Definition 2.2. A fuzzy set λ in a fuzzy topological space X is called

1. a fuzzy AB-set [11] if $\lambda \in$ FAB(X)) = $\{\alpha \wedge \beta: \alpha \in \tau, \text{int}(\text{cl}(\beta)) \leq \beta \leq \text{cl}(\text{int}(\beta))\}$,
2. a fuzzy A-set [5] if $\lambda \in$ FA(X) = $\{\alpha \wedge \beta: \alpha \in \tau, \beta = \text{cl}(\text{int}(\beta))\}$,
3. a fuzzy B-set [9] if $\lambda \in$ FB(X) = $\{\alpha \wedge \beta: \alpha \in \tau, \text{int}(\text{cl}(\beta)) \leq \beta\}$,
4. a fuzzy LC-set [8] if $\lambda \in$ F LC(X) = $\{\alpha \wedge \beta: \alpha \in \tau, \text{cl}(\beta) = \beta\}$.

The collection of all fuzzy AB-set(resp. fuzzy A-set, fuzzy B-set, fuzzy LC-set) in a fuzzy topological space X will be denoted by FAB(X)(resp. FA(X), FB(X), F LC(X)).

Definition 2.3. A function $f: X \to Y$ is called

1. fuzzy α-continuous [2] if $f^{-1}(\lambda)$ is fuzzy α-open in X, for each fuzzy open set λ in Y,
2. fuzzy pre-continuous [2] if $f^{-1}(\lambda)$ is fuzzy pre-open in X, for each fuzzy open set λ in Y,
3. fuzzy semi- continuous [4] if $f^{-1}(\lambda)$ is fuzzy semi-open in X, for each fuzzy open set λ in Y,
4. fuzzy A-continuous [5] if $f^{-1}(\lambda)$ is fuzzy A-set in X, for each fuzzy open set λ in Y,
5. fuzzy B-continuous [8] if $f^{-1}(\lambda)$ is fuzzy B-set in X, for each fuzzy open set λ in Y,
6. fuzzy AB-continuous [10] if $f^{-1}(\lambda)$ is fuzzy AB-set in X, for each fuzzy open set λ in Y,
7. fuzzy LC-continuous [7] if $f^{-1}(\lambda)$ is fuzzy LC-set in X, for each fuzzy open set λ in Y.

3. Fuzzy η-sets and fuzzy $\eta\zeta$-sets

Definition 3.1. A fuzzy set λ of a fuzzy topological space X is called a fuzzy η-set if $\lambda = \alpha \wedge \beta$, where α is fuzzy open and β is a fuzzy α-closed set in X.

Definition 3.2. A fuzzy set λ of a fuzzy topological space X is called a fuzzy $\eta\zeta$-set if $\lambda = \alpha \wedge \beta$,where α is fuzzy open and β is a fuzzy clopen set in X. The collection of all fuzzy η-sets(resp. fuzzy ηζ-sets) in a fuzzy topological space X will be denoted by Fη (X)(resp. F ηζ(X)).

Definition 3.3. A fuzzy set λ in a fuzzy topological space X is called a fuzzy generalized α-closed (Written as fuzzy gα-closed) in X if $cl_\alpha(\lambda) \leq \alpha$ whenever $\lambda \leq \alpha$ and α is fuzzy open in X.

Theorem 3.4. For a fuzzy set λ in a fuzzy topological space X, the following are hold:

1. Every fuzzy ηζ-set is a fuzzy A-set.
2. Every fuzzy A-set is a fuzzy LC-set.

Proof: 1. Let λ be fuzzy ηζ-set. Then $\lambda = \alpha \wedge \beta$, where α is fuzzy open and β is fuzzy clopen. Since every fuzzy clopen set is fuzzy regular closed, β is fuzzy regular closed. Hence λ is fuzzy A-set in X.

3. Let λ be fuzzy A-set. Then $\lambda = \alpha \wedge \beta$, where α is fuzzy open and β is fuzzy regular closed. Since every fuzzy regular closed set is closed, β is closed. Hence λ is a fuzzy LC-set in X.

Theorem 3.5. For a fuzzy set λ in a fuzzy topological space X, the following are hold:

1. Every fuzzy LC-set is a fuzzy η-set.
2. Every fuzzy η-set is a fuzzy B-set.

Proof: 1.Let λ be fuzzy LC-set. Then $\lambda = \alpha \wedge \beta$, where α is fuzzy open and β is fuzzy closed. Since, every fuzzy closed set is fuzzy α-closed, β is fuzzy α-closed. Hence λ is a fuzzy η-set in X.

2. Let λ be fuzzy η-set. Then $\lambda = \alpha \wedge \beta$, where α is fuzzy open and β is fuzzy α-closed. Since every fuzzy α-closed of fuzzy semi-closed, β is fuzzy semi-closed. Hence λ is fuzzy B-set in X.

For the sets defined above, we have the following implications:

$$\begin{tikzcd}
F\eta\zeta(X) \arrow[r] & FA(X) \arrow[r] \arrow[dd] & FLC(X) \arrow[d] \\
 & & F\eta(X) \arrow[d] \\
 & FAB(X) \arrow[r] & FB(X)
\end{tikzcd}$$

None of these implications are reversible as shown from the following examples.

Example 3.6. Let X = {a, b, c}. Define $\alpha_1, \alpha_2 : X \to [0, 1]$ by

$\alpha_1(a) = 0.6 \quad \alpha_2(a) = 0.2$

$\alpha_1(b) = 0.5$ $\alpha_2(b) = 0.5$
$\alpha_1(c) = 0.2$ $\alpha_2(c) = 0.4$
Let $\tau = \{0, \alpha_1, 1\}$. Then (X, τ) is a fuzzy topological space. Now, α_2 is a fuzzy η-set but not a fuzzy AB-set and not a fuzzy LC-set.

Example 3.7. Let $X = \{a, b, c\}$. Define $\alpha_1, \alpha_2 : X \to [0, 1]$ by
$\alpha_1(a) = 0.3$ $\alpha_2(a) = 0.4$
$\alpha_1(b) = 0.3$ $\alpha_2(b) = 0.5$
$\alpha_1(c) = 0.5$ $\alpha_2(c) = 0.5$
Let $\tau = \{0, \alpha_1, 1\}$. Then (X, τ) is a fuzzy topological space. Now, α_2 is a fuzzy AB-set but not a fuzzy η-set.

Example 3.8. Let $X = \{a, b, c\}$. Define $\alpha_1, \alpha_2 : X \to [0,1]$ by
$\alpha_1(a) = 0.3$ $\alpha_2(a) = 0.7$
$\alpha_1(b) = 0.4$ $\alpha_2(b) = 0.6$
$\alpha_1(c) = 0.4$ $\alpha_2(c) = 0.6$
Let $\tau = \{0, \alpha_1, 1\}$. Then (X, τ) is a fuzzy topological space. Now, α_2 is a fuzzy A-set but not a fuzzy $\eta\zeta$-set.

Remark 3.9. Example 3.6 and Example 3.7 show that the notions of fuzzy η-sets and fuzzy AB-sets are independent.

Theorem 3.10. For a fuzzy set λ of a fuzzy topological space (X,τ), the following are equivalent:
1. λ is fuzzy α-closed,
2. λ is a fuzzy η-set and fuzzy $g\alpha$-closed.

Proof : (1) $\Rightarrow$ (2). Since every fuzzy α-closed set is fuzzy $g\alpha$-closed and every fuzzy α-closed set is a fuzzy η-set.
(2) $\Rightarrow$ (1). Since λ is a fuzzy η-set, then $\lambda = \alpha \wedge cl_\alpha(\lambda)$, where α is fuzzy open and $cl_\alpha(\lambda)$ is fuzzy α-closed. So $\lambda \leq \alpha$ and since λ is fuzzy $g\alpha$-closed then $cl_\alpha(\lambda) \leq \alpha$. Therefore, $cl_\alpha(\lambda) \leq \alpha \wedge cl_\alpha(\lambda) = \lambda$. But $\lambda \leq cl_\alpha(\lambda)$. Therefore, $\lambda = cl_\alpha(\lambda)$. Hence, λ is fuzzy α-closed.

Theorem 3.11. A fuzzy set λ in a fuzzy topological space X is fuzzy open if and only if it is a fuzzy pre-open set and a fuzzy B-set.
Proof: Necessity is trivial.
Sufficiency: Since λ is fuzzy B-set, we have $\lambda = \alpha \wedge \beta$, where α is a fuzzy open set and $int(cl(\beta)) \leq \beta$, which implies $int(cl(\beta)) \leq int(\beta)$. But we know $int(\beta) \leq int(cl(\beta))$. Since λ is a fuzzy pre-open, We have,
$$\begin{aligned}\lambda &\leq int(cl(\lambda)),\\ &= int(cl(\alpha \wedge \beta)),\\ &\leq int(cl(\alpha) \wedge cl(\beta)),\end{aligned}$$

$= int(cl(\alpha)) \wedge int(cl(\beta)),$

$= int(cl(\alpha)) \wedge int\ \beta.$

Now, $\lambda = \alpha \wedge \beta,$

$= (\alpha \wedge \beta) \wedge \alpha,$

$\leq (int\ (cl\ (\alpha)) \wedge int\ (\beta)) \wedge \alpha,$

$= (int\ (cl(\alpha)) \wedge \alpha) \wedge int\ (\beta),$

$= \alpha \wedge int\ (\beta).$

Now $\lambda = \alpha \wedge \beta \geq \alpha \wedge int\ (\beta)$, we have

$\lambda = \alpha \wedge int\ (\beta) = int(\alpha) \wedge int\ (\beta) = int(\alpha \wedge \beta),$

$= int\ (\lambda)$. Therefore, λ is fuzzy open.

Theorem 3.12. Let X be a fuzzy topological space. Then a fuzzy set λ is fuzzy open if and only if it is both fuzzy α-open and a fuzzy A-set.

Proof: Necessity is trivial.

Sufficiency: Let $\lambda = \alpha \wedge \beta$ be a fuzzy A-set. Where α is fuzzy open and $\beta = cl(int(\beta))$. Since λ is a fuzzy α-open set,

$\lambda = \alpha \wedge \beta \leq int(cl(int(\alpha \wedge \beta))),$

$= int(cl(int(\alpha) \wedge int(\beta))),$

$= int(cl(\alpha \wedge int(\beta))),$

$\leq int(cl(\alpha) \wedge cl(int(\beta)),$

$= int(cl(\alpha) \wedge \beta),$

$= int(cl(\alpha)) \wedge int(\beta).$

Since $\alpha \leq int(cl(\alpha))$, we have

$\alpha \wedge \beta = (\alpha \wedge \beta) \wedge \alpha,$

$\leq int(cl(\alpha)) \wedge int(\beta) \wedge \alpha = \alpha \wedge int(\beta).$

Since $\alpha \wedge \beta \geq \alpha \wedge int(\beta)$, and so

$\lambda = \alpha \wedge \beta = int(\alpha) \wedge int(\beta) = int(\alpha \wedge \beta)$. Therefore λ is a fuzzy open set.

Theorem 3.13. For a fuzzy set λ of a fuzzy topological space (X, τ), the following are equivalent:

1. λ is fuzzy open,
2. λ is fuzzy ηζ-set,
3. λ is fuzzy α-open and a fuzzy A-set,
4. λ is fuzzy pre-open and a fuzzy A-set,
5. λ is fuzzy α-open and a fuzzy η-set,
6. λ is fuzzy α-open and fuzzy LC set,

7. λ is fuzzy pre-open and fuzzy LC set,
8. λ is fuzzy pre-open and a fuzzy η-set,
9. λ is fuzzy pre-open and a fuzzy B-set.

Proof : (1) $\Rightarrow$ (2). Since λ is fuzzy open and $\lambda = \lambda \wedge 1$, where the fuzzy set 1 is fuzzy clopen, then λ is a fuzzy $\eta\zeta$-set.
(2) $\Rightarrow$ (3). The proof follows from Theorem 3.12.
(3) $\Rightarrow$ (4). The result is trivial.
(4) $\Rightarrow$ (5). Every fuzzy A-set is a fuzzy η-set. Since λ is a fuzzy A-set, then λ is fuzzy semi-open. Now both fuzzy semi-open and fuzzy pre-open are fuzzy α-open.
(5) $\Rightarrow$ (6). Since λ is a fuzzy η-set, then $\lambda = \alpha \wedge cl_{\alpha}(\lambda)$ where α is fuzzy open and $cl_{\alpha}(\lambda)$ is fuzzy α-closed. Since λ is fuzzy α-open, it is fuzzy β-open set. Therefore,$\lambda \leq cl(int(cl(\lambda)))$.
Now,

$$
\begin{aligned}
cl(cl_{\alpha}(\lambda)) &= cl(\lambda \vee (cl(int(cl(\lambda))))), \\
&= cl(cl(int(cl(\lambda)))), \\
&= cl(int(cl(\lambda))), \\
&= cl_{\alpha}(\lambda).
\end{aligned}
$$

Hence, $cl_{\alpha}(\lambda)$ is fuzzy closed and λ is fuzzy locally closed.
(6) $\Rightarrow$ (7). This is trivial.
(7) $\Rightarrow$ (8). Let λ be fuzzy pre-open and fuzzy locally closed by the definition of fuzzy locally closed $\lambda \in LC(X) = \{\alpha \wedge \beta : \alpha \in \tau, cl(\beta) = \beta\}$.
Since every fuzzy closed set is fuzzy α-closed, β is fuzzy α-closed this implies λ is a fuzzy η-set.
(8) $\Rightarrow$ (9). Assume that λ is a fuzzy pre-open and a fuzzy η-set. Since λ is a fuzzy η-set, $\lambda = \alpha \wedge \beta$, where α is fuzzy open and β is fuzzy α-closed. Since every fuzzy α-closed set is fuzzy semi-closed, λ is a fuzzy B-set.
(9) $\Rightarrow$ (1). The proof follows from Theorem 3.11.

4.Fuzzy η-continuity and fuzzy $\eta\zeta$-continuity

Definition 4.1. A function $f : X \to Y$ is said to be fuzzy η-continuous(resp. fuzzy $\eta\zeta$-continuous, fuzzy generalized α-continuous) if $f^{-1}(V)$ is an fuzzy η –set(resp. fuzzy $\eta\zeta$-set, fuzzy $g\alpha$-closed set) in X for every fuzzy open set V of Y.

$$
\begin{tikzcd}
F\eta\zeta\text{-continuity} \arrow[r] & FA\text{-continuity} \arrow[r] \arrow[dd] & F\ LC\text{-continuity} \arrow[d] \\
 & & F\ \eta\ \text{-continuity} \arrow[d] \\
 & FAB\text{-continuity} \arrow[r] & FB\text{-continuity}
\end{tikzcd}
$$

None of these implications are reversible as shown from the following examples.

Example 4.2. Let $X = \{a, b, c\}$ and α_1, α_2 are fuzzy sets defined by
$\alpha_1(a) = 0.3$ $\alpha_2(a) = 0.7$
$\alpha_1(b) = 0.4$ $\alpha_2(b) = 0.6$
$\alpha_1(c) = 0.4$ $\alpha_2(c) = 0.6$
Let $\tau_1 = \{0, \alpha_1, 1\}$, $\tau_2 = \{0, \alpha_2, 1\}$. Then the identity mapping $f : (X, \tau_1) \to (X, \tau_2)$ is fuzzy A-continuous but not a fuzzy ηζ-continuous.

Example 4.3. Let $X = \{a, b, c, d\}$ and α_1, α_2 are fuzzy sets defined by
$\alpha_1(a) = 0.3$ $\alpha_2(a) = 0.7$
$\alpha_1(b) = 0.4$ $\alpha_2(b) = 0.6$
$\alpha_1(c) = 0.5$ $\alpha_2(c) = 0.5$
$\alpha_1(d) = 0.6$ $\alpha_2(d) =0.4$
Let $\tau_1 = \{0, \alpha_1, 1\}$, $\tau_2 = \{0, \alpha_2, 1\}$. Then the identity mapping $f : (X, \tau_1) \to (X, \tau_2)$ is fuzzy LC-continuous but not a fuzzy A-continuous.

Example 4.4. Let $X = \{a, b, c\}$ and α_1, α_2 are fuzzy sets defined by
$\alpha_1(a) = 0.6$ $\alpha_2(a) = 0.2$
$\alpha_1(b) = 0.5$ $\alpha_2(b) = 0.5$
$\alpha_1(c) = 0.2$ $\alpha_2(c) = 0.4$
Let $\tau_1 = \{0, \alpha_1, 1\}$, $\tau_2 = \{0, \alpha_2, 1\}$. Then the identity mapping $f : (X, \tau_1) \to (X, \tau_2)$ is fuzzy η-continuous but neither fuzzy LC-continuous nor fuzzy AB-continuous.

Remark 4.5. Example 4.4 we can see that fuzzy η-continuity and fuzzy AB-continuity are independent.

Theorem 4.6. For a function $f : (X, \tau) \to (Y, \sigma)$, the following conditions are equivalent:
1. f is fuzzy continuous,
2. f is fuzzy ηζ -continuous,
3. f is fuzzy α-continuous and a fuzzy A-continuous,
4. f is fuzzy pre-continuous and a fuzzy A-continuous,
5. f is fuzzy α-continuous and a fuzzy η-continuous,
6. f is fuzzy α-continuous and a fuzzy LC-continuous,
7. f is fuzzy pre-continuous and a fuzzy LC-continuous,
8. f is fuzzy pre-continuous and a fuzzy η-continuous,
9. f is fuzzy pre-continuous and a fuzzy B-continuous.

Proof: The proof follows from Theorem 3.13.

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